An Algorithm for Stopping a Class of Underactuated Nonlinear Mechanical Robotic Systems

Jason Nightingale and Bill Goodwine

Abstract— **We provide a constructive global discontinuous control law with state dependent switches for a class of underactuated nonlinear mechanical robotic systems that will drive the system to an arbitrarily small neighborhood of rest from all initial configurations and velocities in arbitrarily small time. Because all physical mobile robotic systems are mechanical in nature, control methodologies which exploit the fact that the system is governed by principles of mechanics which are particularly important for robotic engineers. The philosophy of the approach is that instead of using control algorithms which start with a completely generic dynamical system, we constrain the structure of the system to be one which is a Lagrangian control system. To the extent the structure of the mechanical system can be exploited, stronger control results are possible to obtain, such as the stopping algorithm in this paper. Specifically, for control of general nonlinear systems, there are many unsolved problems for the case when the system is not at an equilibrium, and the results in this paper are an initial contribution to this area. The robot is assumed to be underactuated by one in the configuration space; hence, in the state space it is underactuated by twice the dimension of the configuration space plus two. Our method can easily be extended to construct a global discontinuous control law with state dependent switches that will drive the system to an arbitrarily small neighborhood of any velocity from any initial configuration and velocity in arbitrarily small time.**

I. Introduction

Nonlinear mechanical control systems form a large and challenging class of control systems and are of particular importance in robotics. The simple example used in this paper is a planar robotic hovercraft model illustrated in Figure 1. We emphasize that while this model has been studied extensively (for example it is differentiably flat [18]), our focus is not to simply control the hovercraft, but rather to illustrate the application of our general results to a specific system. That model has three configuration variables, position and orientation in the plane, and hence in the state space, *i.e.*, converted to a system of first-order equations, is six-dimensional. It has two forces as inputs. Differential and Riemannian geometry provide an elegant framework for modeling, analysis and control for such systems. This framework has given rise to powerful insights into nonlinear controllability in the *zero velocity setting* motivating stabilization, tracking and motion planning algorithms [2]. A vexing and persistent problem in nonlinear control, including control of mechanical robotic systems, is extending such results to the non-zero velocity case.

J. Nightingale is a graduate of the Department of Aerospace and Mechanical Engineering and the Department of Mathematics, University of Notre Dame, 356 Fitzpatrick Hall, Notre Dame, IN USA jnightin@nd.edu.

B. Goodwine is an Associate Professor in the Department of Aerospace and Mechanical Engineering, University of Notre Dame, 376 Fitzpatrick Hall, Notre Dame, IN USA bill@controls.ame.nd.edu.

The partial support of the National Science Foundation under the CPS Large Grant No. CNS-1035655 is gratefully acknowledged.

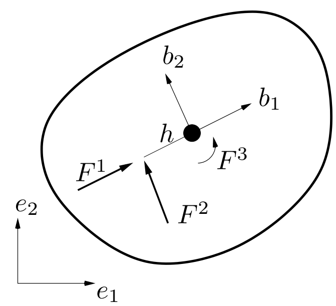

Fig. 1. The planar rigid body hovercraft robot in Section V.

We have extended this research on the analysis and control of underactuated mechanical robotic systems to the *non-zero velocity setting*. Our short term goal has been to develop necessary and sufficient conditions for reaching zero velocity [9] and [10]. We have identified an intrinsic vector-valued symmetric bilinear form that can be associated with an underactuated mechanical control system and developed computable tests dependent upon the definiteness of the symmetric form to determine if the system can or cannot be driven to rest. We have also developed an iterative stopping algorithm that takes advantage of the underlying quadratic structure [11].

As will be shown in this paper, our results are useful design tools which provide constructive strategies for actuator assignment and help to make the control scheme robust to actuator failure [15]. For fully actuated mechanical systems, it is possible to provide a comprehensive solution to the problem of trajectory tracking [2]. In contrast, motion planning algorithm for underactuated mechanical systems is still not well understood [13]. Due to the challenging nature of these problems, many of the existing results have been limited for example to ad hoc gait generation algorithms [7] [3], ad hoc configuration to configuration algorithms with zero-velocity transitions between feasible motions [2] and numerically generated optimal trajectories [12]. The constructive algorithm presented in this paper is formulated in a general setting and applicable to any member of the general class of robotic systems underactuated by one control that satisfy the sufficient conditions given in Theorem 4.5 in Section IV.

For effective control of robotic systems, control methods must take into account the underlying nonlinear geometry of the geometric model, which is made difficult by the nonzero drift. In general, these systems are not full-state feedback linearizable and thus not amendable to standard techniques in control theory [6]. The last decades have shown a great interest in the analysis and control of nonlinear mechanical control systems designed to be underactuated by one control. Such systems include underactuated ships [4], gymnastic robots [17], the Harrier which is a planar vertical/short take-off and landing aircraft in the absence of gravity [14], a hovercraft type vehicle [16] and a planar rigid body with two thrusters moving on a flat horizontal plane [8].

