Fractional-Order Dynamics in a Random, Approximately Scale-Free Network of Agents

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***Abstract*—Differential equations with fractional-order derivatives, *e.g.*, the "one-half" derivative, have a long history in mathematics, but have not yet attained mainstream use in engineering and applied science. While applications do exist in modeling specific phenomena such as visco-elasticity and other types of difficult-to-model phenomena, and extensions to control such as in fractional-order PID do exist, everyday use of fractional order modeling is uncommon. A subset of complex systems called Cyber-Physical Systems (CPS) is receiving much emphasis in the research community. In this paper we show examples of networked system models which exhibit fractional-order dynamic responses. This suggests that fractional-order dynamics may be prevalent in CPS and hence may be an important and useful modeling tool in that area. We particularly focus on a *scale free* networked system.**

I. Introduction

This paper investigates fractional-order modeling for networked Cyber-Physical Systems (CPS). We show that for distinct types of linear systems with integer-order component dynamics, the interaction among the components lead to fractional-order dynamics. While it is the subject of continued investigation, we believe that fractional-order dynamics may be very common in formation control of systems of mobile robots and other complex and cyber-physical systems. The examples in this paper make it evident that such effects may be commonplace, and hence if tractable models and accurate descriptions of system dynamics are necessary, then fractional-order models and system identification may be necessary in CPS.

Recognizing this fractional-order nature of the dynamics is important for several reasons. First, it leads to a deeper understanding of the system and broadens the "toolbox" of control possibilities for multi-robot systems. Second, it provides for substantial model reduction and computational savings for modeling and controlling the system. Third, when considering loop shaping, large frequency ranges characterized by non-integer order dynamics (non-integer magnitude slopes and non-multiple of 90° phases) may need to be addressed by fractional-order control methods.

Control of multi-robot systems is a well-studied area in robotics and control with many significant contributions (see for example, [1, 10, 12, 16, 26] and survey papers [8, 23]). Some of the author's prior work is directed toward exact model reduction for symmetric systems [14, 20, 21]. Fractional calculus has a much longer history. As a mathematical subject, it dates back to near the foundations of calculus, and it has been used in engineering applications for at least several decades. Books on the mathematics and engineering applications include [2, 24] and there are a number of review articles [17, 25]. A closely related study is [6, 7] which studied formation control of fractional systems. While involving fractional-order systems and formation control, that paper considered a different problem in that the individual components were fractional in nature; whereas, in this paper, the fractional dynamics arise from the structure of the interaction among the agents. Other related studies include [27] (walking robots), [11, 28] (flexible manipulators), [9] (time delays) and control using fractional-order PID control [22, 28]. Studies in other areas such as visco-elastic phenomena can be found in [15, 19].

The type of system considered in this paper is a *scale-free* network. Scale-free networks are such that a relatively small number of nodes have a very high degree (degree of connectivity to other notes) while most nodes have a relatively small degree. *Self-similarity* is a common characteristic of scale-free networks, and we will make use of that fact in the subsequent analysis. The literature on scale-free networks is vast, but notable papers include [3, 4] and the book [5].

II. Dynamics of an Example Scale-Free Network

We consider a network of agents. Each agent is connected to some of the other agents and the network is configured initially with few agents all connected. As additional agents are added, they preferentially connect to the agents with a relatively large number of connectected agents. Specifically we consider 200 agents. Initially four agents are created and all four of the agents are connected to the other three. Then 196 agents are added one at a time. Each of these 196 agents are connected to three other agents when they are added to the network, and they are preferentially connected to agents with a large degree. Specifically, we construct an adjacency matrix, A with a 1 in the (n, m) position if agents n and m are connected. Because we will model the interconnections as mechanical components, we consider an undirected graph representation and hence A is symmetric. Specifically, the algorithm in Table I (Ocatave syntax) generated the network studied in this paper.

A system created by this algorithm is illustrated in Figure 1.[1] Obviously we represent the system with a graph, where the nodes represent individual agents and an edge

[1]The illustrated graph was created using the gephi visualization package, `http://gephi.org`.

