Controllability with Unilateral Control Inputs

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Abstract

Many control systems contain control inputs which are constrained to be non–negative. Unfortunately, the unilateral nature of such inputs makes them similar to a drift term, and correspondingly complicates any attempt to determine controllability. We present a controllability test in which Lie brackets are characterized as "good" or "bad," from which controllability follows from the relationship between these "good" and "bad" brackets. This test is relatively straightforward to apply and its application is illustrated with an example.

1 Introduction

Many interesting and useful control systems have control inputs which are constrained to be non–negative; yet, the issue of *controllability* of such systems has not been fully explored. For example, a control input corresponding to some sort of thruster can only produce a zero or positive thrust (unless the thrusters are arranged in opposing pairs). Similarly, a control input corresponding to the normal force produced by physical contact clearly must also be non–negative. Such unilateral forces arising from physical contact naturally arise in many locomotion problems, wherein an object, such as an animal or robot, moves through its environment via physical contact with surrounding objects.

To our knowledge, the few fundamental results pertaining to controllability for such restricted control inputs appear in work by Sussmann in [1, 2] and by Lynch and Mason in [3]. The results by Sussmann in references [1, 2] related to the case where the inputs are restricted to be non–negative, are low order tests, *i.e.*, the Lie algebra rank condition must be satisfied by vector fields of sufficiently low order. We present a result which is distinct to the extent that it allows the Lie algebra rank condition to be satisfied by vector fields of arbitrarily high order. However, this comes at the expense of a more restrictive condition on the first order vector fields.

2 Unilateral Control Inputs

We will consider control systems of the form

$$\dot{x} = f(x) + h_i(x)v^i + g_j(x)u^j \qquad (1)$$

where $i = 1, \ldots, m$, $j = 1, \ldots, n$, $v^i \in [0, 1)$ $\forall i$, $u^j \in (-1, 1)$ $\forall j$, $x \in M$, M a manifold and f, h_i and g_j vector fields on M; that is, the control inputs v^i are restricted to be non–negative.

Let $\mathcal{U}$ be the set of admissible controls. Given an open set $V \subseteq M$, define $R^V(x_0, T)$ to be the set of states x such that there exists $(v^i, u^j) : [0, T] \to \mathcal{U}$ that steers the control system from $x(0) = x_0$ to $x(T) = x_f$ and satisfies $x(t) \in V$ for $0 \le t \le T$. Define

$$R^V(x_0, \le T) = \bigcup_{0 < \tau \le T} R^V(x_0, \tau)$$

to be the set of states reachable up to time T. A system is *small time locally controllable* ("STLC") if $R^V(x_0, \le T)$ contains a neighborhood of x_0 for all neighborhoods V of x_0 and $T > 0$.

Let $\mathcal{C}$ be the smallest subalgebra of $V^\infty(M)$ (the Lie algebra of smooth vector fields on M) that contains $f, h_1, \ldots, h_m, g_1, \ldots, g_n$, and let C be the *accessibility distribution* generated by $\mathcal{C}$:

$$C(x) = \text{span}\{X(x) : X \in \mathcal{C}\}, \quad x \in M.$$

If $\dim C(x_0) = \dim M$, then the system satisfies the *Lie Algebra Rank Condition* ("LARC") at x_0.

For a given Lie bracket X, consider the *degree of a bracket with respect to a vector field* f, h_i or g_j, denoted by $\delta^f(X)$, $\delta^{h_i}(X)$ and $\delta^{g_j}(X)$, respectively, to be the number of times that the superscripted vector field appears in the bracket X. Now consider the *total degree*, $\delta(X)$, to be

$$\delta(X) = \delta^f(X) + (1 + \epsilon) \sum_{i=1}^{m} \delta^{h_i}(X) + \sum_{j=1}^{n} \delta^{g_j}(X),$$

where $0 < \epsilon \ll 1$. This concept of degree and total degree is presented rigorously in Section 4, but for now we take it to mean that we can determine the degree of a bracket by counting the number of times that a particular vector field appears in the bracket.

Now, we will call a bracket, X "bad" if δ^{g_j} is even (including 0) for each j, $\delta^f(X) + \sum_{i=1}^{m} \delta^{h_i}(X)$ is odd and $\sum_{i=1}^{m} \delta^{h_i}(X) \ne 1$. Otherwise, call the bracket "good."

We can now state our main result.

