Controllability with Unilateral Control Inputs

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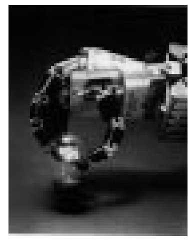
In this talk we present a controllability test for systems which may have inputs which are constrained to be non–negative. This problem has not been fully investigated, but nonetheless is of great practical importance. Our result is based on a general result by Sussmann [3], but is formulated in simpler terms and is easy to apply to *engineering* type problems.

Motivating Examples

Many control systems have inputs which must be *positive*.



Unilateral inputs are problematic for most controllability tests.

Previous results: Sussmann† and Lynch and Mason‡.

†Hector J. Sussmann. A general theorem on local controllability. *Siam J. Control and Optimization*, 25(1):158–194, 1987.

‡K. M. Lynch and M. T. Mason. Stable pushing: Mechanics, controllability and planning. *International Journal of Robotics Research*, to appear.

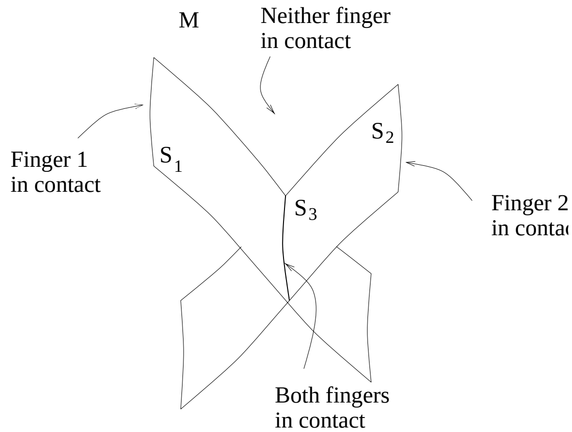
Slide 1

In both the linear as well as the nonlinear context, controllability tests assume that control inputs can be both positive and negative. This is usually implicitly assumed because the test ultimately relies upon a set of vectors *spanning* a linear space. Unfortunately, in the nonlinear context, the spanning requirement can not simply be changed to a positive spanning or convex hull type requirement because Lie brackets can not simply reformulated in such a manner.

As illustrated, at a minimum there are two important classes of examples where control inputs are constrained to be non–negative: "thruster" type problems, and manipulation via physical contact.

Unilateral inputs may arise frequently in problems where there is *intermittent contact*, such as robotic grasping or pushing problems.

Consider a simple two finger grasping problem:



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The authors have done previous work regarding controllability and trajectory generation for so–called *stratified systems*. Such systems are characterized by their configuration space containing submanifolds upon which the system is subjected to constraints which are not present off of the submanifold.

Problem formulation

- We consider problems of the form

$$\dot{x} = f(x) + h_i(x)v^i + g_j(x)u^j. \tag{1}$$

- $f(x)$ is the *drift term.*
- The v^i are the *unilateral inputs*, *i.e.*, $v^i \in [0, 1)$.
- The u^j are regular inputs, *i.e.*, $u^j \in (-1, 1)$.

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This slide mainly establishes notation. Also it illustrates the fact that we consider a very general class of problems. We consider control systems vector fields with no inputs (the drift term), vector fields with inputs restricted to be non–negative (the unilateral inputs) and vector fields with regular inputs that can be both positive and negative.

Our result is based upon a general result due to Sussmann.†

$$
\begin{tikzcd}
\underline{\text{Vector Fields}} & & \underline{\text{Indeterminates}} \\
\left.\begin{array}{c} \{f, g_1, \dots, g_i\} \\ \dot{x} = f(x) + g_i(x)u^i \end{array}\right\} \arrow[rr] & & \left\{\begin{array}{c} \{X_0, X_1, \dots, X_i\} \\ \dot{S} = S(X_0 + X_i u^i) \end{array}\right. \arrow[d] \\
& & \text{“Lie group” } \hat{S}(\mathbf{X}) \subset \hat{\mathbf{G}}(\mathbf{X}) \arrow[d] \\
& & \begin{array}{l} \text{“Input symmetries”} \\ \text{“Dilations”} \end{array} \arrow[d] \\
\text{Controllability} & & \text{Controllability} \arrow[ll, "\mathrm{Ev}(\cdot)"']
\end{tikzcd}
$$

†Hector J. Sussmann. A general theorem on local controllability. *Siam J. Control and Optimization*, 25(1):158–194, 1987.

