Analysis and Control of Unstable Rolling Wheel Dynamics

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ABSTRACT

Many vehicle systems contain rolling elements which exhibit unstable rolling motion, called shimmying, which may lead to disastrous results. The *classical shimmying wheel* is a simple model which captures the essential dynamics of such systems. In particular, the uncontrolled stability of the rolling wheel is characterized by a subcritical Hopf bifurcation when certain parameter values are varied. As such, the equilibrium state for the system may be unstable, and even if it is stable, sufficiently non-local initial conditions will lead to unstable dynamics. To control this unstable behavior, a particular geometric structure of this model can be exploited which provides a simple means to design a globally stabilizing controller by the means of feedback linearization.

1 INTRODUCTION

A schematic drawing of the model is shown in Figure 1. The rotational angle of the wheel with radius r is given by ϕ. The caster length, or the offset of the axis of the wheel with respect to the vertical center of rotation of the wheel (the kingpin) is l, and the angle of rotation of the wheel assembly with respect to the "straight" position is given by θ. We will consider the kingpin to be massless. Call the mass of the connecting assembly m_c and the mass of the wheel m_w. For the control problem, we will consider the control input to be a torque, u, about the vertical center of rotation of the wheel assembly.

In this study, the simplest possible mechanical model is considered, with the lowest number of degrees of freedom which still exhibits the shimmying instability. This goal of simplicity perhaps makes the model less similar to a particular example, *e.g.*, less like an automobile suspension. On the other hand, reducing the problem to the simplest possible model serves a two–fold purpose. First, the problem becomes tractable, allowing the geometry of the dynamics to be explored. Second, by considering the simplest possible model, we hope to reduce the rather general phenomena, present in many different applications, to its essential elements.

The main simplification of this model is that the elastic nature of the system is modeled by springs;

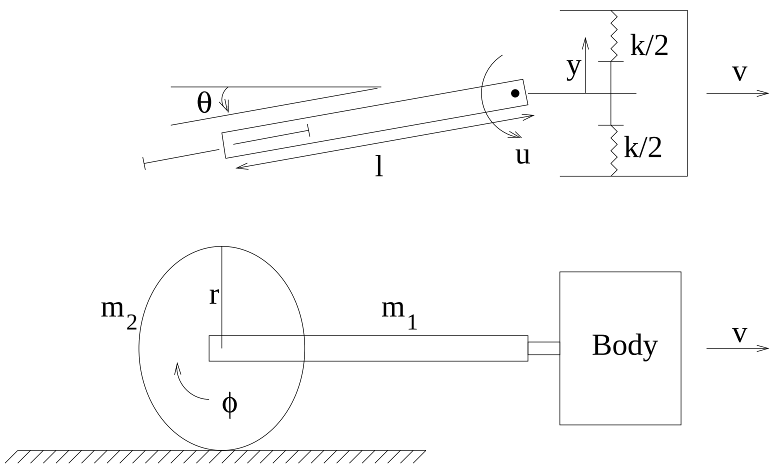

Figure 1. Classical shimmying wheel model.

whereas, the more sophisticated models mentioned previously directly attempt to model the infinite dimensional elastic nature of pneumatic tires, or possibly reduce the problem to a finite dimensional representation by only considering the lower order modes [Sharp and Jones, 1980]. However, there are some cases where our simplified model may be a more accurate model than the more complicated ones. For example, on an aircraft with a relatively tall landing gear structure, the elastic effect of the tire may be small relative to the lateral elastic properties of the landing gear strut. Another case is whenever there is a small contact region, or if the wheel is rigid (as in a shopping cart). Regardless, here we model the elastic element by two springs, each with spring constant $\frac{k}{2}$. The kingpin is constrained to deflect laterally (so it can not deflect "forwards and backwards", but only "side to side"), and the amount of deflection is represented by the variable y.