PROPOSITION 2.1 *Consider the control system described by Equation 1. Assume that the system satisfies the LARC and that there exist coefficients λ_i and α_j such that*

$$\sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{n} \alpha_j g_j(x) = 0 \qquad \forall x \in \text{nbd}(x_0), \quad (2)$$

where $\lambda_i \in (0,1)$ and $\alpha_j \in \mathbb{R}$. Assume further that any bad bracket can be written as a linear combination of brackets of lower total degree. Then the system is STLC at x_0.

The proof of this Proposition appears in Section 5. We note that in the proof of this result, we provide an alternative result given by Proposition 5.1 which is more general and thus less restrictive; however, in practice, this more general test will be more difficult to apply than Proposition 2.1, above.

Although technically difficult to prove, the intuition behind the restriction expressed by Equation 2 is simple. Due to the control input restrictions, none of the control inputs v^i can be negative. However, Equation 2 can be solved for one $-h_i$ in terms of the other h_j's with positive coefficients and the g_k's with arbitrary coefficients. Thus, allowable control inputs ($v^j > 0$, $j \neq i$) effect the same result as a disallowed control input (one $v^i < 0$).

3   Example

Before we present the proof of Proposition 2.1, we will illustrate its application with an example. Here we consider the controllability of rigid body with "thrusters." Initially consider the body to be centered about orthogonal coordinate axes, and let two thrusters be placed at the points where the x–axis intersects the surface of the body and two more thrusters placed at the points where the y–axis intersects the surface. Let the force of the thrusters on the x–axis be in the negative z–direction, and the force of the thrusters on the y–axis be in the positive z–direction. As a fifth control input, let the thrusters rotate by a small angle, ψ (which can be both positive and negative) about their respective axes so that the thrusters aligned on the x– axis rotate in opposite directions so that if they are both "thrusting" they both contribute to a positive torque about the z–axis, and let the thrusters aligned on the y–axis rotate in opposite directions so that both contribute a negative torque about the z–axis. We will consider a spherical body with unit radius and mass $\frac{5}{2}$ (so that the inertia tensor is the identity) as illustrated in Figure 1; although, for a non–spherical body, the following controllability analysis still holds. In Figure 1, for clarity, the coordinate system is shown displaced from the center of mass of the sphere, but for the calculations, the origin of the coordinate system is assumed to initially coincide with its center of mass.

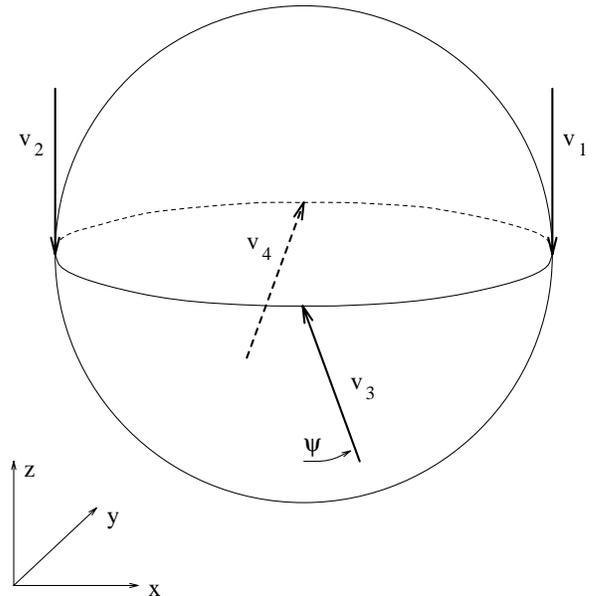Parameterize the configuration space for the system $SE(3) \times S^1$ by the coordinates $X = (x, y, z)$, which

Figure 1. Rigid Body with Thrusters

are the displacements of the center of mass from the fixed inertial frame, $\Phi = (\phi_1, \phi_2, \phi_3)$, which are the "roll, pitch, yaw" rotations about the x– y– and z– axes, respectively and ψ, which is the rotational angle of the thrusters. Thus, a point in the phase space is given by $q = (X, \Phi, \dot{X}, \dot{\Phi}, \psi)$.

