University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
Third Exam

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ID Number: SOLUTIONS

NAME: ____________________

- Do not start or turn the page until instructed to do so.
- You have 50 minutes to complete this exam.
- This is an open book exam. You may consult the course text and anything you have written in it, but nothing else.
- You may **not** use a calculator or other electronic device.
- There are three problems. Problems 1 and 2 are worth 35 points and Problem 3 is worth 30 points.
- The second part of Problem 1 requires some thought. It may be best to save that for last (so that you have plenty of time to savor the experience).
- Your grade on this exam will constitute 15% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in the space provided on each page. If you need more space, use the back of the pages or use additional sheets of paper as necessary.
- If you do not have a stapler, do not take the pages apart.

Great minds discuss ideas; Average minds discuss events; Small minds discuss people.
–Eleanor Roosevelt

1. Consider a vibrating string described by the one dimensional wave equation

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) = \hat{T}\frac{\partial^2 u}{\partial x^2}(x,t),$$

where $\rho = 2$, $\hat{T} = 4$, $L = 3$, $u(0,t) = 0$, $u(L,t) = 0$ and

$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 1, & a < x < b \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < a < b < L$. This models an impact on the string along the length of the string between a and b.

(a) Determine the solution. (25 points)

(b) Figure 1 illustrates $\sin(n\pi x/L)$ (left) and $\cos(n\pi x/L)$ (right), for $n = 1$ and $n = 10$. Consider the two cases

i. $a = 0.45$ and $b = 0.55$

ii. $a = 0.1$ and $b = 0.2$.

In which case will mode 1 be larger? In which case will mode 10 be larger? Explain your answer by specifically referring to features from the plots in Figure 1. (10 points)

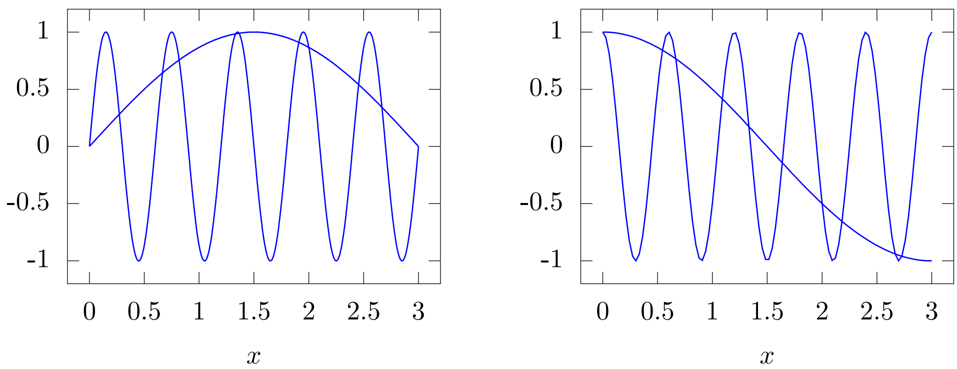

Figure 1. Plots of $\sin(n\pi x/L)$ (left) and $\cos(n\pi x/L)$ (right) for Problem 1 with $n = 1$ and $n = 10$.

Solution:

(a) This is the usual wave equation, so the solution is given in the book, for example in Section 11.1.4, by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\left(a_n \sin\frac{\alpha n\pi t}{L} + b_n \cos\frac{\alpha n\pi t}{L}\right)\right]$$

where

$$a_n = \frac{2}{\alpha n\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

Since $f(x) = 0$, $b_n = 0$. Evaluating the sin coefficients with $\alpha = \sqrt{2}$ and $L = 3$ gives

$$a_n = \frac{\sqrt{2}}{n\pi}\int_a^b \sin n\pi x 3 dx = -\frac{3\sqrt{2}}{n^2\pi^2}\left(\cos\frac{n\pi b}{3} - \cos\frac{n\pi a}{3}\right).$$

So

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{3\sqrt{2}}{n^2\pi^2}\left(\cos\frac{n\pi a}{3} - \cos\frac{n\pi b}{3}\right)\sin\frac{n\pi x}{L}\sin\frac{\alpha n\pi t}{L}.}$$

(b) The difference between a and b is the same in each case. Hence, the size of the Fourier coefficient will be largest when the slope of cos is largest.