We consider the system to be moving with a constant velocity, v. This assumption further reduces the dimension of the phase space, thus helping to further simplify the problem. Such an assumption is justified in cases where the body is massive relative to the mass of the wheel and the associated structure (as in an airplane), or where some external control keeps the overall structure moving with constant velocity (such as a truck trailer or shopping cart). Finally, we note that many structures such as an aircraft landing gear system or automobile suspension also include significant vertical elastic elements, *e.g.*, the shock absorbers. Clearly, a model including these elements would be more realistic and could possibly alter the dynamics of the system. However, since the simpler, planar model we consider exhibits the phenomenon we wish to control, we will restrict our attention to the simpler model for the reasons set forth above.

2 THE DYNAMICS OF THE CLASSICAL SHIMMYING WHEEL

The equations of motion for this system with the ideal nonholonomic constraining forces and control input u are given by:

$$\begin{aligned}
\ddot{\theta} &= \frac{-\frac{v}{l}\left(\sec^2\theta - \frac{1}{2} + \frac{3m_w}{2m_c}\tan^2\theta\right)\dot{\theta}}{\left(\frac{1}{3}+\tan^2\theta\right)\cos\theta + \frac{m_w}{4m_c}\left(\frac{r^2}{l^2}\cos\theta + 6\tan^2\theta\cos\theta\right)} \\
&- \frac{\frac{k}{lm_c}y + \left(1+\frac{3m_w}{2m_c}\right)\frac{\tan\theta}{\cos\theta}\dot{\theta}^2 - \frac{\cos\theta}{l^2 m_c}u}{\left(\frac{1}{3}+\tan^2\theta\right)\cos\theta + \frac{m_w}{4m_c}\left(\frac{r^2}{l^2}\cos\theta + 6\tan^2\theta\cos\theta\right)} \\
\dot{y} &= l\dot{\theta}\cos\theta + \left(v + l\dot{\theta}\sin\theta\right)\tan\theta \\
\dot{\phi} &= \frac{\sec\,\theta\left(v + l\dot{\theta}\sin\theta\right)}{r}.
\end{aligned} \tag{1}$$

A complete study of the dynamics of this system is presented by [Stépán, 1991]. In particular, it was shown that the linearized stability of the system about the $\theta = 0$ and $y = 0$ position was governed by the following condition:

$$\frac{l}{r} > \sqrt{\frac{3m_w}{2m_c}} \tag{2}$$

If the above condition is satisfied, the equilibrium point is asymptotically stable, and the equilibrium point is unstable otherwise. Additionally, it was shown, that when the above local stability condition is satisfied, an unstable limit cycle exists around the stable stationary motion. This stability condition does not contain the velocity term, which may seem to contradict intuition because in vehicle dynamics, the notion of a "critical speed" is often utilized. However, in the case presented here, equation 2 does not contain a velocity term because we have not included viscous damping in the equations of motion.

If we set the control input to zero, we can numerically verify and observe the above results. For $m_c = 1.5$kg, $m_w = 2.75$kg, $l = 0.2$m, $r = 0.1$m, $k = 75$N/m and $v = 1$m/s, the value of the critical caster length is $l_{cr} \approx 0.1658$m, so the length of the caster is greater than the critical length, and so the equilibrium solution is stable. Figure 2 shows this stable equilibrium solution. For a decreased caster length, $l = 0.152\text{m} < l_{cr}$, Figure 3 shows the local instability of the equilibrium solution. This

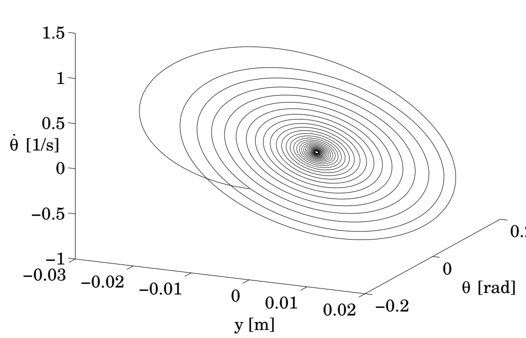

Figure 2. Locally stable rolling system.