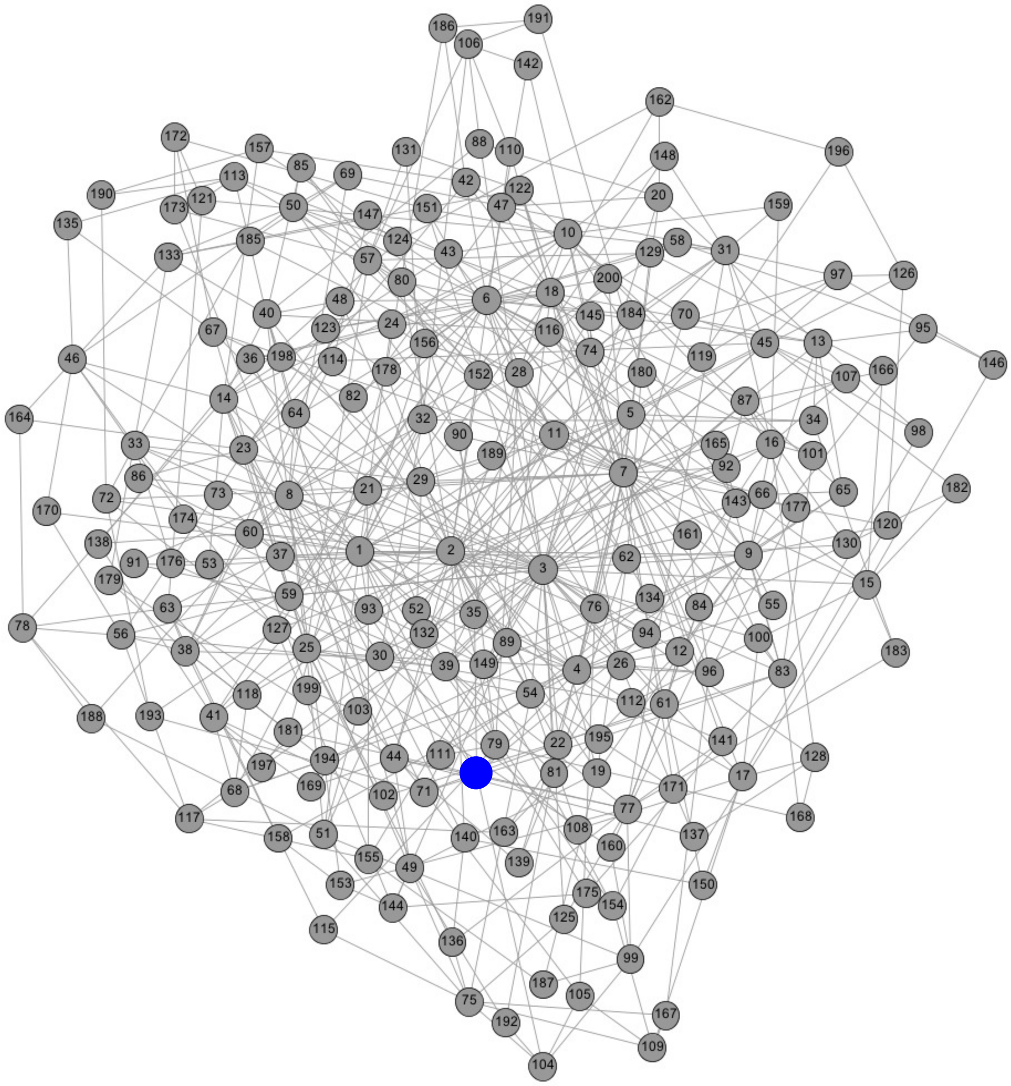

Fig. 1. Scale-free network.

TABLE I
ALGORITHM FOR CONSTRUCTING THE NETWORK.

```
N = 200;
micon = 3;
A = zeros(N,N);
A(1:mincon+1,1:mincon+1) =...
  ones(mincon+1,mincon+1)-eye(mincon+1);
for n=5:N
  adj = sum(A');
  for i=1:mincon
    flag = 0;
    while(flag<1)
      target = floor(rand()*(n-1))+1;
      if(adj(target) > rand()*(n+mincon)...
          && target != n && A(target,n) != 1)
        A(target,n) = 1;
        A(n,target) = 1;
        flag = 1;
      end
    end
  end
end
```

between nodes represents connectedness. This network is, at least approximately, scale-free. Figure 2 plots the degree of a node versus the number of agents with that degree, which is approximately a power law, indicated by the nearly straight line on the log-log plot.

