Homework 2 Solution

1. Using Riccati equation, there is a specific solution $y = x$ and $y' = (-x^2)y^2 + (-x^2)y + (x^4 + x^3 + 1)$, set $y = x + \frac{1}{z(x)}$, we could get $\frac{dz}{dx} + (-2x^3 - x^2)z = x^2 \Rightarrow e^{\int(-2x^3-x^2)dx} = e^{-\frac{x^4}{2}-\frac{x^3}{3}} \Rightarrow z = \frac{\int x^2 e^{-\frac{x^4}{2}-\frac{x^3}{3}}dx + C}{e^{-\frac{x^4}{2}-\frac{x^3}{3}}} \Rightarrow y = x + \frac{e^{-\frac{x^4}{2}-\frac{x^3}{3}}}{\int x^2 e^{-\frac{x^4}{2}-\frac{x^3}{3}}dx + C}$

2. Using Bernoulli equation $\dot{x} - 2tx = te^{-t^2}x \Rightarrow u = x^{-1} \Rightarrow u' + 2tu = -te^{-t^2} \Rightarrow e^{\int 2tdt} = e^{t^2} \Rightarrow e^{t^2}u = \int -te^{-t^2}e^{t^2}dt = -\int tdt = -\frac{1}{2}t^2 + C \Rightarrow u = -\frac{1}{2}t^2e^{-t^2} + Ce^{-t^2} \Rightarrow x = u^{-1} = \frac{2}{2C - t^2}e^{t^2}$.

3. $P = 3x^2y^2, Q = 2x^3y$ and $\frac{\partial P}{\partial y} = 6x^2y = \frac{\partial Q}{\partial x} \Rightarrow$ this equation is exact. $\frac{\partial F}{\partial x} = 3x^2y^2 \Rightarrow F(x,y) = x^3y + A(y) \Rightarrow \frac{\partial F}{\partial y} = x^3 + \frac{dA}{dy} = 2x^3y \Rightarrow \frac{dA}{dy} = 2x^3y - x^3 \Rightarrow A = x^3y^2 - x^3y - C \Rightarrow F = x^3y^2 - C = 0 \Rightarrow x^3y^2 = C$

4. $(x-y)dx + (-x-y)dy = 0, P = x - y, Q = -x - y, \frac{\partial P}{\partial y} = -1 = \frac{\partial Q}{\partial x} \Rightarrow$ this equatio is exact. $\frac{\partial F}{\partial x} = x - y \Rightarrow F(x,y) = \frac{x^2}{2} - xy + A(y) \Rightarrow \frac{\partial F}{\partial y} = -x + \frac{dA}{dy} = -x - y \Rightarrow A = -\frac{y^2}{2} - C \Rightarrow F = \frac{x^2}{2} - xy - \frac{y^2}{2} - C = 0 \Rightarrow \frac{x^2}{2} - xy - \frac{y^2}{2} = C$

5. If $y' = x \Rightarrow y = \frac{x^2}{2} + C$.
   If $y' \neq x \Rightarrow y'' = -2 \Rightarrow y = -x^2 + C_1x + C_2$

6. Let $u = y', y'' = u' \Rightarrow u' = -2\frac{u}{x} + 1 = f(\frac{u}{x})$. Let $v = \frac{u}{x} \Rightarrow u' = v + xv' \Rightarrow v = \frac{1}{3} - \frac{C}{3|x|^2} \Rightarrow y' = \frac{x}{3} - C\frac{x}{3|x|^3} \Rightarrow y = \frac{x^2}{6} + C_1\frac{1}{3x} + C_2$. Given $y(1) = 1, y'(1) = 1 \Rightarrow C_1 = -2, C_2 = 1.5 \Rightarrow y = \frac{x^2}{6} - \frac{2}{3x} + 1.5$

7. Let $u = y', y'' = u\frac{du}{dy} \Rightarrow u\frac{du}{dy} = 2yu, u = 0$ is a trivial solution. For $u \neq 0, \frac{dy}{dx} = y^2 + C_1$. If $C_1 > 0, x = \frac{1}{\sqrt{C_1}}\arctan\frac{y}{\sqrt{C_1}} + C_2$, if $C_1 < 0, x = \frac{1}{2\sqrt{-C_1}}\ln|\frac{\sqrt{-C_1}+y}{\sqrt{-C_1}-y}| + C_2$. Given $y(0) = 0, y'(0) = 3 \Rightarrow C_1 = 3, C_2 = 0 \Rightarrow x = \frac{1}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}}$ or $y = \sqrt{3}\tan(\sqrt{3}x)$

