Homework 1 Solution

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1. Given $f = z^3 + zx + x^4y - 3y = 0$
(a) $\because \frac{\partial f}{\partial x} = z + 4x^3y, \frac{\partial f}{\partial y} = x^4 - 3, \frac{\partial f}{\partial z} = 3z^2 + x$
$\therefore \frac{\partial z}{\partial x}|_y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{z+4x^3y}{3z^2+x}, \frac{\partial z}{\partial y}|_x = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{x^4-3}{3z^2+x}$
(b)For $(x, y) = (1, 2) \Rightarrow z = -0.6894 \pm 1.5575i, 1.3788$,
$\frac{\partial z}{\partial x}|_y = -\frac{z+8}{3z^2+1}, \frac{\partial z}{\partial y}|_x = \frac{2}{3z^2+1}$

8. Find the length of the shortest curve between two points with cylindrical coordinates $(r, \theta, z) = (a, 0, 0)$ and $(r, \theta, z) = (a, \Theta, Z)$ along the surface of the cylinder $r = a$.
$\begin{cases} \xi^1 &= r\cos\theta \\ \xi^2 &= r\sin\theta \\ \xi^3 &= z \end{cases}$. So the Jacobian matrix $J = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the metric tensor $G = J^T \cdot J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
So the distance between $(a, 0, 0)$ and (a, Θ, Z) is $\begin{pmatrix} 0 & \Theta & \text{Z} \end{pmatrix} \cdot G \cdot \begin{pmatrix} 0 \\ \Theta \\ \text{Z} \end{pmatrix} = \sqrt{r^2\Theta^2 + Z^2}$

10. Given $I = \int_0^1 (x^2y'^2 + 40x^4y)dx$, using Euler equation, get $40x^4 - \frac{d}{dx}(2x^2y') = 0 \Rightarrow 2x^2y' = 8x^5 + c_1$. For $x \neq 0$, get $y' = 4x^3 + \frac{c_1}{2x^2} \Rightarrow y = x^4 - \frac{c_1}{2x} + c_2$. For $x = 0 \Rightarrow c_1 = 0$. Given $y(0) = 0, y(1) = 1 \Rightarrow c_2 = 0 \Rightarrow$ the extremum $y = x^4$

13. Given $u = \frac{x+y}{x-y}, v = \frac{xy}{(x-y)^2}$,
$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ -\frac{y(x+y)}{(x-y)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix} = 0 \Rightarrow u, v$ dependent

16. Given $\begin{matrix} \xi^1 &= \cosh x^1 \cos x^2 \\ \xi^2 &= \sinh x^1 \sin x^2 \\ \xi^3 &= x^3 \end{matrix}$

- $J = \begin{pmatrix} \sinh x^1 \cos x^2 & -\cosh x^1 \sin x^2 & 0 \\ \cosh x^1 \sin x^2 & \sinh x^1 \cos x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $G = J^T \cdot J$
$= \begin{pmatrix} \frac{\cosh 2x^1 - \cos 2x^2}{2} & 0 & 0 \\ 0 & \frac{\cosh 2x^1 - \cos 2x^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $\because \cosh x^1 \cdot \sinh x^1 = 1$,
$(\frac{\xi^1}{\cosh x^1})^2 - (\frac{\xi^1}{\sinh x^1})^2 = \cos 2x^2 = \sqrt{1 - \xi^1\xi^2}$
$(\frac{\xi^1}{\cosh x^1})^2 + (\frac{\xi^1}{\sinh x^1})^2 = 1$,
Inverse Transformation is
$\begin{cases} x^1 &= \texttt{arccosh}\frac{\xi^1}{\cos x^2} = \texttt{arccosh}\frac{\xi^1}{\cos(\frac{1}{2}\arcsin 2\xi^1\xi^2)} \\ x^2 &= \frac{1}{2}\arcsin 2\xi^1\xi^2 \\ x^3 &= \xi^3 \end{cases}$
- As for plot, if x^1 =constant, $\frac{\xi^2}{\xi^1} = c_1 \tan x^2$, where $c_1 = \tanh x^1$; if x^2 =constant, $\frac{\xi^2}{\xi^1} = c_2 \tanh x^1$, where $c_2 = \tan x^2$

17. Similar with the example 1.6 in textbook.

$\because \nabla \cdot u = \frac{\partial u^i}{\partial \xi^i} + \Gamma^i_{il} u^l$

$\Gamma^i_{il} u^l = \frac{\partial^2 \xi^p}{\partial x^i \partial x^l} \frac{\partial x^i}{\partial \xi^p} u^l$

$= \frac{\partial^2 \xi^1}{\partial x^1 \partial x^1} \frac{\partial x^1}{\partial \xi^1} u^1 + \frac{\partial^2 \xi^1}{\partial x^1 \partial x^2} \frac{\partial x^1}{\partial \xi^1} u^2 + \frac{\partial^2 \xi^1}{\partial x^2 \partial x^1} \frac{\partial x^2}{\partial \xi^1} u^1 + \frac{\partial^2 \xi^1}{\partial x^2 \partial x^2} \frac{\partial x^2}{\partial \xi^1} u^2 +$
$\frac{\partial^2 \xi^2}{\partial x^1 \partial x^1} \frac{\partial x^1}{\partial \xi^2} u^1 + \frac{\partial^2 \xi^2}{\partial x^1 \partial x^2} \frac{\partial x^1}{\partial \xi^2} u^2 + \frac{\partial^2 \xi^2}{\partial x^2 \partial x^1} \frac{\partial x^2}{\partial \xi^2} u^1 + \frac{\partial^2 \xi^2}{\partial x^2 \partial x^2} \frac{\partial x^2}{\partial \xi^2} u^2$

