Homework 4 Solution

1.a Given $x^2y'' - 2xy' + (2x-3)y = 0$, $x = 0$ is a regular singular point.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - 2\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + (2x-3)\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty} a_n ((n+r)^2 - 3(n+r) - 3) x^{n+r} + 2\sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$
$$\sum_{n=0}^{\infty} a_n ((n+r)^2 - 3(n+r) - 3) x^{n+r} + 2\sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0$$

so the indicical equation is $r^2 - 3r - 3 = 0 \Rightarrow r_1 = \frac{3+\sqrt{21}}{2}, r_2 = \frac{3-\sqrt{21}}{2}$ and $r_1 - r_2 \neq$ integer
For $r_1 = \frac{3+\sqrt{21}}{2}, a_n = -\frac{2a_{n-1}}{n^2+\sqrt{21}n}, y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$
For $r_2 = \frac{3-\sqrt{21}}{2}, b_n = -\frac{2b_{n-1}}{n^2-\sqrt{21}n}, y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$

$$y = C_1 a_0 x^{\frac{3+\sqrt{21}}{2}} \left(1 - \frac{2x}{1+\sqrt{21}} + \frac{2^2x^2}{(2^2+2\sqrt{21})(1+\sqrt{21})} - \cdots\right) + C_2 b_0 x^{\frac{3-\sqrt{21}}{2}} \left(1 - \frac{2x}{1-\sqrt{21}} + \frac{2^2x^2}{(2^2-2\sqrt{21})(1-\sqrt{21})} - \cdots\right)$$

$\because y(0) < \infty \Rightarrow C_2 = 0$
$\because y(1) = 1 \Rightarrow C_1 a_0 = \frac{1}{1-\frac{2}{1+\sqrt{21}}+\frac{2^2}{(2^2+2\sqrt{21})(1+\sqrt{21})}-\cdots} \Rightarrow y = x^{\frac{3+\sqrt{21}}{2}} \frac{1-\frac{2x}{1+\sqrt{21}}+\frac{2^2x^2}{(2^2+2\sqrt{21})(1+\sqrt{21})}-\cdots}{1-\frac{2}{1+\sqrt{21}}+\frac{2^2}{(2^2+2\sqrt{21})(1+\sqrt{21})}-\cdots}$

1.b Given $xy'' + 2y' + x^2y = 0$, $x = 0$ is a regular singular point.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + 2\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$\sum_{n=0}^{\infty} a_n((n+r)^2 + (n+r))x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

$$a_0(r^2+r)x^{r-1} + a_1((r+1)^2 + (r+1))x^r + a_2((r+2)^2 + (r+2))x^{r+1} + \sum_{n=0}^{\infty}(a_{n+3}((n+r+3)^2 + (n+r+3)) + a_n)x^{n+r+2} = 0$$

Since $y(0) = 1 \Rightarrow r = 0$. $y'(0) < \infty \Rightarrow a_1 < \infty$
$\therefore a_0 \neq 0, a_1 = a_2 = 0, r_1 = r_2 = 0, a_{n+3} = -\frac{a_n}{(n+3)^2+(n+3)}$ for $n \geqslant 0$

$$y_1 = a_0\left(1 - \frac{x^3}{3^2+3} + \frac{x^6}{(6^2+6)(3^2+3)} - \cdots\right)$$

$$y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n$$

$y = C_1y_1 + C_2y_2$ and $y(0) = 1 \Rightarrow C_1a_0 = 1, C_2 = 0 \Rightarrow y = 1 - \frac{x^3}{3^2+3} + \frac{x^6}{(6^2+6)(3^2+3)} - \cdots$

5. Given $\frac{d^2x}{dt^2} + \sin x = 0$ with $x(0) = \epsilon$, $\frac{dx}{dt}(0) = 0$. Looking solutions valid all of the time, let $t = (1 + c_1\epsilon + c_2\epsilon^2 + \cdots)\tau$, so

$$\dot{x} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1} = \frac{dx}{d\tau}(1 + c_1\epsilon + c_2\epsilon^2 + \cdots)^{-1}$$

$$\ddot{x} = \frac{d^2x}{d\tau^2}(1 - 2c_1\epsilon + (3c_1^2 - 2c_2)\epsilon^2 + \cdots)$$

$$x = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots$$

, and
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
$= (x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^2)) - \frac{1}{3!}(x_0^3 + 3\epsilon x_0^2 x_1 + 3\epsilon^2(x_0x_1^2 + x_0^2x_2) + O(\epsilon^2)) + \frac{1}{5!}(x_0^5 + 5\epsilon x_0^4 x_1 + 10\epsilon^2 x_0^3 x_1^2 + 5\epsilon^2 x_0^4 x_2 + O(\epsilon^2)) - \frac{1}{7!}(x_0^7 + 7\epsilon x_0^6 x_1 + 21\epsilon^2 x_0^5 x_1^2 + 7\epsilon^2 x_0^6 x_2 + O(\epsilon^2)) + \cdots$
$= (x_0 - \frac{x_0^3}{3!} + \frac{x_0^5}{5!} - \frac{x_0^7}{7!} + \cdots) + \epsilon(x_1 - \frac{x_0^2x_1}{2} + \frac{x_0^4x_1}{4!} - \frac{x_0^6x_1}{6!} + \cdots) + \epsilon^2(x_2 - \frac{x_0x_1^2+x_0^2x_2}{2} + \frac{2x_0^3x_1^2+x_0^4x_2}{4!} - \frac{3x_0^5x_1^2+x_0^6x_2}{6!} + \cdots) + O(\epsilon^2)$
$= \sin x_0 + \epsilon x_1 \cos x_0 + \epsilon^2\left(x_2\cos x_0 - \frac{x_1^2}{2}\sinh x_0\right) + O(\epsilon^2)$, so