Now, we can write the equations of motion as

$$\begin{pmatrix} \dot{X} \\ \dot{\Phi} \\ \ddot{X} \\ \ddot{\Phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \dot{X} \\ \dot{\Phi} \\ 0 \\ -Q^{-1}\dot{Q}\dot{\Phi} \\ 0 \end{pmatrix} + F \cdot v, \qquad (3)$$

where Q is the local mapping which takes the derivatives of the roll, pitch and yaw coordinates we chose for our parameterization and gives the body angular velocities, which given by

$$Q = \begin{pmatrix} \cos\phi_2 \cos\phi_3 & \sin\phi_3 & 0 \\ -\cos\phi_2 \sin\phi_3 & \cos\phi_3 & 0 \\ \sin\phi_2 & 0 & 1 \end{pmatrix}.$$

If we let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ represent the standard unit vectors in the x–, y– and z–directions fixed in the body, R_x and R_y represent the usual rotation matrices representing a rotation by an angle ψ about the x–axis and the y–axis, respectively and R represent the 3×3 rotation matrix that takes body coordinates to spatial

coordinates, then

$$F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{5}{2}RR_x\hat{e}_3 & -\frac{5}{2}RR_x\hat{e}_3 \\ Q^{-1}((R_x\hat{e}_3) \times \hat{e}_2) & Q^{-1}((-R_x\hat{e}_3) \times \hat{e}_2) \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{5}{2}RR_y\hat{e}_3 & \frac{5}{2}RR_y\hat{e}_3 & 0 \\ Q^{-1}((R_y\hat{e}_3) \times e_1) & (Q^{-1}(-R_y\hat{e}_3) \times e_1) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $v = (v^1, \dots, v^4, u)^T$. We will refer to the first four columns in the matrix on the right hand side of the equation for F as $h_1, \dots, h_4$, the fifth column as g and the first column on the right hand side of Equation 3 as f, to notationally correspond to Equation 1, *i.e.*, the thruster forces, v_j are restricted to be non–negative, and the thruster rotation angle, ψ can be either positive or negative.

First we must check that the system satisfies the LARC. Tedious calculations show that the following collection of vector fields spans T_xM everywhere except for the parameterization singularity at $\phi_2 = \frac{\pi}{2}$:

$$\{h_1, h_2, h_3, g, [g, h_1], [g, h_2], [h_1, f], [h_2, f], [h_3, f],$$
$$[g, h_3], [[g, h_1], f], [[g, h_2], f], [[g, h_3], f]\}.$$

Now, we note that the hypothesis of Proposition 2.1 expressed by Equation 2 is satisfied because

$$\sum_j h_j(x) = 0, \quad \forall x \in M.$$

The LARC is satisfied by brackets with total degree $\leq 3 + \epsilon$, so we need to show that all bad brackets with total degree $\leq 3 + \epsilon$ are spanned by brackets with lower total degree. Note, that the only bad bracket with one element is $f(x)$. However, $f(x) = 0$ if $\dot{X} = \dot{\Phi} = 0$. For brackets with three elements, we note that any bad bracket must have zero or two occurrences of the vector field g. If there are zero g's, there must be one or more h's (since $[f, [f, f]] = 0$). If there is only one h, then it is not a bad bracket. If there are two or more h's, then the total degree of the bracket is greater than $3 + \epsilon$. If there are two occurrence of g, then there must either be two g's and one h_i or two g's and one f. In the first case, since there is one h_i, that bracket is not a bad bracket. In the second case, we note that in this example $[g, f](x) = 0$, so that bad bracket can be written as a linear combination of lower order elements. Therefore the system is STLC from any position with zero velocity.

3.1 Simulation Results

In this section we present simulation results which demonstrate the controllability properties of the rigid body example. The following graphs are intended to illustrate that, after a sequence of control inputs, and possibly after a complicated series of gyrations, the system, to leading order, has undergone a net motion in

a particular direction. We only present results for motion in two directions, but note that it is possible to do so for all directions in the 13–dimensional phase space.

For example, consider motion in the x–direction. We note that $\dot{x} = \frac{1}{2}[[g, h_1], f] - \frac{1}{2}[[g, h_2], f]$. Figure 2 illustrates a sequence of motions that the system may undergo to move in the x–direction. Note that the system is momentarily displaced in various states other than purely the x–direction; however, the end result is pure displacement in the x–direction.
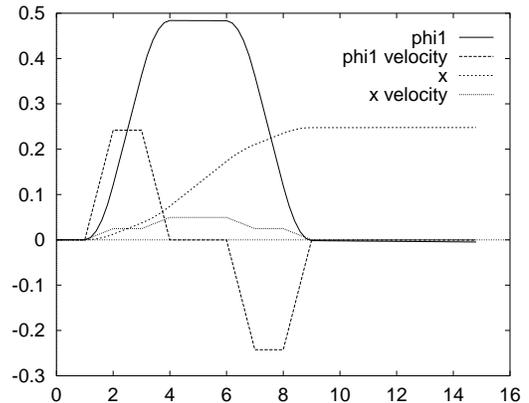
Figure 2. Control Inputs for x Direction