- For the first mode, the slope is largest in the middle and small near the ends, hence the first mode will be largest for case (a).
- For the tenth mode, the slope in the middle is small and the slope between $x = 0.1$ and $x = 0.2$ is fairly large. Hence, mode 10 will be largest for case (b).

2. Consider the one-dimensional wave equation with damping

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + b\frac{\partial u}{\partial t}(x,t), \tag{1}$$

where $u(0,t) = u(L,t) = 0$ and

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

(a) Assume a solution of the form $u(x,t) = X(x)T(t)$ and determine the ordinary differential equations that $X(x)$ and $T(t)$ satisfy.

(b) Determine the solutions for $X(x)$ $T(t)$.
- You only have to find a solution for $X(x)$ which satisfies the boundary consitions. You do not need to work exhaustively through all the possible cases.
- Also assume light damping that satisfies $(bL)^2 < 4\pi$.

(c) Write the solution $u(x,t)$ as an infinite series, which satisfies Equation 1 and the boundary conditions.

(d) Determine an expression for any constants that appear in your solution from the previous part.

(e) Indicate in your solution the feature that corresponds to adding damping and why it would have an effect that would be expected from damping. Does the damping affect the lower or higher modes more?

Solution

(a) Assuming $u(x,t) = X(x)T(t)$ and substituting gives

$$X''(x)T(t) = X(x)T''(t) + X(x)T'(t) \quad \Longleftrightarrow \quad \frac{X''(x)}{X(x)} = \frac{T''(t) + bT'(t)}{T(t)} = -\lambda$$

where λ is a constant. Hence

$$\boxed{X''(x) + \lambda X(x) = 0, \qquad T''(t) + bT'(t) + \lambda T(t) = 0.}$$

(b) Assume that to satisfy the homogeneous bounday conditions, $\lambda > 0$ and hence

$$X(x) = c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x.$$

The boundary condition at $x = 0$ gives $c_2 = 0$. The boundary conditon at $x = L$ gives

$$\sqrt{\lambda}L = n\pi \quad \Longrightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2$$

for $n = 1, 2, \ldots, n$. Hence, for each different n there is a solution

$$\boxed{X_n(x) = c_n \sin\frac{n\pi x}{L}.}$$

Substituting λ into the equation for $T(t)$ gives

$$T''(t) + bT'(t) + \left(\frac{n\pi}{L}\right)^2 T(t) = 0.$$

Assuming a solution of the form

$$T(t) = e^{rt}$$

gives the characteristic equation

$$r^2 + br + \left(\frac{n\pi}{L}\right)^2 = 0$$

so

$$r = -\frac{b}{2} \pm i\frac{\sqrt{4n^2\pi^2 - b^2L^2}}{2L}$$

and hence

$$\boxed{T_n(T) = e^{-bt/2}\left(a_n \sin\omega t + b_n \cos\omega t\right),}$$

where

$$\boxed{\omega_n = \frac{\sqrt{4n^2\pi^2 - b^2L^2}}{2L}.}$$

(c) Since the equation is linear, any linear combination of $X_n(x)T_n(t)$ is also a solution; hence,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} e^{-bt/2}\left(a_n \sin\omega_n t + b_n \cos\omega_n t\right)}$$

where the constant in the solution for $X_n(x)$ was absorbed into a_n and b_n.

(d) At $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

so

$$\boxed{b_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L} f(x)dx}$$

which is the same.