The class of quadratically coupled mechanical systems underactuated by one control is closely related to the systems considered in [5] where a homogeneous, discontinuous, stabilising feedback controller is designed based on the underlying quadratic structure. Our constructive algorithm can also be applied to their example system. In general, quadratic or symmetric bilinear structures can be found in a variety of areas in control theory which has motivated a new initiative to understand the underlying geometry [1] [2].

We focus our analysis on such systems whose underlying quadratic structure is indefinite and independent of the configuration. We demonstrate the applicability of our stopping algorithm by applying it to the underactuated planar rigid body [2] whose linearization is not controllable and it is not full-state feedback linearizable. We provide a schematic drawing, the geometric model, our alternative representation and explicit control law for this system.

II. Geometric Model

A. Mechanical Control System

We consider a simple mechanical control system with no potential to be comprised of an n-dimensional configuration manifold M; a Riemannian metric $\mathbb{G}$ which represents the kinetic energy; m linearly independent one forms $F^1, \ldots, F^m$ on M which represents the input forces; and $U = \mathbb{R}^m$ which represents the set of inputs. We represent the input forces as one forms and use the associated dual vector fields $Y_a = \mathbb{G}^\sharp(F^a),\ a = 1, \ldots, m$ in our computations. Formally, we denote the control system by the tuple $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, U\}$ where $\mathcal{Y} = \{Y_a \mid Y_a = \mathbb{G}^\sharp(F^a)\ \forall\ a\}$ is the input distribution. Note we restrict our attention to control systems where the input forces are dependent upon configuration and independent of velocity and time. A thorough description of simple mechanical control systems is provided by Bullo and Lewis [2].

It is well known that the Lagrange-d'Alembert principle can be used to generate the equations of motion for a forced simple mechanical system in coordinate invariant form. If we set the Lagrangian equal to the kinetic energy, then the equations are given by

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = u^a(t) Y_a(\gamma(t)) \tag{1}$$

where ∇ is the Levi-Civita connection corresponding to $\mathbb{G}$, u is a map from $I \subset \mathbb{R} \mapsto \mathbb{R}^m$, $\gamma : I \to M$ is a curve on M and $t \in I$. Therefore, a *controlled trajectory* for Σ is taken to be the pair (γ, u) where γ and u are defined on the same interval $I \subset \mathbb{R}$. The usual summation notation will be assumed over repeated indices throughout this paper.

A critical tool used to analyze distributions and mechanical control systems is the symmetric product [2]. Given a pair of vector fields X, Y, their symmetric product is the vector field defined by

$$\langle X : Y\rangle = \nabla_X Y + \nabla_Y X$$

where $\nabla_X Y$ is the covariant derivative of Y with respect to X. In coordinates we have

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^i + \Gamma^i_{jk} X^j Y^k\right)$$

where $X = X^i \frac{\partial}{\partial q^i}$ and $Y = Y^i \frac{\partial}{\partial q^i}$. The Christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{1}{2}\mathbb{G}^{il}\left(\frac{\partial \mathbb{G}_{lj}}{\partial q^k} + \frac{\partial \mathbb{G}_{lk}}{\partial q^j} - \frac{\partial \mathbb{G}_{jk}}{\partial q^l}\right)$$

where $\mathbb{G}^{ij}$ is the inverse of the matrix $\mathbb{G}_{ij}$ that represents $\mathbb{G}$.

B. Alternative Representation

An input distribution $\mathcal{Y}$ on M is a subset $\mathcal{Y} \subset TM$ having the property that for each $q \in M$ there exists a family of vector fields $\{Y_1, \ldots, Y_m\}$ on M so that for each $q \in M$ we have

$$\mathcal{Y}_q \equiv \mathcal{Y} \cap T_q M = \mathrm{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}.$$

We refer to the vector fields $\{Y_1, \ldots, Y_m\}$ as generators for $\mathcal{Y}$ and $\{W_1, \ldots, W_m\}$ as the orthonormal generators for $\mathcal{Y}$ where $\mathbb{G}(W_a, W_p) = 1$ when $a = p$ and $\mathbb{G}(W_a, W_p) = 0$ when $a \neq p$. Let $\mathcal{Y}^\perp$ denote an orthonormal frame $\{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ that generates the $\mathbb{G}$-orthogonal complement of the input distribution $\mathcal{Y}$. Note that even though $\mathcal{Y}^\perp$ is canonically defined, we must choose an orthonormal basis. It is clear that $\{\mathcal{Y}_q, \mathcal{Y}_q^\perp\}$ forms a basis for $T_q M$ at each $q \in M$. Note that $\mathcal{Y} = \{W_1, \ldots, W_m\}$ is a set of m orthonormal vector fields while $\mathcal{Y}^\perp = \{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ is a set of $n-m$ orthonormal vector fields.