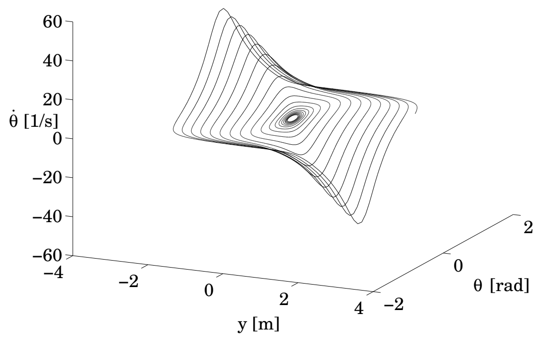

Figure 3. Locally unstable rolling system.

unstable solution appears to be growing unbounded in all variables except θ, which is bounded by $\pm\frac{\pi}{2}$.

The previous simulations verify that with certain parameter values, the equilibrium point can be locally unstable. Next, we numerically verify that even if the equilibrium point is locally stable, there exists an unstable limit cycle around it. Figure 4 shows two solutions, with initial conditions which are "close" together, one of which is stable, the other of which is unstable. The first solution has initial conditions leading inside the limit cycle, and the second solution has initial conditions leading outside the limit cycle. In this simulation, we use the same physical parameters for the system as for the simulation demonstrating the local stability of the equilibrium solution, except $l = 0.171\text{m} > l_{\text{cr}}$, which still satisfies the local stability condition expressed by equation 2. In both cases, the initial conditions are all zero, except for the solution leading inside the limit cycle, the initial angle is $\theta_0 = -0.24$ and the initial angular velocity is $\dot{\theta}_0 = 0.4\text{s}^{-1}$. For the solution leading outside the limit cycle, $\theta_0 = -0.24$ and $\dot{\theta}_0 = 0\text{s}^{-1}$. See the bifurcation analysis in [Stépán, 1991] for more details.

3 FEEDBACK LINEARIZATION

This section constructs a feedback linearizing controller for the classical shimmying wheel. First, we review the basics of feedback linearization. Consider the control system described by

$$\Sigma_0 = \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{cases}$$

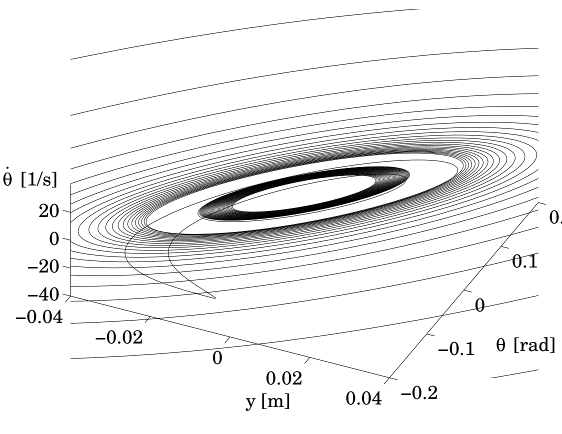

Figure 4. Unstable limit cycle.

where h is called the *output function* and $\dim(x) = n$. This system has *relative degree* r at a point x_0 if

1. $L_g L_f^k h(x) = 0 \; \forall x$ in a neighborhood of x_0, and all $k < r - 1$, and
2. $L_g L_f^{r-1} h(x_0) \neq 0$,

where $L_g h$ is the Lie derivative of the function h along the vector field g.