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The general result in [3] is based upon associating with the vector fields in the original expression for the control system to *indeterminates*. One would like to think of the control system as a sort of “group action” on its state space. In order to make this rigorous, we work with the *free Lie algebra* in the indeterminates. Along with the free Lie algebra, we have, among other things,

- the free associative algebra generated by the indeterminates,
- formal power series in the indeterminates,
- the exponential map and its inverse, log,
- formal brackets,
- Lie series in the indeterminates,
- the group of exponential Lie series,
- the evaluation map,
- input symmetries, and
- dilations.

The main results in [3] are formulated in terms of tools from the left–hand (“Indeterminates”) column. The main result in this talk is formulated in terms of the vector fields in the right–hand column, and this can be applied without knowledge of all the algebraic machinery from the left–hand column.

Good and Bad brackets

- Let δ^f, δ^{h_i} and δ^{g_j} denote the number of times that f, h_i and g_j appear in a given bracket.
- We designate brackets as "good" or "bad" as follows:
  - A bracket is "bad" if δ^{g_j} is even for each j, $\sum_{i=1}^{m} \delta^{h_i} \neq 1$ and, in total, has an odd number of terms
  - Otherwise a bracket is "good."
- Examples:

f	bad	h_i	good
$[\cdot,\cdot]$	good	$[g_k,[g_j,h_i]]$	good
$[g_j,[g_j,f]]$	bad	$[g_j,[g_j,g_k]]$	good

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Anyone familiar with the "good" and "bad" bracket formulation for normal systems with drift should find our definitions of good and bad brackets familiar. Essentially, we are treating unilateral input like drift terms, *except* when there is only one unilateral input in a bracket.

The main result

Consider the control system described by Equation 1. Assume that the system satisfies the LARC and that there exist coefficients λ_i and α_j such that

$$\sum_i \lambda_i h_i(x_0) + \sum_j \alpha_j g_j(x_0) = 0, \qquad (2)$$

where $\lambda_i \in (0,1)$ and $\alpha_j \in \mathbb{R}$. Assume further that any bad bracket can be written as a linear combination of brackets of lower total degree.

Then the system is STLC at x_0.

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An important point here is that we do *not* require that the unilateral inputs be spanned by the normal inputs. What we require is that only one particular positive combination of them can be expressed as a combination of the ordinary vector fields. Then, if all the bad brackets can be expressed by lower order good brackets, then we have controllability.

Idea of proof

- In the indeterminate formulation, the "bad" brackets are the brackets that are fixed under the action of a group of *input symmetries*. In our case, this group is generated by
  - $\sigma_i : \{g_1, \ldots, g_i, \ldots, g_m\} \mapsto \{g_1, \ldots, -g_i, \ldots, g_m\}$,
  - $\pi_m \in S_n : g_j \mapsto g_{\pi_n(j)}$, and
  - $\pi_n \in S_n : h_j \mapsto h_{\pi_n(j)}$.
- There is some flexibility in the notion of *degree*. In this case, we use the dilation defined by

$$\Delta(\rho) : (X_0, \ldots, X_{m+n}) \mapsto (\rho X_0, \rho^\theta X_1, \ldots, \rho^\theta X_m, \rho X_{m+1}, \ldots, \rho X_{m+n}). \tag{3}$$

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The proof of the proposition is far too long to present in detail. Here we just highlight two features of the proof, *input symmetries* and *dilations*.