Given a basis of $\mathbb{G}$-orthonormal vector fields $\{X_1, \ldots, X_n\}$ on M, the *generalized Christoffel symbols* are defined by the n^3 functions $\hat\Gamma^k_{ij} : M \to \mathbb{R}$ where

$$\nabla_{X_i} X_j = \hat\Gamma^k_{ij} X_k.$$

Definition 2.1: We define the **generalized symmetric Christoffel symbols** for ∇ with respect to the basis of $\mathbb{G}$-orthonormal vector fields $\{X_1, \ldots, X_n\}$ on M as the n^3 functions $\tilde\Gamma^k_{ij} : M \to \mathbb{R}$ defined by

$$\tilde\Gamma^k_{ij} X_k = \frac{1}{2}\left(\hat\Gamma^k_{ij} + \hat\Gamma^k_{ji}\right) X_k = \frac{1}{2}\mathbb{G}\left(\langle X_i : X_j\rangle, X_k\right) X_k.$$

We may define the velocity vector $\dot\gamma(t) = v^i(t)\frac{\partial}{\partial q^i}$ of the curve $\gamma(t)$ in terms of the family of vector fields $\{\mathcal{Y}, \mathcal{Y}^\perp\}$. The new expression for $\dot\gamma(t)$ is in the form

$$\dot\gamma(t) = w^a(t) W_a(\gamma(t)) + s^b(t) Y_b^\perp(\gamma(t)) \tag{2}$$

where

$$w^a = \mathbb{G}(\dot\gamma(t), W_a(\gamma(t))) \tag{3}$$

$$s^b = \mathbb{G}(\dot\gamma(t), Y_b^\perp(\gamma(t))). \tag{4}$$

Proposition 2.2: Let $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, U\}$ be a simple mechanical control system defined above. The following holds along the curve $\gamma(t)$ satisfying (1):

$$\frac{d}{dt}w^l = -\tilde{\Gamma}^l_{ap}w^a w^p - 2\tilde{\Gamma}^l_{ar}w^a s^{r-m} - \tilde{\Gamma}^l_{rk}s^{r-m}(t)s^{k-m} + u^a g(Y_a, W_l) \tag{5}$$

$$\frac{d}{dt}s^{b-m} = -\tilde{\Gamma}^b_{ap}w^a w^p - 2\tilde{\Gamma}^b_{ar}w^a s^{r-m} - \tilde{\Gamma}^b_{rk}s^{r-m}s^{k-m} \tag{6}$$

where $a, l, p \in \{1, \ldots, m\}$, $b, k, r \in \{m+1, \ldots, n\}$.

Proof: We proceed by substituting Equation 2 into Equation 3 and Equation 4 then differentiating and taking advantage of the compatibility associated with the Levi-Civita connection. ■

C. Partial Feedback Linearization

By inspection we see that Equation 5 can be written as

$$\frac{d\mathbf{w}}{dt} = \underbrace{\left[\begin{array}{cc} \mathbf{w} & \mathbf{s} \end{array}\right]\left[\tilde{\boldsymbol{\Gamma}}\right]\left[\begin{array}{c} \mathbf{w} \\ \mathbf{s} \end{array}\right]}_{\mathbf{F}} + \mathbf{G}\mathbf{u}. \tag{7}$$

We perform feedback linearization on Equation 7 by defining our input to be

$$\mathbf{u} = -\mathbf{G}^{-1}\mathbf{F} + \mathbf{G}^{-1}\mathbf{z}$$

where we assume that $\mathbf{G}$ is full rank. Now substituting $\mathbf{u}$ into Equation 7 to get

$$\frac{d\mathbf{w}}{dt} = \mathbf{z}.$$

III. Quadratically Coupled Control System

Let us consider a class of nonlinear mechanical systems where $n - m = 1$ and the right-hand-side of Equation 6 is independent of q and s. The governing equations in local coordinates are

$$\frac{d\mathbf{q}}{dt} = \mathbf{v}, \qquad \frac{d\mathbf{w}}{dt} = \mathbf{z}, \qquad \frac{ds}{dt} = Q(\mathbf{w}) \tag{8}$$

where the configuration and pseudo-velocity states are $(q, \mathbf{w}, s) \in \mathcal{Y} \times \mathcal{Y}^\perp$, $\mathbf{z} \in Z \subseteq \mathbb{R}^m$ is the control, and $Q : \mathcal{Y}_q \to \mathbb{R}$ is a quadratic map, *i.e.*, $Q(\lambda\mathbf{w}) = \lambda^2 Q(\mathbf{w})$ for all $\mathbf{w} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$. We refer to $\mathbf{w}$ and s to be pseudo-velocities because in general $(\mathbf{w}, s) \neq \mathbf{v}$. This class of system is clearly not linearly controllable. Let us define the control system to be the tuple $\Delta = (TM, \frac{d\mathbf{w}}{dt}, \frac{ds}{dt}, V)$ that defines the state space, governing equations of motion and the set of available inputs. A basic property of $\mathbb{R}$-valued quadratic forms is that indefiniteness is equivalent to the existence of a basis $\mathcal{V}$ for $\mathbb{R}^m$ such that the diagonal entries in the matrix representation of Q are all zero [2]. In other words, if Q is indefinite then exists a basis $\mathcal{V}$ for $\mathbb{R}^m$ such that the expansion