Now we consider the dynamics of the system. Motivated by formation control, each agent has a unit mass and one degree of freedom. Each edge in the network is randomly assigned either a spring or viscous dashpot with equal probability (this assignment is not illustrated in the graph in Figure 1). The equation of motion for agent i is

$$m\ddot{x}_i = \sum_{j \in \mathcal{N}} f_{i,j}(x_j, \dot{x}_j) \tag{1}$$

where $\mathcal{N}$ represents the set of neighbors of agent i,

$$f_{i,j}(x_j, \dot{x}_j) = \begin{cases} kx_j, & \text{edge } (i,j) \text{ is a spring} \\ b\dot{x}_j, & \text{edge } (i,j) \text{ is a dashpot} \end{cases}$$

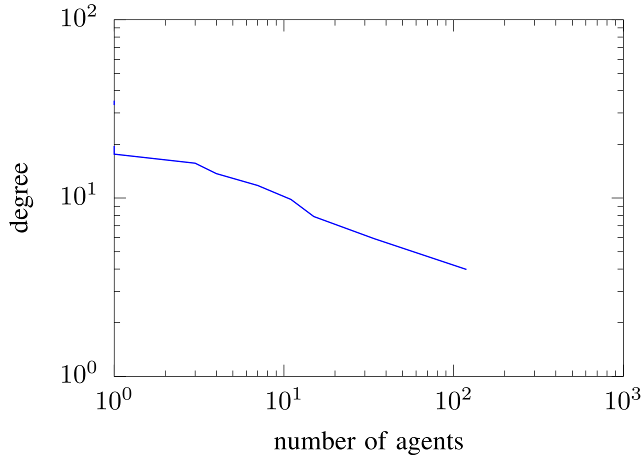

Fig. 2. Scale free network.

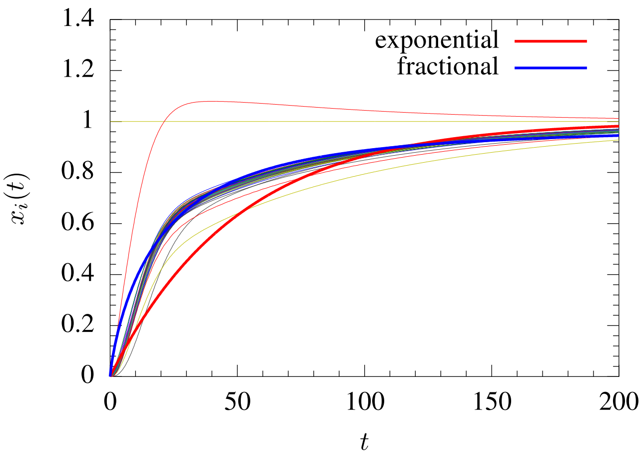

Fig. 3. Step response of scale free network.

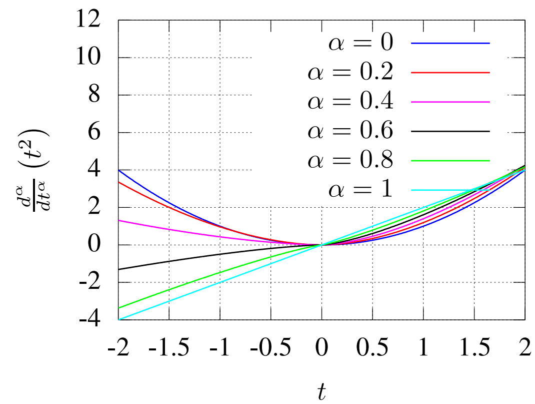

Fig. 4. Fractional-order derivatives for $f(t) = t^2$ for various orders between 0 and 1. Note that the zeroth derivative is the parabola, the first derivative the expected straight line and the fractional derivatives between these two vary in a reasonably expected manner.

where $k = 1$, $b = 10$ and $m = 1$. All agents have zero initial conditions except one agent (selected randomly) which has an initial value of one and initial velocity of zero.