8. Set $y_2 = \frac{u(x)}{x}$, then get $\frac{u''}{x} + u'(2(\frac{1}{x})' + \frac{3}{x}\frac{1}{x}) = 0 \Rightarrow |\frac{dy}{dx}| = \frac{C}{|x|}$. The sign can be integrated to C, so $y_2 = C\frac{\ln|x|}{x}$

9. a. $y'\tan y + \sin 2x + \ln x = 0 \Rightarrow \ln|\cos y| = -\frac{1}{2}\cos 2x + x\ln x + x + C$
   b. $x = 0$ is a trivial solution. For $x \neq 0, y' = (\frac{1}{x})y^2 + (\frac{2}{x})y + x^3$ and $y = ix^2$ is a specific solution, so let $y = ix^2 + \frac{1}{z} \Rightarrow \frac{dz}{dx} + (2ix + \frac{2}{x})z = -\frac{1}{x} \Rightarrow \frac{d}{dx}(x^2e^{ix^2}z) = x^2e^{ix^2}(-\frac{1}{x}) = -xe^{ix^2} \Rightarrow x^2e^{ix^2}z = \frac{1}{2}ie^{ix^2} + C \Rightarrow z = \frac{i}{2x^2} + \frac{C}{x^2e^{ix^2}}$
   c. $\cot y dy = -\tan x dx \Rightarrow \sin y = C\cos x$
   d. $\because e^{\int \cot s ds} = e^{\ln|\sin x|} = |\sin x| \Rightarrow \frac{d(\sin x)y}{dx} = e^x \sin x \Rightarrow (\sin x)y = \frac{1}{2}e^x(\sin x - \cos x) + C$
   e. $y' + \frac{1}{x^5}y = e^{x^2}(\frac{1}{x^5} - x)y^3$, Bernoulli eqn. type with $n = 3$. Let $u = \frac{1}{y^2} \Rightarrow u' - \frac{2}{x^5}u = -2e^{x^2}(\frac{1}{x^5} - x)$. $\because e^{\int -\frac{2}{x^5}dx} = e^{\frac{1}{2x^4}} \Rightarrow \int_1^x \frac{d(e^{\frac{1}{2x^4}}u)}{dx}dx = \int_1^x e^{x^2 + \frac{1}{2x^4}}dx, \because u(1) = \frac{1}{y^2(1)} = e \Rightarrow e^{\frac{1}{2x^4}}u - e^{3/2} = e^{x^2+\frac{1}{2x^4}} - e^{3/2} \Rightarrow u = e^{x^2} = \frac{1}{y^2} \Rightarrow y = e^{-\frac{x^2}{2}}$
   f. $y' = -y^2 + xy + 1$ and one specific solution $y = x, \Rightarrow y = x + \frac{1}{z} \Rightarrow \frac{dz}{dx} - xz = 1 \Rightarrow y = x + \frac{1}{e^{\frac{x^2}{2}}(z_0 + \int_0^x e^{-\frac{t^2}{2}}dt)}$

g. $ydx - (x + y^2)dy = 0$, not an exact eqn, find integrate factor $u = \frac{1}{y^2} \Rightarrow (\frac{y}{x})dx - (1 + \frac{y^2}{x})dy = 0$. Now $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -\frac{\partial 1}{\partial y^2} \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial 1}{\partial y} \Rightarrow F = \frac{x}{y} + A(y) \Rightarrow \frac{\partial F}{\partial y} = -\frac{x}{y^2} + \frac{dA}{dy} = -1 - \frac{x}{y^2} \Rightarrow A = y - C \Rightarrow \frac{x}{y} + y = C$

h. $(x + 2y - 5)dx + (2x + y - 4)dy = 0, \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2 \Rightarrow$ this is an exact eqn. $\frac{\partial F}{\partial x} = x + 2y - 5 \Rightarrow F = \frac{x^2}{2} + 2xy - 5x + A(y) \Rightarrow \frac{\partial F}{\partial y} = 2x + \frac{dA}{dy} = 2x + y - 4 \Rightarrow \frac{dA}{dy} = y - 4 \Rightarrow A = \frac{y^2}{2} - 4y - C \Rightarrow \frac{x^2}{2} + 2xy + \frac{y^2}{2} - 5x - 4y = C$

i. $|y| = C\exp(x - \frac{x^2}{2})$