$$\frac{ds}{dt} = Q_{ij}w^i w^j \tag{9}$$

$$= 2Q_{12}w^1w^2 + \cdots + 2Q_{1m}w^1w^m + \cdots + 2Q_{23}w^2w^3 + \cdots + 2Q_{2m}w^2w^m + \cdots + 2Q_{m-1m}w^{m-1}w^m \tag{10}$$

where indices $i, j = 1, \ldots, m$ and $Q_{ij} = 0$ when $i = j$. It is also clear that if Q is indefinite then there exists at least 1 symmetric term $Q_{ij} \neq 0$ where $i \neq j$. *In order to simplify the derivation and presentation of our algorithm we will refer to the first coefficient $Q_{ij} \neq 0$ in in Equation 10 as Q_c and the corresponding w parameters as w^1_c and w^2_c.*

IV. Algorithm

This section contains the main result of this paper. We provide a global discontinuous control law with state dependent switches for a class of mechanical control systems that will drive the system to an arbitrarily small neighborhood of $(\mathbf{q}, \mathbf{0}) \in TM$ from all initial configurations and initial velocities in arbitrarily small time. This class of mechanical control systems is characterized by a quadratic coupling Q between the equations of motion for the actuated velocity states and the unactuated velocity state. Our results are applicable to the case when the quadratic coupling is indefinite. Our method can easily be extended to construct a global discontinuous control law with state dependent switches that will drive the system to an arbitrarily small neighborhood of any velocity from any initial configuration and velocity in arbitrarily small time. Our procedure for demonstrating the existence of such a control law involves three basic steps.

1) Drive $\mathbf{w} \to 0$ in time κ.
2) Drive $s \to 0$ in time γ.
3) Drive $\mathbf{w} \to 0$ while keeping $s \in B(0, \epsilon)$ in time τ.

We begin by introducing a series of lemmas which contain the control laws corresponding to each step in our basic procedure. The lemmas will be used to prove our main result.

Lemma 4.1: For all $\mathbf{w}(t_0)$ and $\kappa > 0$ if $\mathbf{z}$ defined on $[t_0, t_1]$ is $\mathbf{z}(t) = -\mathbf{w}(t_0)/\kappa$ where $t_1 = t_0 + \kappa$ then $\mathbf{w}(t_1) = 0$.

Proof: We substitute the control law $\mathbf{v}(t)$ into Equation 8 to get

$$\frac{d\mathbf{w}}{dt} = -\frac{\mathbf{w}(t_0)}{\kappa}.$$

We can apply separation of variables and integration to get

$$\mathbf{w}(t) = \mathbf{w}(t_0) - \frac{\mathbf{w}(t_0)}{\kappa}(t - t_0)$$

on $[t_0, t_1]$. It's clear from $\mathbf{w}(t)$ that $\mathbf{w}(t_1) = 0$ holds. ■

Remark 4.2: For the remainder of the proofs, we will only need the first nonzero term Q_c in the expansion of Equation 9 and the corresponding w terms w^1_c and w^2_c. Once the $\mathbf{w}$ is driven to zero then we will only utilize the $\mathbf{z}$ inputs z^1_c and z^2_c corresponding to Q_c, w^1_c and w^2_c. The remaining $\mathbf{z}$ inputs will be set to zero. Again, this simplifies our derivation and presentation of the stopping algorithm.

Lemma 4.3: For all $s(t_1)$, $\mathbf{w}(t_1) = 0$ and $\gamma > 0$ if $Q : \mathbb{R}^m \to \mathbb{R}$ is indefinite and $\mathbf{z}$ defined on $[t_1, t_2]$ is

$$z^1_c = -\operatorname{sgn}(Q_c)|M|, \qquad z^2_c = \operatorname{sgn}(s(t_1))|M|$$

where $t_2 = t_1 + \gamma$ and

$$M = \left(\frac{3}{2}\frac{s(t_1)}{Q_c\gamma^3}\right)^{1/2}$$

then $s(t_2) = 0$.

Proof: We substitute the control law $\mathbf{z}(t)$ into Equation 8 to get

$$\frac{dw_1}{dt} = -\operatorname{sgn}(Q_c)|M|, \qquad \frac{dw_2}{dt} = \operatorname{sgn}(s(t_1))|M|$$

with

$$M = \left(\frac{3}{2}\frac{s(t_1)}{Q_c\gamma^3}\right)^{1/2}.$$

We can apply separation of variables and integration to get

$$w_c^1(t) = -\operatorname{sgn}(Q_c)|M|(t-t_1)$$
$$w_c^2(t) = \operatorname{sgn}(s(t_1))|M|(t-t_1)$$

on $[t_1, t_2]$. We also know that

$$s(t_2) = s(t_1) + A\left(\frac{ds}{dt}\right)\gamma$$

where

$$A\left(\frac{ds}{dt}\right) = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} 2Q_c w_c^1(t) w_c^2(t)dt = \frac{2Q_c z_c^1 z_c^2}{\gamma}\int_0^\gamma t^2 dt = \frac{2Q_c z_c^1 z_c^2}{3}\gamma^2.$$

