Homework 3 Solution

2. $w = \begin{vmatrix} \sin x & x\cos x & x \\ \cos x & \cos x - x\sin x & 1 \\ -\sin x & -x\cos x - 2\sin x & 0 \end{vmatrix} = -x^2 + 2\sin^2 x - x\sin x\cos x$. Only when $x = 0, w = 0 \Rightarrow y_i$ is linearly independent.
For these three linearly independent solutions, the homogeneous equations should be 3rd order ODE, say, $L(y) = A_3 y''' + A_2 y'' + A_1 y' + A_0 y = 0 \Rightarrow$
$$\begin{aligned} A_3(-\cos x) + A_2(-\sin x) + A_1(\cos x) + A_0 \sin x &= 0 \\ A_3(-3\cos x + x\sin x) + A_2(-2\sin x - x\cos x) + A_1(\cos x - x\sin x) + A_0 x\cos x &= 0 \\ A_1 + A_0 x &= 0 \end{aligned}$$
Set $a_i = \frac{A_i}{A_3}, i = 0, 1, 2 \Rightarrow L(y) = y''' + a_2 y'' + a_1 y' + a_0 y = 0$
$\Rightarrow \begin{pmatrix} -\sin x & \sin x - x\cos x \\ -2\sin x - x\cos x & x^2 \sin x \end{pmatrix} \begin{pmatrix} a_2 \\ a_0 \end{pmatrix} = \begin{pmatrix} \cos x \\ 3\cos x - x\sin x \end{pmatrix}$
Using Cramer's rule, $a_2 = \dfrac{\begin{vmatrix} \cos x & \sin x - x\cos x \\ 3\cos x - x\sin x & x^2\sin x \end{vmatrix}}{\begin{vmatrix} -\sin x & \sin x - x\cos x \\ -2\sin x - x\cos x & x^2 \sin x \end{vmatrix}} = \frac{x\sin^2 x + 3x\cos^2 x - 3\sin x\cos x}{-x^2 - x\sin x\cos x + 2\sin^2 x}$,
$a_0 = \dfrac{\begin{vmatrix} -\sin x & \cos x \\ -2\sin x - x\cos x & 3\cos x - x\sin x \end{vmatrix}}{\begin{vmatrix} -\sin x & \sin x - x\cos x \\ -2\sin x - x\cos x & x^2\sin x \end{vmatrix}} = \frac{x - \sin x\cos x}{-x^2 - x\sin x\cos x + 2\sin^2 x} \Rightarrow$
$a_1 = -a_0 x = \frac{-x^2 + x\sin x\cos x}{-x^2 - x\sin x\cos x + 2\sin^2 x}$
So, $L(y) = y''' + a_2 y'' + a_1 y' + a_0 y = 0$

3. Using characteristic equation $r^2 + Cr + 4 = 0$ with $y(0) = 1, y'(0) = -3$.

   I. $C = 6, r^2 + 6r + 4 = 0, r = -3 \pm \sqrt{5} \Rightarrow y = ae^{(-3+\sqrt{5})t} + be^{(-3-\sqrt{5})t} \Rightarrow a = b = \frac{1}{2} \Rightarrow y = \frac{1}{2}(e^{(-3+\sqrt{5})t} + e^{(-3-\sqrt{5})t})$.
   II. $C = 4, r^2 + 4r + 4 = 0, r_1 = r_2 = -2 \Rightarrow y = ae^{-2t} + bte^{-2t} \Rightarrow a = 1, b = -1 \Rightarrow y = e^{-2t} - te^{-2t}$.
   III. $C = 3, r^2 + 3r + 4 = 0, r = \frac{-3 \pm \sqrt{7}i}{2} \Rightarrow y = ae^{-1.5t}\cos\sqrt{7}t + be^{-1.5t}\sin\sqrt{7}t \Rightarrow a = 1, b = -\frac{1.5}{\sqrt{7}} \Rightarrow y = e^{-1.5t}(\cos\sqrt{7}t - \frac{1.5}{\sqrt{7}}\sin\sqrt{7}t)$.

5. Given $\frac{d^2y}{dx^2} - 4y = \cosh 2x$

   I. Variation of Parameters $\because \cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$, set $y_1 = e^{2x}, y_2 = e^{-2x}$ and have
   $$\begin{aligned} u_1' e^{2x} + u_2' e^{-2x} &= 0 \\ 2u_1' e^{2x} - 2u_2' e^{-2x} &= \frac{e^{2x} + e^{-2x}}{2} \end{aligned} \Rightarrow \begin{aligned} u_1 &= \frac{x}{8} - \frac{e^{-4x}}{32} \\ u_2 &= -\frac{x}{8} - \frac{e^{4x}}{32} \end{aligned}$$
   $\Rightarrow y_p = u_1 y_1 + u_2 y_2 = (\frac{x}{8} - \frac{e^{-4x}}{32})e^{2x} + (-\frac{x}{8} - \frac{e^{4x}}{32})e^{-2x}$
   $= (\frac{x}{8} - \frac{1}{32})e^{2x} - (\frac{x}{8} + \frac{1}{32})e^{-2x}$
   II. Undetermined Coefficients set $y = ae^{2x} + be^{-2x} \Rightarrow y'' = 4ae^{2x} + 4be^{-2x} = 4y$, so this is not a good assumption. Try again $y = (a_1 + a_2 x)e^{2x} + (b_1 + b_2 x)e^{-2x} \Rightarrow y'' - 4y = 4a_2 e^{2x} - 4b_2 e^{-2x} = \frac{e^{2x} + e^{-2x}}{2} \Rightarrow a_2 = \frac{1}{8}, b_2 = -\frac{1}{8}, a_1, b_1$ arbitrary. $\Rightarrow y = \frac{x}{8}(e^{2x} - e^{-2x})$.