Also, consider motion in the ϕ_3 direction. Note that $\dot{\phi}_3 = \frac{1}{5}([[g, h_1], f] + [[g, h_2], f])$. Figure 3 illustrates a sequence of motions that the system may undergo to move in the ϕ_3–direction. Again, note that the system is momentarily displaced in various states other than purely the ϕ_3–direction; however, the end result is pure displacement in the ϕ_3–direction.
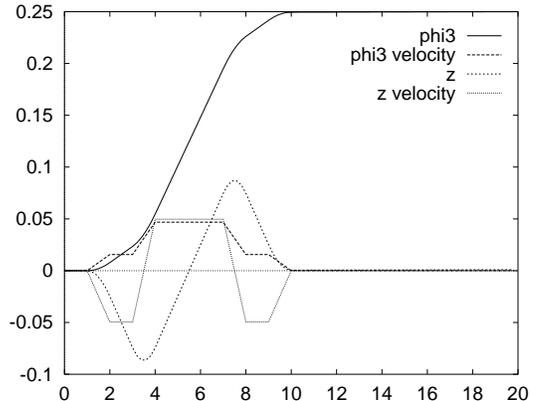
Figure 3. Control Inputs for ϕ_3 Direction

Rather than specifically illustrate how we obtained the sequence of control inputs used for the above simulations, we will discuss the heuristic synthesis technique we used in more general terms. This technique is merely presented to illustrate the means by which we obtained the control inputs for the above examples, and is *not* presented as part of a rigorous theory. (A possible rigorous approach to this synthesis problem would be to extend the averaging results of Leonard

and Krishnaprasad [4] to the context of the problem we are considering.)

Recall that if we denote the flow of a vector field f at time t starting at a point x_0 by $\phi_t^f(x_0)$, we can write

$$\phi_\epsilon^{-g_2} \cdot \phi_\epsilon^{-g_1} \cdot \phi_\epsilon^{g_2} \cdot \phi_\epsilon^{g_1}(x_0) = x_0 + \epsilon^2[g_1, g_2] + \mathcal{O}(\epsilon^3),$$

where $\phi_\epsilon^g \cdot \phi_\epsilon^f$ denotes the composition of the two flows, *i.e.*, the flow along an integral curve of f followed by the flow along an integral curve of g. Thus to "flow" in the direction of a Lie bracket (to leading order), we simply modulate the control inputs associated with two vector fields to execute the sequence of flows illustrated.

Now consider, for example, the third order bracket $[f, [f, g]]$. Writing this in terms of its approximation by flows we have

$$[f, [f, g]] \sim$$
$$\phi_\epsilon^{-f} \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^f \cdot \phi_\epsilon^g \cdot \phi_\epsilon^{-f} \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^{-f} \cdot \phi_\epsilon^g \cdot \phi_\epsilon^f \cdot \phi_\epsilon^f.$$

All the $-f$ terms appearing throughout the above equation are clearly problematic. However, it is actually possible to rewrite the bracket in a manner that will allow it to be executed. First, consider

$$[f, [f, g]] = [-f, [-f, g]] \sim \qquad (4)$$
$$\phi_\epsilon^f \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^{-f} \cdot \phi_\epsilon^g \cdot \phi_\epsilon^f \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^f \cdot \phi_\epsilon^g \cdot \phi_\epsilon^{-f} \cdot \phi_\epsilon^{-f}.$$

Now, the first two $-f$ terms do not affect the flow since we assume that $f(x_0) = 0$. There still is one remaining $-f$ flow. However, we note that it corresponds to a set of four flows which are approximating the bracket $[g, -f]$. However, $[g, -f] = [-f, -g]$ and $[-f, g] \sim \phi_\epsilon^g \cdot \phi_\epsilon^f \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^{-f}$, and when this is substituted into Equation 4, we have

$$[f, [f, g]] = [-f, [-f, g]] \sim$$
$$\phi_\epsilon^g \cdot \phi_\epsilon^f \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^{-f} \cdot \phi_\epsilon^f \cdot \phi_\epsilon^{-g} \cdot \phi_\epsilon^f \cdot \phi_\epsilon^g \cdot \phi_\epsilon^{-f} \cdot \phi_\epsilon^{-f},$$

and so the composed flow $\phi_\epsilon^{-f} \cdot \phi_\epsilon^f = 0$, and thus has no effect on the net flow. Finally, to flow along the negative direction of an integral curve of one of the h_i, we simply flow along the positive direction of the vector field $\sum_{j \neq i} h_j$.