This gives us

$$s(t_2) = s(t_1) + \frac{2Q_c z_c^1 z_c^2}{3}\gamma^3.$$

Now substitute for z_c^1 and z_c^2 to get

$$s(t_2) = s(t_1) + \frac{2Q_c(-\operatorname{sgn}(Q_c)|M|)(\operatorname{sgn}(s(t_1)))|M|}{3}\gamma^3.$$

Now substitute for M and simplify to get

$$s(t_2) = s(t_1) - \frac{2|Q_c|\left|\left(\frac{3}{2}\frac{s(t_1)}{Q_c\gamma^3}\right)\right|(\operatorname{sgn}(s(t_1)))}{3}\gamma^3$$
$$s(t_2) = s(t_1) - |s(t_1)|\operatorname{sgn}(s(t_1))$$
$$s(t_2) = 0.$$

■

Lemma 4.4: Let us assume that

$$w_c^1(t_2) = -\operatorname{sgn}(Q_c)|M|\gamma$$

and

$$w_c^2(t_2) = \operatorname{sgn}(s(t_1))|M|\gamma.$$

For all $s(t_1)$, $\epsilon, \gamma > 0$ and $s(t_2) = 0$, if

$$z_c^1 = \frac{\operatorname{sgn}(Q_c)}{\tau}|M|, \qquad z_c^2 = -\frac{\operatorname{sgn}(s(t_1))}{\tau}|M|$$

where

$$\tau < \frac{\epsilon\gamma^3}{|s(t_1)|} \quad \text{and} \quad M = \left(\frac{3}{2}\frac{s(t_1)}{Q_c\gamma^3}\right)^{1/2}$$

then $(\mathbf{w}(t_3), s(t_3)) \in B(0, \epsilon)$.

Proof: We substitute the control law $\mathbf{z}(t)$ into Equation 8 to get

$$\frac{dw_1}{dt} = \frac{\operatorname{sgn}(Q_c)}{\tau}|M|, \qquad \frac{dw_2}{dt} = -\frac{\operatorname{sgn}(s(t_1))}{\tau}|M|$$

with

$$\tau < \frac{\epsilon\gamma^3}{|s(t_1)|} \quad \text{and} \quad M = \left(\frac{3}{2}\frac{s(t_1)}{Q_c\gamma^3}\right)^{1/2}.$$

We can apply separation of variables and integration to get

$$w_c^1(t) = w_c^1(t_2) + \frac{\operatorname{sgn}(Q_c)}{\tau}|M|(t-t_2)$$
$$w_c^2(t) = w_c^2(t_2) - \frac{\operatorname{sgn}(s(t_1))}{\tau}|M|(t-t_2)$$

on $[t_2, t_3]$. It is clear that $\mathbf{w}(t_3) = 0$. Now we need to check that $|s(t_3)| < \epsilon$. Since $s(t_2) = 0$ then we also know that

$$s(t_3) = A\left(\frac{ds}{dt}\right)\tau$$

where

$$A\left(\frac{ds}{dt}\right) = \frac{1}{t_3 - t_2}\int_{t_2}^{t_3} 2Q_c w_c^1(t) w_c^2(t)dt = \frac{2Q_c z_c^1 z_c^2}{\tau}\int_0^\tau t^2 dt = \frac{2Q_c z_c^1 z_c^2}{3}\tau^2.$$

This gives us

$$s(t_3) = \frac{2Q_c z_c^1 z_c^2}{3}\tau^3.$$

Now substitute for z_c^1 and z_c^2 to get

$$s(t_3) = \frac{2Q_c(\frac{\operatorname{sgn}(Q_c)}{\tau}|M|)(-\operatorname{sgn}(s(t_1)))|M|}{3}\tau^3.$$

Now substitute for M and simplify to get

$$|s(t_3)| = |\left|\frac{2Q_c}{3}\left(\frac{3}{2}\frac{|s(t_1)|}{Q_c\gamma^3}\right)\tau\right||$$
$$|s(t_3)| = |s(t_1)|\frac{\tau}{\gamma^3}.$$

It is clear that if

$$\tau = \frac{\epsilon\gamma^3}{|s(t_1)|}$$

then $|s(t_3)| = \epsilon$. Recall by construction we pick

$$\tau < \frac{\epsilon\gamma^3}{|s(t_1)|}$$

which implies $|s(t_3)| < \epsilon$. ■

Theorem 4.5: Given the control system $\Delta = (TM, \frac{d\mathbf{w}}{dt}, \frac{ds}{dt}, Z)$. For all initial conditions $\mathbf{w}(t_0)$ and $s(t_0)$ and constants $\epsilon, \delta > 0$ if Q is indefinite and $\mathbf{z}(t)$ is defined on $[t_0, t_3]$ to be