The group of input symmetries act on the inputs in a manner that maps solutions to solution. Examples of inputs given in [3] include: interchanging two inputs, multiplying an input by -1 if its range of allowable values permits it, and time reversal. In our case, the group of input symmetries is generated by the group of permutations acting on the set of unilateral inputs, the group of permutations acting on the regular inputs, and the map that takes a regular input to the same input with opposite sign. Time reversal is also an input symmetry, but is implicitly incorporated into the result from [3] upon which we base our proof.

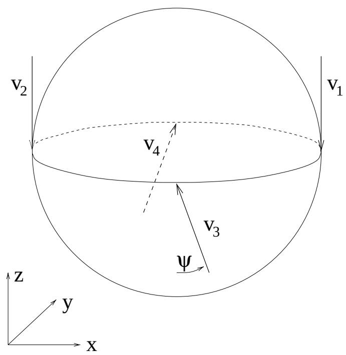
We also are able to manipulate the notion of degree by using *dilations*. We consider the dilation that assigns to each unilateral vector field a slightly higher degree than the drift vector field and the normal inputs.

Finally, we note the requirement that a particular sum of the unilateral inputs can be expressed as a sum of the regular inputs is equivalent to requiring that when a single unilateral input appears in a bracket, then, under the action of the group of input symmetries, the fixed element will contain the *sum* of the unilateral inputs. Since we assume that this sum can be expressed by a sum of ordinary inputs, each of this type of bracket will automatically be spanned by brackets of lower degree.

We note if there is more than one h_i in a bracket, this will not hold because the action of the symmetrization operator will not result in a sum of brackets that can be combined together to contain one sum of all the unilateral inputs.

Example

Consider a spherical rigid body with four "thrusters."



- Roll–Pitch–Yaw Euler angles give:

$$\dot{x} = f(x) + h_i(x)v^i + g_j(x)u^j.$$

- Tedious calculations show

$$\begin{aligned} T_xM = \operatorname{span}\{&h_1, h_2, h_3, g, \\ &[g,h_1],[g,h_2],[h_1,f],[h_2,f], \\ &[h_3,f],[g,h_3],[[g,h_1],f], \\ &[[g,h_2],f],[[g,h_3],f]\}. \end{aligned}$$

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Here we consider a simple example of a spherical rigid body with thrusters. If we parameterize the configuration manifold with the x– y– and z–displacements of the center of mass of the body as well as the "roll," "pitch" and "yaw" Euler angles, we can write the equations of motion in the form the we require.

It's fairly straightforward to show that the collection of brackets listed span the phase space for the system.

- From the collection

$$\{h_1, h_2, h_3, g, [g, h_1], [g, h_2], [h_1, f], [h_2, f], [h_3, f], \\ [g, h_3], [[g, h_1], f], [[g, h_2], f], [[g, h_3], f]\}\,.$$

  the highest order bracket has degree $3 + \epsilon$.

- Degree 1 "bad" brackets: $f(x)$.
- Degree 2 brackets are automatically good.
- Degree 3 bad brackets: must have 0 or 2 g's.
  - If there are zero g's, there must be one or more h_i's.
    * One $h_i \Longrightarrow$ not bad.
    * Two or more h_i's, $\Longrightarrow$, degree $\geq 3 + \epsilon$.
  - If there are two g_i's, there must be one h_i or one f.
    * One $h_i \Longrightarrow$ not bad.
    * One f: $[g, f](x) = 0 \Longrightarrow$ annihilated.

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There are far too many details here to absorb in the talk, but it illustrates the type of analysis necessary to determine controllability. Basically, we need to determine the maximum degree of brackets need to span the tangent space, and then make sure that all the bad brackets of lower degree are killed off.