If $r = n$ for all x, consider the change of coordinates

$$\begin{aligned}
\xi_1 &= h(x) \\
\xi_2 &= \dot{\xi}_1 = L_f h(x) \\
\xi_3 &= \dot{\xi}_2 = L_f^2 h(x) \\
&\vdots \\
\xi_n &= \dot{\xi}_{n-1} = L_f^{n-1} h(x) \\
&\dot{\xi}_n = L_f^n h(x) + L_g L_f^{n-1} h(x) u.
\end{aligned}$$

Since $L_g L_f^{n-1} h(x) \neq 0 \; \forall x$, we can define

$$u = \frac{1}{L_g L_f^{n-1} h(x)} (-L_f^n h(x) + v)$$

so that $\dot{\xi}_n = v$. In this manner, the nonlinear control system Σ_0 is transformed into a controllable linear system. If $h_d(t)$ is the desired "trajectory," choose

$$v = h_d^{(n)} + \alpha_{n-1}(h_d^{(n-1)} - \xi_n) + \cdots + \alpha_0(h_d - \xi_1),$$

as a feedback control law where the α_i are such that $s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$ is a Hurwitz polynomial. For a complete explanation see [Isidori, 1989], and [Nijmeijer and der Schaft, 1990].

For an arbitrary system, there is no general method for constructing an output function, h, which generates the coordinate transformation under which the system is rendered linear. In the case of the classical shimmying wheel (the purely rolling case) the ϕ–coordinate is cyclic, so we can consider $x = (\theta, y, \dot{\theta})$ and write the system as$f(x) + g(x)u$, where $g(x)$ is simply the terms in $\ddot{\theta}$ which contain the control input term, u, and $f(x)$ contains all the other terms in the equations of motion.

If we consider $h_1 = y$ as a candidate output function,

$$\begin{aligned}
L_g h_1 &= 0 \\
L_f h_1 &= l\dot{\theta} \cos\theta + (v + l\dot{\theta} \sin\theta) \tan\theta \\
L_g L_f h &\neq 0.
\end{aligned}$$

Since $L_g L_f h \neq 0$, h_1 is not an output function which renders the system feedback linearizable. Note, however, that if another output function, h_2 were purely a function of θ, then

$$L_f h_2 = \frac{dh_2}{d\theta} \dot{\theta},$$

since the θ component of f is simply $\dot{\theta}$. Since $L_f h_1$ is linear in $\dot{\theta}$, and otherwise only a function of θ, we can differentiate it with respect to $\dot{\theta}$, integrate it with respect to θ, and subtract h_1 from the result. If we denote the resulting function by h, we have

$$h(x) = -y - l \, \log(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}) + l \, \log(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}),$$

where h renders the system feedback linearizable via the preceding construction.

The efficacy of this controller can be verified via numerical simulation. The results of a simulation which illustrates that the controller stabilizes the system when the physical parameters do not satisfy the linear stability criterion are presented in Figure 5. In this simulation, $m_2 = 5.75$ and the initial conditions are $(\theta, y, \phi, \dot{\theta}) = (-0.75, 0, 0, 0)$.

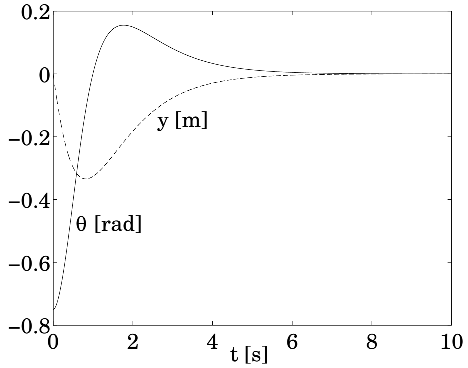

Figure 5. Controlled Pure Rolling System

4 CONCLUSIONS

This paper analyzed the dynamics of he classical shimmying wheel and presented a control design methodology which is effective in controlling its unstable dynamics. Of course, there are many more avenues available for investigation. Obviously, more work is required to determine the nature of the chaotic attractor and its relationship to the purely rolling regimes where the controller is guaranteed to stabilize the system. Another avenue of study would be to study the efficacy of the controller designed here on a more realistic model of an elastic tire. Also, investigating the geometric nature of a more realistic tire model, such as presented in [Barta and Stépán, 1995] may yield insights into controller design for practical implementation.

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