In this example, agent 27 (colored in blue in Figure 1) was randomly selected, so the dynamics of the system are described by the set of 200 second-order differential equations given in Equation 1 with $x_i(0) = \dot{x}_i(0) = 0$ for all i except $x_{27}(0) = 1$ and $\dot{x}_{27}(0) = 0$. Thus, this is a type of step response where agent 27 is the input. The response of the system is illustrated in Figure 3. All 200 agent responses are plotted with the thin lines. The thicker red and blue lines are exponential and fractional-order solutions described subsequently.

We emphasize that while fractional-order dynamics are *present* in this problem and therefore important to understand, it is *not* the case that the step response with other nodes selected as the input are necessarily fractional-order in nature. The contribution of this paper is to highlight the fractional-order nature of some of the dynamics which should be considered in control and analysis of such problems. Indeed, integer-order dynamics may even predominant, but a full understanding of the problem requires consideration of both fractional- and integer-order dynamics.

III. Dynamics of Fractional-Order Systems

This section reviews fractional-order derivatives, integrals and differential equations, and generally follows the development from the references cited in the Introduction.

It is natural to ask, given a function, $f(t)$ with a first derivative, $f^{(1)}(t)$ and second derivative, $f^{(2)}(t)$, *etc.*, whether there are operators "in between" the integer order derivatives such as

$$\frac{d^{\frac{1}{2}}f}{dt^{\frac{1}{2}}}(t) = f^{\left(\frac{1}{2}\right)}(t).$$

To begin, consider $f(t) = t^k$, and observe that

$$\frac{d^n}{dt^n}t^k = \frac{k!}{(k-n)!}t^{k-n} \tag{2}$$

when n is an integer. The most common generalization of the factorial function is the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt.$$

Note that in the case where α is an integer, this can be integrated by parts multiple times to eliminate the t-term in the integrand and it is clear that $\Gamma(n) = (n-1)!$. Replacing the factorials in Equation 2 with gamma functions gives

$$\frac{d^\alpha}{dt^\alpha}t^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha},$$

which is illustrated in Figure 4 for several $\alpha \in [0, 1]$ for $f(t) = t^2$. The intermediate-order derivatives between 0 and 1 are such that they provide an intuitively acceptable interpolation between the two integer-order derivatives.

To extend this notion beyond simple polynomials, we use Cauchy's formula for repeated integration, which is given by

$$\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} f(\tau_n)\, d\tau_n d\tau_{n-1} \cdots d\tau_1 = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau, \tag{3}$$

and is easily proven by induction. One interpretation of this formula is that "integrating the function, f, n times" is given by the single integral on the right-hand side of Equation 3. In that expression, the number of integrations, n, only appears in the factorial function and in the exponent in the integrand. Of these two, only the factorial function requires n to be an integer. Hence, if we denote n such integrations by $f^{-n}(t)$, we can write

$$f^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha} f(\tau) d\tau, \tag{4}$$

which provides a means for fractional-order integration, from which fractional-order derivatives immediately follow because if we want, for example, the $3/4$ derivative, we can integrate $1/4$ times and then differentiate once (one time, integer order).

Note that, unlike integer-order derivatives, fractional-order derivatives require more than local information. In fact, it is apparent from the integral in the definition in Equation 4, that *all* past values of a function enter into the computation for the fractional derivative. This imposes some significant computational cost on evaluating fractional-order derivatives. It is worth noting, however, that for differential equations most contexts implicitly assume analytic solutions, which also effectively incorporate non-local information by way of all the derivatives of the function under consideration.

Here we will take a standard linear control-theoretic approach and assume that all initial conditions are zero and also that all the history for all signals for negative times are zero as well. While closed-form solutions for fractional-order differential equations do exist, we also must resort to numerical approximations. To that end, if we consider the first and second derivatives of a function to be defined as

$$\frac{df}{dt}(t) = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t}$$
$$\frac{d^2 f}{dt^2}(t) = \lim_{\Delta t \to 0} \frac{f(t) - 2f(t-\Delta t) + f(t-2\Delta t)}{(\Delta t)^2}$$

or in general for an integer n

$$\frac{d^n f}{dt^n}(t) = \lim_{\Delta t \to 0} \frac{\sum_{0 \le m \le n} (-1)^m \binom{n}{m} f(t + (n-m)\Delta t)}{(\Delta t)^n},$$

where the usual binomial coefficient is given by

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!},$$

which, consistent with what we have done so far is easily generalized to non-integers by gamma functions