These observations allowed us to determine sequences of control inputs which produced displacements in all 13 states of the system, similar to the results illustrated in Figures 2 and 3

4 Background Material

In this Section, we briefly outline the necessary mathematical objects for the proof. In Section 5, we present the proof for the main result. This presentation is necessarily brief. Complete definitions and more complete explanations of these can be found in Sussmann [5] or Lewis [6]. Other algebraic material is from [7, 8].

Denote the free associative algebra over $\mathbb{R}$ generated by the set of non–commuting indeterminates $\mathbf{X} = \{X_0, X_1, \ldots, X_{m+n}\}$ by $A(\mathbf{X})$ and its homogeneous components of degree N by $A^{N,hom}(\mathbf{X})$. Let $L(\mathbf{X})$ be the free Lie algebra generated by $\mathbf{X}$, and

denote its homogeneous components of degree N by $L^{N,hom}(\mathbf{X})$.

Let $\hat{A}(\mathbf{X})$ denote the set of all *formal power series* $\sum_I a_I X_I$, where $X_I = X_{i_1} X_{i_2} \cdots X_{i_k}$ for the multi–index $I = (i_1, i_2, \ldots, i_k)$ and let $\hat{A}_0(\mathbf{X})$ denote the set of formal power series for which $a_\emptyset = 0$. The *exponential map* is the well defined bijection

$$\exp : \hat{A}_0(\mathbf{X}) \to 1 + \hat{A}_0(\mathbf{X})$$

whose inverse is denoted by log, both of which are defined by their usual series definitions.

Also, let $\hat{L}(\mathbf{X}) \subseteq \hat{A}_0(\mathbf{X})$ be the set of all formal sums $\sum_{N=1}^\infty S_N$ such that each S_N is in $L^{N,hom}(\mathbf{X})$. Note that the exponential map is well defined on $\hat{L}(\mathbf{X})$. The elements of $\hat{A}(\mathbf{X})$ that are of the form $\exp(S)$ for some $S \in \hat{L}(\mathbf{X})$ are the *exponential Lie series in* $X_0, \ldots, X_m$. The set of all such series is denoted by $\hat{G}(\mathbf{X})$, which is a group. The exponential map, restricted to $\hat{L}(\mathbf{X})$ is a bijection from $\hat{L}(\mathbf{X})$ to $\hat{G}(\mathbf{X})$.

Consider the differential equation

$$\dot{S} = S \left(X_0 + \sum_{i=1}^m u_i X_i \right) \qquad (5)$$

for an $\hat{A}(\mathbf{X})$–valued function $t \to S(t)$, with the initial condition $S(0) = 1$. Sussmann [5] notes that the solution to this differential equation exists and is unique, and is given by

$$S(t) = \sum_I \left(\int_0^t u_I \right) X_I \qquad (6)$$

where

$$\int_0^t u_I = \int_0^t \int_0^{\tau_k} \int_0^{\tau_{k-1}} \cdots$$
$$\int_0^{\tau_2} u_{i_k}(\tau_k) u_{i_{k-1}}(\tau_{k-1}) \cdots u_{i_1}(\tau_1) d\tau_1 \cdots d\tau_k.$$

The series $S(T(u(\cdot)))$, with $t \to S(t)$ as above is the *formal power series associated with the control* $u(\cdot)$, and will be denoted by $\mathrm{Ser}(u(\cdot))$, where $T(u(\cdot))$ is the terminal time of $u(\cdot)$.

Let $\mathcal{U}$ be the set of all functions $u(\cdot)$ whose domain is a compact interval of the form $[0, T]$ such that $u(\cdot)$ takes values in $\mathbb{R}^m$ and is Lebesgue integrable on $[0, T]$. Now, if K is an arbitrary subset of $\mathbb{R}^m$, then we can consider $\mathcal{U}_m(K)$, the subsemigroup of $\mathcal{U}_m$ whose elements are the K–valued controls. The image of $\mathcal{U}_m(K)$ under Ser will be denoted by $\hat{S}(\mathbf{X}, K)$.

If V is a linear space over $\mathbb{R}$, a *group of dilations* of V is a mapping $\rho \to \Delta(\rho)$ that assigns to every real $\rho > 0$ a linear endomorphism $\Delta(\rho) : V \to V$, in such a way that

1. $\Delta(1) = $ identity,
2. $\Delta(\rho_1)\Delta(\rho_2) = \Delta(\rho_1\rho_2) \quad \forall \rho_1, \rho_2$,
3. V has a direct sum decomposition $V = \oplus V_j$ such that the subspaces V_j are invariant under the $\Delta(\rho)$, and the action of $\Delta(\rho)$ on each V_j is given by multiplication by ρ^{α_j} for some $\alpha_j \geq 0$.