Time Interval	$z_c^1(t)$	$z_c^2(t)$
$[t_0, t_1)$	$-\frac{w_c^1(t_0)}{\kappa}$	$-\frac{w_c^2(t_0)}{\kappa}$
$[t_1, t_2)$	$-\operatorname{sgn}(Q_c)\|M\|$	$\operatorname{sgn}(s(t_1))\|M\|$
$[t_2, t_3]$	$\frac{\operatorname{sgn}(Q_c)}{\tau}\|M\|$	$-\frac{\operatorname{sgn}(s(t_1))}{\tau}\|M\|$

where

$$M = \left(\frac{3}{2} \frac{s(t_1)}{Q_c \gamma^3} \right)^{1/2} \quad \text{and} \quad \tau < \frac{\epsilon \gamma^3}{|s(t_1)|}$$

then $(\mathbf{w}(t_3), s(t_3)) \in B(\mathbf{0}, \epsilon)$ and $|t_3 - t_0| < \delta$.

Proof: It follows from Lemma 4.1 that the control law defined on the interval $[t_0, t_1)$ will give rise to $\mathbf{w}(t_1) = 0$. Now we apply Lemma 4.3 to see that given $\mathbf{w}(t_1) = 0$ and any $s(t_1)$ that the control law defined on the interval $[t_1, t_2)$ will result in $s(t_2) = 0$, $w_c^1(t_2) = -\operatorname{sgn}(Q_C)|M|\gamma$ and $w_c^2(t_2) = \operatorname{sgn}(s(t_1))|M|\gamma$. Finally, we can appeal to Lemma 4.4 to show that given $s(t_2) = 0$, $w_c^1(t_2) = -\operatorname{sgn}(Q_c)|M|\gamma$ and $w_c^2(t_2) = \operatorname{sgn}(s(t_1))|M|\gamma$ that the control law defined on the final interval $[t_2, t_3]$ will drive the system to $(\mathbf{w}(t_3), s(t_3)) \in B(0, \epsilon)$. In order to ensure that $|t_3 - t_0| < \delta$ pick κ, γ, τ such that $\kappa + \gamma + \tau < \delta$. ■

Remark 4.6: Theorem 4.5 can be easily extended to construct a global discontinuous control law with state dependent switches that will drive the system to an arbitrarily small neighborhood of any velocity from any $(\mathbf{q}, \mathbf{w}, s) \in TM$ in arbitrarily small time. In addition, it can be shown that as long as the controls for Δ take values in a subset $Z \subset \mathbb{R}^m$ for which $0 \in \operatorname{int}(\operatorname{conv}(U))$ then we can still drive Δ to an arbitrarily small neighborhood of rest.

V. Example: Forced Planar Rigid Body Hovercraft Robot

In this section we review the geometric model, construct the alternative representation of the equations of motion, perform partial feedback linearization and construct the stopping algorithm for the planar rigid body (Figure 1). The configuration manifold for the system is the Lie group $SE(2)$. Use coordinates (x, y, θ) for the planar robot where (x, y) describes the position of the center of mass and θ describes the orientation of the body frame $\{b_1, b_2\}$ with respect to the inertial frame $\{e_1, e_2\}$. In these coordinates, the Riemannian metric is given by

$$\mathbb{G} = m dx \otimes dx + m dy \otimes dy + J d\theta \otimes d\theta,$$

where m is the mass of the body and J is the moment of inertia about the center of mass. Let us analyze the set of inputs that consist of the force F^1 applied to a point and a torque F^3 about the center of mass. We assume that the point of application of the force is a distance $h > 0$ from the center of mass along the b_1 body-axis. The input force can represent a variable-direction thruster on the body which can be resolve into components along the b_1 and b_2 directions. The control inputs are given by

$$F^1 = \cos\theta dx + \sin\theta dy, \qquad F^3 = d\theta.$$

Problem Statement 5.1: Given the planar rigid body with input forces $\{F^1, F^3\}$, construct the control law $\mathbf{z}(t)$ that will drive the system from any $(\mathbf{q}(t_0), \mathbf{w}(t_0), s(t_0)) \in TM$ to $(q, B(0, \epsilon)) \in TM$ in time δ.

Solution 5.2: We use the metric $\mathbb{G}$ to compute the input vector fields $Y_a = \mathbb{G}^\sharp(F^a)$ to be

$$Y_1 = \frac{1}{m}\cos\theta \frac{\partial}{\partial x} + \frac{1}{m}\sin\theta \frac{\partial}{\partial y}, \qquad Y_3 = \frac{1}{J}\frac{\partial}{\partial \theta}.$$

Now let us construct the orthonormal basis vector fields W_1, W_2 that generate $\mathcal{Y}$. It's clear that Y_1 and Y_3 are orthogonal with respect to $\mathbb{G}$. All we need to do is normalize Y_1 and Y_3 with respect to $\mathbb{G}$ and set them equal to W_1 and W_2 respectively. This gives us

$$W_1 = \frac{\sqrt{m}}{m}\cos\theta \frac{\partial}{\partial x} + \frac{\sqrt{m}}{m}\sin\theta \frac{\partial}{\partial y}, \qquad W_2 = \frac{\sqrt{J}}{J}\frac{\partial}{\partial \theta}.$$