Simulations/Intuition

- For the satellite,suppose we want a ϕ_3 displacement.

$$\dot{\phi}_3 = \frac{1}{5}([[g,h_1],f] + [[g,h_2],f]) = [[g,h_1+h_2],f].$$

- Expanding as flows:

$$g \cdot (h_1+h_2) \cdot -g \cdot -(h_1+h_2) \cdot f \cdot (h_1+h_2) \cdot g \cdot -(h_1+h_2) \cdot -g \cdot -f.$$

- Problems: $-(h_1+h_2)$ and $-f$.
- But $-(h_1+h_2) = (h_3+h_4)$.
- Also, $[[g,h_1+h_2],f] = [-f,[g,(h_1+h_2)]]$, so we have

$$g \cdot (h_1+h_2) \cdot -g \cdot (h_3+h_4) \cdot f \cdot -f \cdot g \cdot (h_1+h_2) \cdot -g \cdot (h_3+h_4).$$

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This slide illustrates what is a fact for the satellite example, and what I suspect is possibly a more general phenomenon. In fact, this may illustrate the original intuition behind the Hermes conjecture [3].

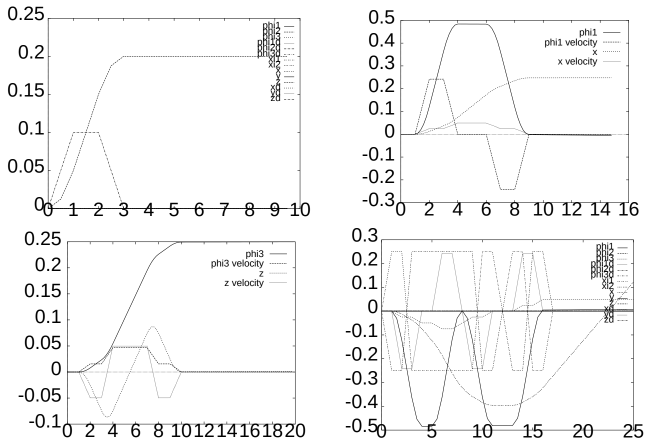
In this, and the following slide, we will illustrate the controllability of the satellite example by constructing control inputs which independently displace the satellite in each of the 12 independent directions in its phase space. To do this, we utilize a simple adaptation of the motion planning algorithm presented by Lafferriere and Sussmann in [1].

Since every bad bracket must be spanned by lower order good brackets, the only brackets for which we need to construct inputs are the good brackets. A simplistic approach to the motion planning problem would be to resolve a desired motion into a Lie bracket direction, and then to "expand" the Lie bracket in terms of flows.

Now, if a Lie bracket containing either the drift term f or one or more of the unilateral inputs h_i is expanded in terms of its flows, there will be terms such as $-f$ or $-h_i$, which are clearly problematic. It turns out, for the satellite example, that every good bracket can be rearranged in a manner that eliminates this problem.

Such a rearrangement is accomplished via two primary mechanisms. One means is to utilize the skew–symmetry of the Lie bracket to rearrange the flows so that the $-f$ term is first. Alternatively, a f and a $-f$ flow can be arranged sequentially so that they effectively cancel.

Doing this for the satellite, we can generate motions in all 12 phase space directions.



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This slide illustrates four of the 12 possible motions necessary to constructively show controllability. In each case, the sequence of control inputs was determined in a manner similar to that discussed on the previous slide.

Conclusions and future work

- We have developed a controllability test for a fairly general class of unilateral input control problems.
- Currently, the test is fairly restrictive. More work must be done to generalize it.
- From an engineering standpoint, a more useful result would be trajectory generation algorithms for such unilateral problems.

References

[1] G. Lafferriere and Hector J. Sussmann. A differential geometric approach to motion planning. In X. Li and J. F. Canny, editors, *Nonholonomic Motion Planning*, pages 235–270. Kluwer, 1993.

[2] K.M. Lynch and M.T. Mason. Stable pushing: Mechanics, controllabilty, and planning. *International Journal of Robotics Research*, 15(6):533–556, December 1996.

[3] Hector J. Sussmann. A general theorem on local controllability. *Siam J. Control and Optimization*, 25(1):158–194, 1987.