Note that any $v \in V$ can be expressed in a unique way as a sum $\sum_j v_j$, $v_j \in V_j$. The Δ–*degree of* v is the largest α_j such that $v_j \neq 0$. A group of dilations is called *strict* if it has no component of degree zero. We say that Δ is *compatible with* $\hat{S}(\mathbf{X}, K)$ if $\Delta(\rho)\,(X_0 + \sum_{i=1}^m u_i X_i)$ is of the form $T\,(X_0 + \sum_{i=1}^m v_i X_i)$ for some $T > 0$, where $(v_1, \ldots, v_m) \in K$, whenever $0 < \rho \leq 1$ and $(u_1, \ldots, u_m) \in K$.

If $\mathbf{f} = (f_0, \ldots, f_m)$ is an $(m+1)$–tuple of C^∞ vector fields on a C^∞ manifold M, then define the map $\mathrm{Ev}(\mathbf{f})$ which assigns to every element of $L(\mathbf{X})$ the vector fields obtained by plugging in each f_i for the corresponding X_i. If $p \in M$, then define the map $\mathrm{Ev}_p(\mathbf{f})$ from $L(\mathbf{X})$ to $T_p M$, given by $\mathrm{Ev}_p(\mathbf{f})(Z) = \mathrm{Ev}(\mathbf{f})(Z)(p)$. If $Z \in L(\mathbf{X})$ is Δ–homogeneous, we say that Z is Δ–*neutralized* for $\mathbf{f}$ at p if $\mathrm{Ev}_p(\mathbf{f})(Z)$ can be expressed as a sum of vectors $\mathrm{Ev}_p(\mathbf{f})(Q_i)$, where the Q_i are elements of $L(\mathbf{X})$ of lower Δ–degree than Z. We will refer to the triple $(M, \mathbf{f}, K)$ as a control system.

The class of controls is embedded as a subsemigroup $\hat{S}(\mathbf{X}, K)$ of the group $\hat{G}(\mathbf{X}) = \{\exp(Z) : Z \in \hat{L}(\mathbf{X})\}$. An automorphism λ of $L(\mathbf{X})$ gives rise to a mapping $\hat{\lambda}$, where, if $Z = \sum_{i=1}^\infty P_i$, where P_i is homogeneous of degree i, then $\hat{\lambda}(Z) = \sum_{i=1}^\infty \lambda(P_i)$. Also, define $\lambda^\#$ from $\hat{G}(\mathbf{X})$ to $\hat{G}(\mathbf{X})$ by letting $\lambda^\#(\exp(Z)) = \exp(\hat{\lambda}(Z))$ for $Z \in \hat{L}(\mathbf{X})$.

An *input symmetry* is an automorphism λ of $L(\mathbf{X})$ such that the corresponding map $\lambda^\#$ maps $\hat{S}(\mathbf{X}, K)$ to $\hat{S}(\mathbf{X}, K)$. A linear map $\lambda : L(\mathbf{X}) \to L(\mathbf{X})$ is *graded* if λ maps $L^{j,hom}(\mathbf{X})$ into $L^{j,hom}(\mathbf{X})$ for each j.

Finally, we say that an element of $L(\mathbf{X})$ is *totally odd* if all its homogeneous components have odd degree.

Our result will follow from the following theorem, which is a corollary to the main result in [2].

THEOREM 4.1 *Let $(M, \mathbf{f}, K)$ be a control system, and let $p \in M$. Assume that $\mathbf{f}$ satisfies the LARC at p, and that there exists (a) an admissible group of dilations Δ of $L^{1,hom}(\mathbf{X})$ which is compatible with $\hat{S}(\mathbf{X}, K)$, (b) a finite group Λ_0 of graded linear maps from $L(\mathbf{X})$ to $L(\mathbf{X})$ that are input symmetries, such that every totally odd Λ_0–fixed element of $L(\mathbf{X})$ is Δ–neutralized for $\mathbf{f}$ at p. Then $(M, \mathbf{f}, K)$ is STLC from p.*

5 Proof of Proposition 2.1

We will prove Proposition 2.1 as a corollary to a more general result.