Also

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} e^{-bt/2}\left[\left(-\frac{b}{2}\left(a_n \sin\omega_n t + b_n \cos\omega_n t\right)\right) + \omega_n\left(a_n \cos\omega_n t - b_n \sin\omega_n t\right)\right].$$

(be sure to use the product rule. and hence

$$u(x,0) = \sum_{n=0}^{\infty} \sin\frac{n\pi x}{L}\left(-\frac{b}{2}b_n + \omega_n a_n\right) = g(x).$$

Multiplying by $\sin(m\pi x/L)$ and integraging with respect to x along the length of the string gives

$$\frac{L}{2}\left(-\frac{b}{2}b_n + \omega_n a_n\right) = \int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

or

$$\boxed{a_n = \frac{2L}{\sqrt{4n^2\pi^2 - b^2L^2}}\left(\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx + \frac{b}{2}b_n\right)}$$

where the value for ω_n was substituted.

(e) The exponential decay term $exp(-bt/2)$ makes the solution decay and is a consequence of the damping. The rate of decay is the same for each mode.

3. Figure 2 illustrates solutions in the phase plane of

$$\ddot{x} + 0.3\dot{x} - 4x + x^2 = 0 \tag{2}$$

for various initial conditions.

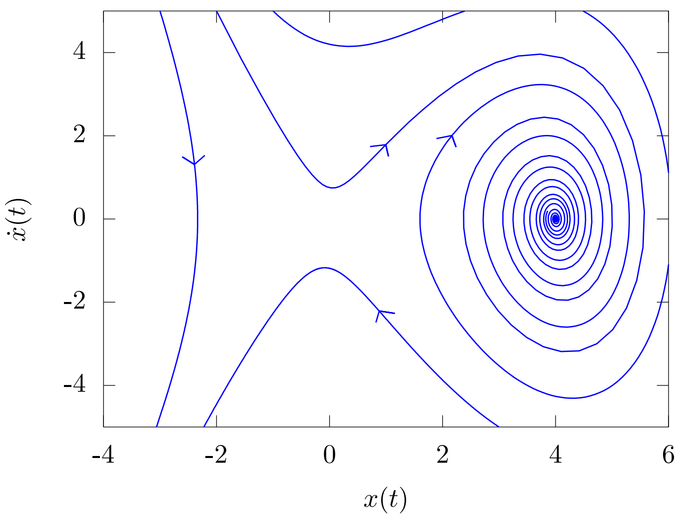

Figure 2. Phase plane solutions to Equation 2.

(a) Determine the equilibrium points. (10 points)

(b) Determine a linear ordinary differential equation that approximates Equation 2 near the point $(4, 0)$. (10 points)

(c) Solve the differential equation you determined in the previous part and indicate the manner to which it corresponds to the solutions in Figure 2. (10 points)

(a) If $\ddot{x} = \dot{x} = 0$, then

$$-4x + x^2 = 0 \quad \Longrightarrow \quad x(x-4) = 0$$

so $x = 0$ and $x = 4$ are the equilibrium points, which corresponds to what appears in Figure 2 as well.

(b) Near $x = 4$ the Taylor eeries for the nonlinear term is

$$x^2 = 16 + 8(x-4) + \cdots . \tag{3}$$

Therefore, the linear differential equation that approximates the nonlinear differential equation near $x = 4$ is

$$\ddot{x} + 0.3\dot{x} - 4x + 16 + 8(x-4) = 0 \quad \Longrightarrow \quad \boxed{\ddot{x} + 0.3\dot{x} + 4x = 16.}$$

(c) This is a linear, constant coefficient differential equaiton. The homogeneous solution is determined by assuming an exponential solution, which has a characteristic equation

$$\lambda^2 + 0.3\lambda + 4 = 0$$

which has roots

$$\lambda = -\frac{0.3}{2} \pm i\frac{\sqrt{15.91}}{2},$$

and thus

$$x_h = e^{-0.15t}\left(c_1 \cos \omega t + c_2 \sin \omega t\right)$$

where

$$\omega = \frac{\sqrt{15.91}}{2}.$$

The particular solution can be determined using undetermined coefficients. Since 16 is a zeroth order polynomial, assume $x_p = A$, which gives $A = 4$. Hence

$$\boxed{x(t) = e^{-0.15t}\left(c_1 \sin \omega t + c_2 \cos \omega t\right) + 4.}$$

These are decaying oscillations about $x = 4$, which corresponds to the spirals in Figure 2.