6. Given $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$ with $y(0) = 0, y(\frac{\pi}{2}) = -1$. Set $u = \frac{dy}{dx}, \frac{d^2y}{dx^2} = u\frac{du}{dy} \Rightarrow u\frac{du}{dy} + yu = 0$. If $u = 0$, this is a trivial solution. For $u \neq 0$, $\frac{du}{dy} + y = 0 \Rightarrow u = \frac{dy}{dx} = -\frac{1}{2}y^2 + C_1 \Rightarrow \frac{dy}{2C_1 - y^2} = \frac{1}{2}x$

I. If $C_1 > 0$, let $a^2 = 2C_1, -\frac{dy}{y^2-a^2} = \frac{1}{2}x \Rightarrow -\frac{1}{2a}\ln|\frac{y-a}{y+a}| = \frac{1}{2}x + C_2$. $\because y(0) = 0, \therefore C_2 = 0; \because y(\frac{\pi}{2}) = -1, \therefore -\frac{1}{2a}\ln|\frac{1-a}{1+a}| = \frac{\pi}{4}$. Now let's disuss the different situations of a.

- $a = 0, \Rightarrow C_1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}y^2 \Rightarrow \frac{2}{3y^3} = x + C_3$ which is conflict with initial conditions.
- $a > 0$, RHS > 0, LHS < 0, conflict
- $a < 0$, RHS < 0, RHS > 0, conflict

$\therefore C_1 > 0$ cannot find the solution

II. If $C_1 < 0$, let $a^2 = -2C_1, \frac{1}{a}\arctan\frac{y}{a} = -\frac{1}{2}x + C_2$. $\because y(0) = 0, \therefore C_2 = 0; \because y(\frac{\pi}{2}) = -1, \therefore \frac{1}{a}\arctan(-\frac{1}{a}) = -\frac{\pi}{4} \Rightarrow a^2 = 1 \Rightarrow C_1 = -\frac{1}{2} \Rightarrow y = -\tan\frac{x}{2}$

10. Given $y'' + y' - 2y = f(x)$ with $y(0) = 0, y(1) = 0$. We have $L = \frac{d^2}{dx^2} + \frac{d}{dx} - 2$. For $g'' + g' - 2g = 0$, using charateristic equation $r^2 + r - 2 = 0 \Rightarrow r_1 = 1, r_2 = -2 \Rightarrow g = C_1e^x + C_2e^{-2x}$.

a. For $x < s, g(0) = C_1 + C_2 = 0 \Rightarrow g(x,s) = C_1(s)(e^x - e^{-2x}), x < s$.

b. For $x > s, g(1) = C_3e + C_4e^{-2} \Rightarrow g(x,s) = C_3(s)(e^x - e^{-2x+3}), x > s$

From continuity, $C_1(e^s - e^{-2s}) = C_3(e^s - e^{-2s+3})$.
From 'jump', $C_3(e^s + 2e^{-2s+3}) - C_1(e^s + 2e^{-2s}) = 1/1 = 1$
$\Rightarrow C_1 = \frac{e^s - e^{-2s+3}}{3(e^{-s+3} - e^{-s})}, C_3 = \frac{e^s - e^{-2s}}{3(e^{-s+3} - e^{-s})}$
$\Rightarrow g(x,s) = \begin{cases} \frac{e^s - e^{-2s+3}}{3(e^{-s+3} - e^{-s})}(e^x - e^{-2x}) & x < s \\ \frac{e^s - e^{-2s}}{3(e^{-s+3} - e^{-s})}(e^x - e^{-2x+3}) & x > s \end{cases}$
So the solution is $y(x) = \int_0^1 f(s)g(x,s)ds$
$= \int_0^x f(s)\frac{e^x - e^{-2x+3}}{3(e^{-x+3} - e^{-x})}(e^s - e^{-2s})ds + \int_x^1 f(s)\frac{e^x - e^{-2x}}{3(e^{-x+3} - e^{-x})}(e^s - e^{-2s+3})ds$
$= \frac{e^x - e^{-2x+3}}{3(e^{-x+3} - e^{-x})}\int_0^x f(s)(e^s - e^{-2s})ds + \frac{e^x - e^{-2x}}{3(e^{-x+3} - e^{-x})}\int_x^1 f(s)(e^s - e^{-2s+3})ds$
For $f(x) = 3\sin x$, $y(x) = \frac{e^{2x} - e^{-x+3}}{e^3 - 1}(\frac{1}{2}e^x(\sin x - \cos x) + \frac{1}{5}e^{-2x}(2\sin x + \cos x) + \frac{3}{10}) + \frac{e^{2x} - e^{-x}}{e^3 - 1}(e(\frac{9}{10}\sin 1 - \frac{3}{10}\cos 1) - \frac{1}{2}e^x(\sin x - \cos x) - \frac{1}{5}e^{-2x+3}(2\sin x + \cos x))$