Let $\mathbf{X} = \{X_0, X_1, \ldots, X_m, X_{m+1}, \ldots X_{m+n}\}$ and $\mathbf{f} = \{f, h_1, \ldots, h_m, g_1, \ldots, g_n\}$ so that $\mathrm{Ev}(f)$ takes X_0 to f, X_i to h_i for $i = 1, \ldots, m$ and X_j to g_{j-m} for $j = m+1, \ldots, m+n$. The f, h_i and g_j above correspond to the vector fields in the Equation 1. Let $\mathrm{Br}(X)$ be the set of "brackets" of elements from $\mathbf{X}$ and $\delta^i(B)$ be the number of occurrences of X_i in $B \in \mathrm{Br}(\mathbf{X})$.

Consider the automorphism generated by σ_i, $i = m+1, \ldots, m+n$ where σ_i sends X_j to X_j if $j \neq i$ and

X_i to $-X_i$. Clearly, a Λ_0–fixed element of $L(\mathbf{X})$ cannot have an odd number of each $X \in \{X_{m+1}, \ldots, X_{m+n}\}$. Thus, we can consider only elements with an even number of each $X \in \{X_{m+1}, \ldots, X_{m+n}\}$, so we will call an element $B \in \mathrm{Br}(\mathbf{X})$ *bad* if δ^b is even for each $b = m+1, \ldots, m+n$ and $\delta^0 + \sum_{a=1}^m \delta^a$ is odd. A bracket is *good* if it is not bad. Note that these definitions of good and bad are slightly different than those that appeared in the paragraph before Proposition 2.1. This difference will be addressed later in the proof. Let S_m denote the permutation group on m symbols. For $\pi_m \in S_m$ and $\pi_n \in S_n$ define $\overline{\pi}(B)$ to be the bracket obtained by fixing X_0, sending X_a to $X_{\pi_m(a)}$ for $i = 1, \ldots m$ and sending X_b to $X_{(\pi_n(b-m))+m}$ for $b = m+1, \ldots, m+n$. Now define the symmetrization operator

$$\beta(B) = \sum_{\pi_n \in S_n} \sum_{\pi_m \in S_m} \overline{\pi}(B). \tag{7}$$

Now, let $\theta \geq 1$ be a real number, and define $\Delta(\rho)$ by

$$\Delta(\rho) : (X_0, \ldots, X_{m+n}) \mapsto \tag{8}$$
$$(\rho X_0, \rho^\theta X_1, \ldots, \rho^\theta X_m, \rho X_{m+1}, \ldots, \rho X_{m+n}).$$

This dilation is compatible with $\hat{S}(X, K)$ by construction. Note also that this dilation makes each $\overline{\pi}$ a graded linear map. The Δ–degree is given by

$$\delta_\theta(B) = \delta^0(B) + \theta \sum_{i=1}^m \delta^{h_i}(B) + \sum_{i=1}^n \delta^{i+m}(B).$$

The following is a slightly simplified formulation of Theorem 4.1.

PROPOSITION 5.1 *Consider the bijection $\phi : \mathbf{X} \to \mathbf{f}$ which sends X_0 to f, X_a to h_a for $a = 1, \ldots, m$, and X_b to g_{b-m} for $b = m+1, \ldots m+n$. Suppose that the system described by Equation 1 is such that every bad bracket $B \in Br(\mathbf{X})$ has the property that*

$$Ev_x(\phi)(\beta(B)) = \sum_{a=1}^k \xi^a Ev_x(\phi)(C_a) \tag{9}$$

where $\xi^i \in \mathbb{R}$ and $\delta_\theta(C_a) < \delta_\theta(B)$ for $a = 1, \ldots k$. Also suppose that 1 satisfies the LARC at x. Then the system described by Equation 1 is STLC at x.

Proof: Provided that the dilation defined by Equation 8 is compatible with $\hat{S}(\mathbf{X}, K)$ and that the collection of all $\overline{\pi}$'s comprise a group which is an input symmetry, then this is simply a restatement of Theorem 4.1. As previously mentioned, the dilation defined by Equation 8 is compatible with $\hat{S}(\mathbf{X}, K)$ by construction.

We need to show that the group comprised of all the $\overline{\pi}$'s is an input symmetry. Define $\overline{\pi}^\#$ by $\overline{\pi}^\#(\exp Z) = \exp(\hat{\overline{\pi}}(Z))$ $Z \in \hat{L}(\mathbf{X})$, where $\hat{\overline{\pi}} : \hat{L}(\mathbf{X}) \to \hat{L}(\mathbf{X})$ is given by $\hat{\overline{\pi}}(Z) = \sum_{i=1}^\infty \overline{\pi}(P_i)$, if $Z = \sum_{i=1}^\infty P_i$, where each P_i is homogeneous of degree i. Clearly, $\overline{\pi}^\#$ simply fixes X_0, sends X_a to $X_{\pi_m(a)}$ for $i = 1, \ldots m$ and sends X_b

to $X_{\pi_n(b-m)+m}$ for $b = m+1, \ldots, m+n$ for each term in the infinite series.