$$\left(\frac{d^2x_0}{d\tau^2} + \epsilon\frac{d^2x_1}{d\tau^2} + \epsilon^2\frac{d^2x_2}{d\tau^2} + \cdots\right)(1 - 2c_1\epsilon + (3c_1^2 - 2c_2)\epsilon^2 + \cdots) + \sin x_0 + \epsilon x_1\cos x_0 + \epsilon^2\left(x_2\cos x_0 - \frac{x_1^2}{2}\sinh x_0\right) = 0$$

$O(\epsilon^0) : \ddot{x}_0 + \sin x_0 = 0, x_0(0) = 0, \dot{x}_0(0) = 0 \quad \Rightarrow x_0(\tau) = 0$

$O(\epsilon^1) : \ddot{x}_1 - 2c_1\ddot{x}_0 + x_1\cos x_0 = 0, x_1(0) = 1, \dot{x}_1(0) = 0 \quad \Rightarrow x_1(\tau) = \cos\tau$

$O(\epsilon^2) : \ddot{x}_2 - 2c_1\ddot{x}_1 + (3c_1^2 - 2c_2)\ddot{x}_0 + x_2\cos x_0 - \frac{x_1^2}{2}\sinh x_0 = 0, x_2(0) = 0, \dot{x}_2(0) = 0 \quad \Rightarrow x_2 = -c_1\tau\sin\tau$

and $\tau = (1 + c_1\epsilon + \cdots)^{-1}t$, so

$$x(t) = x_0 + \epsilon x_1 + \epsilon^2 x_2$$

$$= \epsilon\cos\left((1 + c_1\epsilon + \cdots)^{-1}t\right) - \epsilon^2 c_1(1 + c_1\epsilon + \cdots)^{-1}t\sin\left((1 + c_1\epsilon + \cdots)^{-1}t\right)$$

13. Given $\dot{x}+2x+\epsilon x^2=0$ with $x(0)=\cosh\epsilon$. Let $x(t)=x_0(t)+\epsilon x_1(t)+\epsilon^2 x_2(t)+\cdots$, then $\cosh\epsilon = 1+\frac{\epsilon^2}{2!}+\frac{\epsilon^4}{4!}+\cdots$, so

$$
\begin{aligned}
O(\epsilon^0) &: \dot{x}_0+2x_0=0, x_0(0)=1 && \Rightarrow x_0=e^{-2t}\\
O(\epsilon^1) &: \dot{x}_1+2x_1+x_0^2=0, x_1(0)=0 && \Rightarrow x_1=\frac{1}{2}(e^{-4t}-e^{-2t})\\
O(\epsilon^2) &: \dot{x}_2+2x_2+2x_0x_1=0, x_2(0)=\frac{1}{2} && \Rightarrow x_2=\frac{1}{4}e^{-6t}-\frac{1}{2}e^{-4t}+\frac{3}{4}e^{-2t}
\end{aligned}
$$

⋄. Given $\ddot{x}-t\dot{x}-x=0$ with $x(0)=1, \dot{x}(0)=0$.

$$
\begin{aligned}
x &= \sum_{n=0}^{\infty} a_n t^n = a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5+a_6t^6+a_7t^7+\cdots\\
x' &= \sum_{n=1}^{\infty} a_n n t^{n-1} = a_1+2a_2t+3a_3t^2+4a_4t^3+5a_5t^4+6a_6t^5+7a_7t^6+\cdots\\
x'' &= \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2} = 2a_2+6a_3t+12a_4t^2+20a_5t^3+30a_6t^4+42a_7t^5+\cdots
\end{aligned}
$$

$$
\begin{array}{rr}
2a_2-a_0=0 & 6a_3-2a_1=0\\
12a_4-3a_2=0 & 20a_5-4a_3=0\\
30a_6-5a_4=0 & 42a_7-6a_6=0\\
\cdots &
\end{array}
$$

$\Rightarrow x = a_0(1+\frac{1}{2}t^2+\frac{1}{8}t^4+\frac{1}{2^3 3!}t^6+\cdots)+a_1t(1+\frac{1}{3}t^2+\frac{1}{15}t^4+\cdots)$
$x(0)=1 \Rightarrow a_0=1$
$x'(0)=0=a_0(t+\frac{1}{2}t^3+\cdots)+a_1(1+\frac{1}{3}t^2+\frac{1}{15}t^4+\cdots)+a_1t(\frac{2}{3}t+\frac{4}{15}t^3+\cdots) \Rightarrow a_1=0$

$$
x=\left(1+\frac{1}{2}t^2+\frac{1}{8}t^4+\frac{1}{2^3 3!}t^6+\cdots\right)=\sum_{0}^{\infty}\frac{1}{2^n n!}t^{2n}
$$