and

$$
\begin{aligned}
\frac{\partial^2 \xi^1}{\partial x^1 \partial x^1} &= \cosh x^1 \cos x^2 \\
\frac{\partial^2 \xi^1}{\partial x^1 \partial x^2} &= -\sinh x^1 \sin x^2 = \frac{\partial^2 \xi^1}{\partial x^2 \partial x^1} \\
\frac{\partial^2 \xi^1}{\partial x^2 \partial x^2} &= -\cosh x^1 \cos x^2 \\
\frac{\partial^2 \xi^2}{\partial x^1 \partial x^1} &= \sinh x^1 \sin x^2 \\
\frac{\partial^2 \xi^2}{\partial x^1 \partial x^2} &= \cosh x^1 \cos x^2 = \frac{\partial^2 \xi^2}{\partial x^2 \partial x^1} \\
\frac{\partial^2 \xi^2}{\partial x^2 \partial x^2} &= -\sinh x^1 \sin x^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial x^1}{\partial \xi^1} &= \frac{\sec(\frac{1}{2}\arcsin(2\xi^1\xi^2)) + \frac{\xi^1\xi^2 \sec(\frac{1}{2}\arcsin(2\xi^1\xi^2))\tan(\frac{1}{2}\arcsin(2\xi^1\xi^2))}{\sqrt{1-4(\xi^1\xi^2)^2}}}{\sqrt{-1+\xi^1\sec(\frac{1}{2}\arcsin(2\xi^1\xi^2))}\sqrt{1+\xi^1\sec(\frac{1}{2}\arcsin(2\xi^1\xi^2))}} \\
\frac{\partial x^2}{\partial \xi^1} &= \frac{(\xi^1)^2 \sec(\frac{1}{2}\arcsin(2\xi^1\xi^2))\tan(\frac{1}{2}\arcsin(2\xi^1\xi^2))}{\sqrt{1-4(\xi^1\xi^2)^2}\sqrt{-1+\xi^1\sec(\frac{1}{2}\arcsin(2\xi^1\xi^2))}\sqrt{1+\xi^1\sec(\frac{1}{2}\arcsin(2\xi^1\xi^2))}} \\
\frac{\partial x^1}{\partial \xi^2} &= \frac{\xi^2}{\sqrt{1-4(\xi^1\xi^2)^2}} \\
\frac{\partial x^2}{\partial \xi^2} &= \frac{\xi^1}{\sqrt{1-4(\xi^1\xi^2)^2}}
\end{aligned}
$$

18. Cylindrical coordinates $\begin{cases} \xi^1 = x^1 \cos x^2 \\ \xi^2 = x^1 \sin x^2 \\ \xi^3 = x^3 \end{cases}$, the contravariant velocity vector is $U^i = \frac{\partial \xi^i}{\partial x^l} u^l$, the covariant derivative of this vector in cylindrical coordinates is $W^i_j = \frac{\partial U^i}{\partial x^j} = \frac{\partial}{\partial x^j}(\frac{\partial \xi^i}{\partial x^l} u^l) = \frac{\partial^2 \xi^i}{\partial x^j \partial x^l} u^l$

$= \frac{\partial^2 \xi^1}{\partial x^1 \partial x^1} u^1 + \frac{\partial^2 \xi^1}{\partial x^1 \partial x^2} u^2 + \frac{\partial^2 \xi^1}{\partial x^2 \partial x^1} u^1 + \frac{\partial^2 \xi^1}{\partial x^2 \partial x^2} u^2 + \frac{\partial^2 \xi^2}{\partial x^1 \partial x^1} u^1 + \frac{\partial^2 \xi^2}{\partial x^1 \partial x^2} u^2 + \frac{\partial^2 \xi^2}{\partial x^2 \partial x^1} u^1 + \frac{\partial^2 \xi^2}{\partial x^2 \partial x^2} u^2$
$= (\cos x^2 - \sin x^2) u^1 + (\cos x^2 - \sin x^2 - x^1(\cos x^2 + \sin x^2)) u^2$

ex. Given $\begin{cases} \theta_1 = \sqrt{x^1 + x^2 + x^3} \\ \theta_2 = \cos^{-1}\left(\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) \\ \theta_3 = \tan^{-1}\left(\frac{x_2}{x_1}\right) \end{cases}$ and the inverse $\begin{cases} x_1 = \theta_1 \sin\theta_2 \cos\theta_3 \\ x_2 = \theta_1 \sin\theta_2 \sin\theta_3 \\ x_3 = \theta_1 \cos\theta_2 \end{cases}$

The Jacobian matrix $J = \frac{\partial(x_1, x_2, x_3)}{\partial(\theta_1, \theta_2, \theta_3)} = \begin{pmatrix} \sin\theta_2\cos\theta_3 & \theta_1\cos\theta_2\cos\theta_3 & -\theta_1\sin\theta_2\sin\theta_3 \\ \sin\theta_2\sin\theta_3 & \theta_1\cos\theta_2\sin\theta_3 & \theta_1\sin\theta_2\cos\theta_3 \\ \cos\theta_2 & -\theta_1\sin\theta_2 & 0 \end{pmatrix}$,

the metric tensor $G = J^T \cdot J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta_1^2 & 0 \\ 0 & 0 & \theta_1^2 \sin^2\theta_2 \end{pmatrix}$