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma(\alpha-m+1)}.$$

Using this we arrive at the *Grünwald - Letnikov derivative*:

$$\frac{d^{\alpha} f}{dt^{\alpha}}(t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(t + (\alpha - j)\Delta t),$$

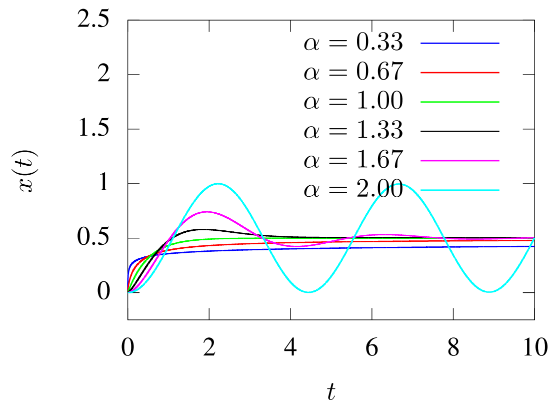

Fig. 5. Solution to Equation 5 using Equation 6.

which, similar to Equation 4 includes all past values of $f(t)$. If $\Delta t \ll 1$ and $t = m\Delta t$, then the time shift by α is small and if $x(t) = 0$ for $t \le 0$, then we have

$$\frac{d^{\alpha} f}{dt^{\alpha}}(t) \approx \frac{1}{(\Delta t)^{\alpha}} \sum_{j=0}^{m} (-1)^j \binom{\alpha}{j} f(t - j\Delta t),$$

which is a useful approximation to solve fractional-order differential equations.

For example, for

$$\frac{d^{\alpha} x}{dt^{\alpha}}(t) + 2x(t) = 1 \tag{5}$$

substituting the finite-difference approximation from the Grünwald - Letnikov definition and letting $t = m\Delta t$, then

$$\frac{d^{\alpha} x}{dt^{\alpha}}(m\Delta t) + 2x(m\Delta t) = 1$$

is approximated by

$$\frac{1}{(\Delta t)^{\alpha}} \sum_{j=0}^{m} (-1)^j \binom{\alpha}{j} x((m-j)\Delta t) + 2x(m\Delta t) = 1.$$

Solving for $x(m\Delta t)$ gives

$$x(m\Delta t) \approx \frac{1 - \frac{1}{(\Delta t)^{\alpha}} \sum_{j=1}^{m} (-1)^j \binom{\alpha}{j} x((m-j)\Delta t)}{2 + \frac{1}{(\Delta t)^{\alpha}}}. \tag{6}$$

Solutions for various $\alpha \in [0.25, 2.0]$ are illustrated in Figure 5. When $\alpha = 1$ and 2 we observe the expected exponential and harmonic solutions, respectively. Intermediate values for the order of the derivative produce reasonably intuitive intermediate responses. Octave code computing these solutions is in Table II.

Observe that the step response for fractional-order systems with orders between zero and one initially increase faster than first order, but then qualitatively turn more sharply and have a slower tail of convergence to the steady-state solution. Referring back to the scale-free networks response in Figure 3, we can observe a similar phenomena. A first-order exponential solution that, by eye, matches the general response of the system fairly well has the same relationship to the system

TABLE II

CODE TO COMPUTE SOLUTIONS TO EQUATION 5 USING EQUATION 6.

```
for alpha = [1/3 2/3 1 4/3 5/3 2]
  x = 0;
  coefs = 0;
  coefs(1) = -bincoeff(alpha,1);
  for i = 2:length(t)
    sum = dot(fliplr(x),coefs);
    x(i) = (1 - sum/(dt^alpha))/(2 + 1/dt^alpha);
    coefs(i) = (-1)^i*bincoeff(alpha,i);
  end
end
```

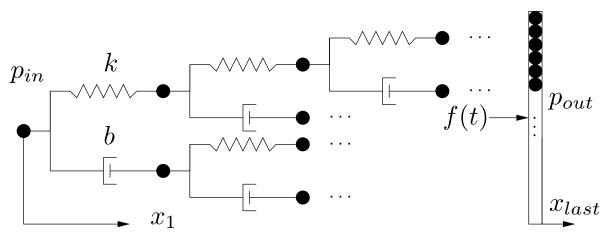

Fig. 6. Structure of robotic formation.

response: initially the system response rises faster, and then crosses the first-order solution and converges to the steady-state solution more slowly. This suggests that a fractional-order model may provide a good reduced-order representation.