Now we construct the single vector field

$$Y^\perp = -\frac{\sqrt{m}}{m}\sin\theta \frac{\partial}{\partial x} + \frac{\sqrt{m}}{m}\cos\theta \frac{\partial}{\partial y}$$

where by inspection we see that it is orthogonal to $\mathcal{Y}$ and normalized. The only non-zero generalized symmetric Christoffel symbols are

$$\tilde{\Gamma}^1_{23} = -\frac{\sqrt{J}}{2J}, \qquad \tilde{\Gamma}^3_{21} = -\frac{\sqrt{J}}{2J}.$$

The resulting equations of motion are

$$\begin{aligned}
\frac{dx}{dt} &= w^1 \frac{\sqrt{m}}{m}\cos\theta - s\frac{\sqrt{m}}{m}\sin\theta, \\
\frac{dy}{dt} &= w^1 \frac{\sqrt{m}}{m}\sin\theta + s\frac{\sqrt{m}}{m}\cos\theta, \\
\frac{d\theta}{dt} &= w^2 \frac{\sqrt{J}}{J}, \\
\frac{dw^1}{dt} &= \frac{\sqrt{J}}{J} w^2 s + u^1 \frac{\sqrt{m}}{m}, \\
\frac{dw^2}{dt} &= u^2 \frac{\sqrt{J}}{J}, \\
\frac{ds}{dt} &= \frac{\sqrt{J}}{J} w^2 w^1.
\end{aligned} \tag{11}$$

Now we set $\mathbf{u}$ equal to

$$u^1 = \frac{m}{\sqrt{m}} z^1 - \frac{m}{\sqrt{m}}\frac{\sqrt{J}}{J} w^2 s, \qquad u^2 = \frac{J}{\sqrt{J}} z^2.$$

Substitute $\mathbf{u}$ into Equation 11 to get

$$\frac{dw^1}{dt} = z^1, \qquad \frac{dw^2}{dt} = z^2, \qquad \frac{ds}{dt} = \frac{\sqrt{J}}{J} w^2 w^1.$$

By inspection, we have $Q_c = \frac{\sqrt{J}}{J}$. Since $J > 0$ we conclude that $Q = \frac{\sqrt{J}}{J} w^2 w^1$ is indefinite.

Time Interval	$z^1(t)$	$z^2(t)$
$[t_0, t_1)$	$-\frac{w^1(t_0)}{\kappa}$	$-\frac{w^2(t_0)}{\kappa}$
$[t_1, t_2)$	$-\lvert M\rvert$	$\operatorname{sgn}(s(t_1))\lvert M\rvert$
$[t_2, t_3]$	$\frac{\lvert M\rvert}{\tau}$	$-\frac{\operatorname{sgn}(s(t_1))}{\tau}\lvert M\rvert$

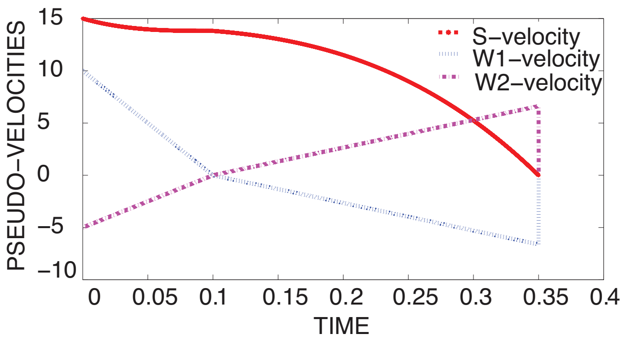

Fig. 2. Application of stopping algorithm to hovercraft robot.

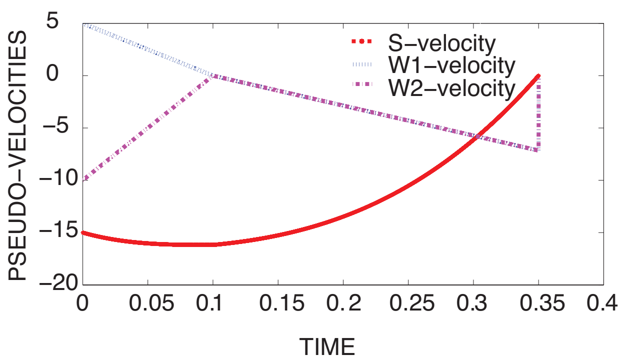

Fig. 3. Application of stopping algorithm to hovercraft robot with a different set of initial conditions.

where

$$M = \left(\frac{3}{2}\frac{Js(t_1)}{\sqrt{J}\gamma^3}\right)^{1/2}, \qquad \tau < \frac{\epsilon\gamma^3}{|s(t_1)|},$$

and we pick κ, γ and τ such that $\kappa + \gamma + \tau < \delta$.