Now, from Equation 6, we can write

$$\overline{\pi}^{\#}(S(t)) = \sum_I \left(\int_o^t u_I \right) X_{\pi(I)}$$

where, for $I = \{i_1, \ldots, i_k\}$, $\pi(I) = \{\pi_{mn}(i_1), \ldots, \pi_{mn}(i_k)\}$, where mn is either m or n, depending upon whether $i \in \{1, \ldots, m\}$ or $i \in \{m+1, \ldots, m+n\}$ respectively. However,

$$\sum_I \left(\int_o^t u_I \right) X_{\pi(I)} = \sum_I \left(\int_o^t u_{\pi^{-1}(I)} \right) X_I$$

since the summation is over all possible multi–indices I. Since π^{-1} maps K to K, it follows that $\overline{\pi}^{\#}$ maps $\hat{S}(X, K)$ to $\hat{S}(X, K)$. ∎

Now, the proof to Proposition 2.1 follows easily.

First, scale the coefficients λ_i and α_j in Equation 2 so that

$$\sum_i |\lambda_i| + \sum_j |\alpha_j| = 1. \tag{10}$$

Now, observe that if the system

$$\dot{x} = f(x) + \tilde{h}_i(x)v^i + \tilde{g}_j(x)u^j \tag{11}$$

where $\tilde{h}_i = \lambda_i h_i$ and $\tilde{g}_j = \alpha_j g_j$, where λ_i and α_j are from Equation 10, is controllable, then so is the system described by Equation 1 (because we have effectively further restricted the set of allowable control inputs).

Note that the definition of "good" and "bad" defined in this Appendix is slightly different from that defined in the paragraph before Proposition 2.1. In particular, Proposition 2.1 required $\sum_{i=1}^m \delta^{h_i}(B) \neq 1$, for B to be bad, but this is not the case for a bad bracket as defined in the Appendix. Otherwise, the definitions of good and bad brackets correspond.

Thus, if B is a bad bracket such that $\sum_{i=1}^m \delta^{h_i}(B) \neq 1$, then Proposition 2.1 is simply a restatement of Proposition 5.1 where, instead of requiring the symmetrization of a bad bracket, defined in Equation 7, to be Δ–neutralized, each term in the sum which defines the symmetrization must be individually Δ–neutralized.

Now if B is such that $\sum_{i=1}^m \delta^{h_i}(B) = 1$, then $\beta(B)$ is of the form

$$\beta(B) = \sum_{\pi_n \in S_n} \overline{\pi}(\tilde{B})$$

where where the single $X \in \{X_1, \ldots, X_m\}$ in B is replaced by $X_1 + \cdots + X_m$ in $\tilde{B}$ by the operation of $\sum_{\pi_m \in S_m} \overline{\pi}(B)$. Recall that, by assumption, $\sum_{i=1}^m \tilde{h}_i = -\sum_{j=1}^n \tilde{g}_j$. Thus, if we let $\hat{B}$ be the bracket $\tilde{B}$ with the term $X_1 + \cdots + X_m$ replaced by $-(X_{m+1} + \cdots + X_{m+n})$, then, we have $\text{Ev}_x(\phi)(\beta(B)) = \text{Ev}_x(\phi)(\sum_{\pi_n \in S_n} \hat{B})$. Since $\delta_\theta(\sum_{\pi_n \in S_n} \hat{B}) < \delta_\theta(\beta(B))$, the hypotheses of Proposition 5.1 is satisfied.

6 Conclusions

In this paper, we have presented and illustrated the application of a controllability test for control systems which may have inputs constrained to be non–negative. Although technically difficult to prove, this result is relatively straightforward to use in applications. We note that the results contained herein can be improved. Roughly, we have treated the vector fields corresponding to the constrained inputs in a manner similar to that for the drift term. Because of the hypothesis expressed in Equation 2, it seems that it should be possible to treat the constrained inputs in a manner more analogous to that of the normal input terms.

Of course, there are many other important related problems, not the least of which is the control synthesis problem. The controllability test presented is useful mainly as a tool for *analysis*. Clearly solutions to problems such as stabilization and trajectory generation for this class of systems are extremely important and should be natural extensions of this work. Also, as previously mentioned, possibly averaging techniques could be used to obtain constructive controllability results.

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