Indeed, if we numerically compute the solution to

$$\frac{d^{0.8}x}{dt^{\frac{4}{5}}}(t) + 0.075x(t) = 0.075$$

with $x(t) = 0$ for $t \leq 0$, we obtain the blue step response plotted in Figure 3. Clearly, this matches the dynamic response of the system better than the first-order exponential response. We emphasize that for neither case, the first-order exponential nor the fractional-order solution, did we utilize an optimized system identification procedure, but rather did the matching "by hand", so better matches may exist. However, in the case of the first-order response, because the system solutions cross the exponential (twice in fact), it is not possible to match the curve with any solution of the form $1 - \exp(-\alpha t)$ regardless of the system identification method used.

IV. Dynamics of a Self-Similar Network

In this section we summarize some prior work which indicates that fractional-order dynamics must be present in a type of self-similar network, and because such self-similarity is present in scale-free networks, the presence of fractional-order dynamics is not surprising. Much of this is a summary from [13] which was motivated by [18, 19].

Consider the system illustrated in Figure 6, which is a fleet of robots arranged in a tree network where in each generation every robot is connected with three other robots, one from the previous generation and two in the subsequent generation. Going to the subsequent generation, one of the robots is connected via a spring and the other by a damper.

In the case where there is an infinite number of generations, this system is self-similar. Consider the transfer function from the input robot, x_1 to the last generation, x_{last} and consider also the transfer function from any other robot, say one in the second generation, to the last robot. In the limit of an infinite number of generations, these transfer functions are equal, which gives recursive relation the transfer function must satisfy, which leads to a repeated fraction representation, which ultimately leads to the transfer function relating the spacing between the first and last generations to the difference in force applied to the first and last robots:

$$\frac{X_1(s) - X_{last}(s)}{F(s)} = \left(\frac{1}{\sqrt{kb}}\right)\frac{1}{\sqrt{s}}.$$

The $\sqrt{s}$ term in the denominator of the transfer function obviously corresponds to a $1/2$-order derivative in the time domain, which is robustly present in the actual system. Even for a relatively small number of generations, such as 6 or 7, the Bode plot for the system is characterized by a wide frequency band with a magnitude plot with slope -10 db/decade and a phase of $-45°$, corresponding to such a half-order derivative. Also numerically the step response almost exactly matches a half-order step response.

The reason to expect fractional-order dynamics in a generic scale-free network follows similar reasoning. Scale invariance is a generic property of self-similar networks. For example, if we select two nodes in the network at random, they will likely be connected nodes with a higher degree and nodes with a lower degree. Because the distribution of degree in a scale-free network follows a power law distribution, at least statistically, the *relative* degree of the neighbors of randomly selected nodes will be characterized by that power law. As such, at least relatively, the recursive structure of the transfer function between elements in our fixed network in Figure 6 will likely also be present in the randomly-generated scale-free network, and hence similar fractional-order dynamics are not unexpected.

V. Conclusions and Future Work

This paper constructed a scale-free network of mechanical agents and studied the dynamic response of the system. By choosing an agent at random, the dynamic response of the rest of the network was computed and it was observed that the nature of the solutions were such that fractional-order dynamics were present. Specifically, by tuning a fractional-order step response by hand, it was determined that the order of the response was approximately, $4/5$, *i.e.*, the dynamics were a solution to a differential equation that had a derivative of $4/5$ order. This was not unexpected, because prior work had indicated that self-similarity was at the core of the analysis indicating that another system was characterized by fractional-order dynamics, and scale-free networks are similarly characterized by self-similarity.

Future work involves several related lines of work. First, using formal system identification methods we may say precisely in what manner the first-order exponential solution is the best fit we can find and thus characterize in a quantifiable way the degree to which it does not model the system well.

Correspondingly, using a fractional-order identification method we can also find the best fractional-order model for the system. Also, the big open question is the second line of inquiry: to what degree can it be stated with certainty that scale-free networks exhibit fractional-order dynamics so that we can be be guaranteed to observe them?

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