Figures 2 and 3 illustrate the application of the algorithm by plotting the controlled and uncontrolled velocity directions for the planar hovercraft for two different sets of initial conditions. In each case the goal was to stop the robot in 0.5 seconds. ♦

VI. Conclusions and Future Work

This paper presented an algorithm to bring a simple mechanical system underactuated by one to rest from an arbitrary initial velocity. The method relies upon the analysis of an intrinsic bilinear symmetric quadratic form which naturally is present in mechanical systems of this form and the resulting control law is discontinuous with state-dependent switches. Because this form represents the degree to which the controlled and uncontrolled degrees of freedom are coupled, when the form is *indefinite* it corresponds to the "ability" of the controlled degrees of freedom to both increase and decrease the uncontrolled velocity, which is needed for arbitrary initial conditions. Our result is an important and novel contribution to the state of the art because in the general nonlinear controls context, analysis and synthesis away from equilibria is generally difficult. Our approach exploits the structure of the system as a mechanical system as the basis for the stopping algorithm.

Some preliminary work has been done to extend our results to systems where Q depends on $\mathbf{w}$, s and $\mathbf{q}$. We are also working on an iterative algorithm under bounded control inputs. Finally, we are less optimistic about systems where Q is vector-valued instead of real-valued. A computationally efficient method for determining the indefiniteness of a vector-valued quadratic form does not exist [2]. This has been given some initial consideration in control literature [1]. Additional lines of future inquiry are into mechanics and control in robotic legged locomotion with the goal of extending analyses such as in [19] to the more general context.

References

[1] F. Bullo, J. Cortes, A. Lewis, and S. Martinez *Vector-valued quadratic forms in control theory*. Sixty Open Problems in Mathematical Systems and Control Theory, V. Blondel and A. Megretski (Eds.), Princeton University Press, 315-320, 2004.

[2] F. Bullo and A. Lewis. *Geometric Control of Mechanical Systems*. Springer Science+Business Media, New York, NY, 2005.

[3] S. Chitta and V.J. Kumar. *Dynamics and generation of gaits for a planar rollerblader*. Proceedings of the 2003 IEEE IROS, 1:860-865, 2003.

[4] K.D. Do *Practical control of underactuated ships*. Ocean Engineering 37(7):1111-1119, 2010.

[5] R.Hirschorn and A.Lewis. *An example with interesting controllability and stabilisation properties*. Proceedings of the 45th IEEE CDC, 2006.

[6] A. Isidori *Nonlinear Control Systems*. Springer-Verlag London Limited, Great Britian, 1995.

[7] A. Lewis, J.P. Ostrowski, R. Murray and J. Burdick. *Nonholonomic mechanics and locomotion: the snakeboard example*. Proceedings of the 1994 IEEE ICRA, 2391-2400, 1994.

[8] R. M'Closkey *Time-varying homogeneous feedback: Design tools for the exponential stabilization of systems with drift*. International Journal of Control, 71(5):837-869, 1998.

[9] J. Nightingale, R. Hind, B. Goodwine. *Geometric analysis of a class of constrained mechanical control systems in the nonzero velocity setting*. Proceedings of the 2008 IFAC, 2008.

[10] J. Nightingale, R. Hind, B. Goodwine. *Intrinsic Vector-Valued Symmetric Form for Simple Mechanical Control Systems in the nonzero velocity setting*. Proceedings of the 2008 IEEE ICRA, 2008.

[11] J. Nightingale, R. Hind, B. Goodwine. *A stopping algorithm for mechanical systems underactuated by one control*. Proceedings of the 2008 WAFR, 2008.

[12] J.P. Ostrowski, J.P. Desai and V. Kumar. *Optimal gait selection for nonholonomic locomotion systems*. The International Journal of Robotics Research 19(5):225-237, 2000.

[13] M. Reyhanoglu, A. van der Schaft, N.H. Mcclamroch, I. Kolmanovsky *Dynamics and control of a class of underactuated mechanical systems*. IEEE Transactions on Automatic Control 44(9):1663-1671, 1999.

[14] S. Sastry *Nonlinear systems: analysis, stability and control*. Springer Science+Business Media, New York, NY, 1999.

[15] M. Tafazoli *A study of on-orbit spacecraft failures*. Acta Astronautica 64(2-3):195-205, 2009.

[16] K. Tanaka, M. Iwasaki, H.O. Wang *Stable switching fuzzy control and its application to a hovercraft type vehicle*. Proceedings of the 9th IEEE International Conference on Fuzzy, 2:804-809, 2000.

[17] X. Xin and M. Kaneda *Swing-Up control for a 3-DOF gymnastic robot with passive first joint: design and analysis*. IEEE Transactions on Robotics, 23(6):1277-1285, 2007.

[18] Richard M. Murray and Muruhan Rathinam and Willem Sluis *Differential Flatness of Mechanical Control Systems: A Catalog of Prototype Systems*. 1995 ASME Internationalal Mechanical Engineering Congress and Expo, San Francisco, CA, 1995.

[19] Hong Zou and James P Schmiedeler *The effect of asymmetrical body-mass distribution on the stability and dynamics of quadruped bounding*. IEEE Transactions on Robotics, 22(4):711-723, 2006.