Control of Stratified Systems with Robotic Applications

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Abstract

Many interesting and important control systems evolve on stratified configuration spaces. Roughly speaking, a configuration manifold is called "stratified" if it contains subspaces (submanifolds) upon which the system had different equations of motion. Robotic systems, in particular, are of this nature. For example, a legged robot has discontinuous equations of motion near points in the configuration space where each of its "feet" comes into contact with the ground. In such a case, when the system moves from one submanifold to another, the equations of motion change in a non-smooth, or even discontinuous manner. In such cases, traditional nonlinear control methodologies are inapplicable because they generally rely upon some form of differentiation. Yet, it is precisely the discontinuous nature of such systems that is often their most important characteristic.

This dissertation presents methods which consider the intrinsic physical geometric structure present in such problems to address nonlinear controllability and motion planning for stratified systems. For both problems, by exploiting this geometric structure of stratified systems, we can extend standard nonlinear control results and methodologies to the stratified case. A related problem addressed by this dissertation is that of controllability of systems where some control inputs are constrained to be non–negative. This problem arises in stratified systems which arise by way of physical contact because the normal force between contacting systems must be nonnegative. For all the results, a basic goal is to generate results which are general. For example, for robotics applications, these results are independent of a particular robot's number of legs, fingers or morphology.

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Chapter 1

Introduction

Many interesting and important control systems are characterized by the fact that the equations of motion describing the system are discontinuous. Such discontinuous equations of motion may arise in several ways. An important subset of such problems arises when constraints on the system are intermittent. Robotic systems are a common example of such systems, and, in fact, were the original motivation for this work. When a legged robot walks about, its "feet" cyclically make and break contact with the ground. While foot is in contact with the ground, the robot's motion will be constrained in some manner. Such constraints may be holonomic, for a point–like foot, or nonholonomic for possible rolling type contact of a more extensive foot. Regardless of the appropriate characterization of the type of constraint, the intermittent nature of the constraints are manifested primarily by way of a discontinuous change in the equations describing the motion of the robot. Another closely related example is that of a robot hand grasping an object where the "fingers" of the hand make and break contact while the robot manipulates the object.

As a concrete example, consider the miniature six–legged hexapod robot illustrated in Figure 1.1 (perhaps a future robot fabricated using MEMS fabrication techniques). A schematic drawing of the hexapod is illustrated in the right figure to illustrate the kinematics of the model. This model will be fully explored as a recurring example throughout this dissertation. At this point, note that each leg has only two degrees of freedom. In particular, the robot can only lift its legs up and

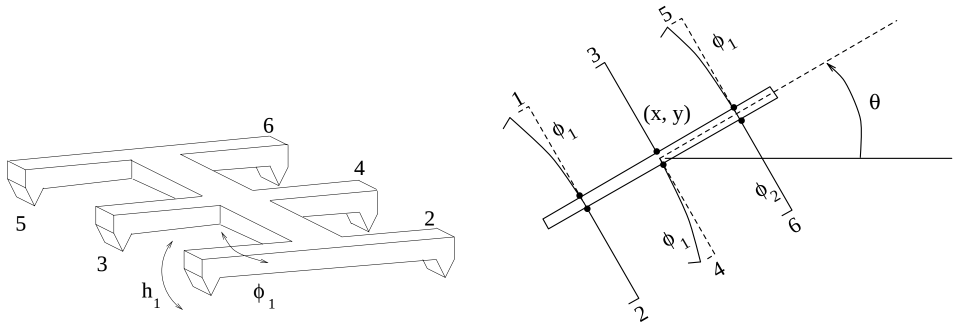


Figure 1.1. Simple hexapod robot.

down and move them forward and backward. Such limited control authority may be desirable in practical situations because it decreases the mechanical complexity of the robot; however, such decreased complexity comes at the cost of requiring more sophisticated control theory. Note that for this model, it is not immediately clear whether the robot can move "sideways," and if it cannot move sideways, then it is not controllable because it cannot move in an arbitrary direction. In this, and other stratified cases, traditional nonlinear controllability analyses are inapplicable because they rely upon differentiation in one form or another. Yet it is the discontinuous nature of such systems that is often their most important characteristic because the system must cyclically move its feet in and out of contact with the ground to be effectively controlled.

The fundamental approach of dissertation is to exploit the geometric structure inherent in such systems. Roughly speaking, we will call a configuration manifold *stratified* if it contains submanifolds upon which the system is subjected to additional constraints (in the legged robot example, these submanifolds correspond to different feet contacting the ground). The two main topics addressed in this thesis are controllability and motion planning.

This dissertation extends standard controllability tests for *smooth* driftless nonlinear systems (in particular, Chow's Theorem and various formulations thereof) to the case where the configuration manifold is stratified. We provide three alternative stratified controllability tests. The first test is based upon the distributions arising from the vector fields in the equations of motion for the system on the different strata. The second controllability test uses methods from exterior differential systems. This approach focuses on the constraint equations, rather than the equations of motion. The third controllability test uses the special geometric structure of a configuration manifold which is a principle fiber bundle. Although this structure is somewhat special, it is a rather generic feature present in many robotic systems. In this case, the configuration manifold can naturally be decomposed into a *shape* space, describing the shape of the mechanism, and the *group* space, describing the mechanism's position and orientation in space. By splitting the configuration manifold in this natural manner, it is possible to reformulate the first test (based on distributions) in terms of *curvature* of *connections*, where a connection maps shape changes to group changes. Additionally, we consider the issue of *nonlinear gait controllability* for legged robots; particularly, whether or not a specified gait can allow the robot to move in any direction.

The issue of controllability is important for two reasons. First, such controllability is a necessary condition for motion planning algorithms. Clearly, if the robot cannot move in all directions, it is then impossible to specify an arbitrary path for the robot to follow. Secondly, controllability is a useful *design* tool. Generally speaking, for autonomous robots, and legged robots in particular, there is a trade off between the complexity of the robot and the associated sophistication of the controller. In other words, if the robot has many degrees of freedom, it is relatively simple to devise a control strategy for it; conversely, if the robot has relatively few degrees of freedom, a control strategy which exploits the particular geometry or other nonlinear features of the robot is necessary.

Another subject of this dissertation is motion planning for stratified systems. In particular, for legged robots, the motion planning problem is the problem of determining control inputs (*i.e.*, mechanism joint variable trajectories) which will steer the robot from a starting configuration to a desired final configuration. We present a general motion planning scheme for a class of kinematic legged robots. The method is independent of the number of legs and other aspects of a robot's morphology. One important feature of this method is that it is distinct from "traditional" legged locomotion control methods in that it is not based on foot placement concepts.

The final main topic in this dissertation is controllability for systems with some inputs constrained to be nonnegative. For a particular subclass of stratified problems, it may be beneficial to consider the contact forces or velocities as inputs rather than consider the problem in the stratified framework. This appears particularly relevant in the area of *nonprehensile manipulation*, or pushing (Lynch (1996)). Not considering this problem as stratified is beneficial for two reasons. First, if the ultimate interest is in manipulating the object, only considering the contact forces or velocities reduces the dimensionality of the problem. Secondly, and more importantly, the most general controllability test for such systems is actually too restrictive if considered as a stratified system and scales poorly with an increasing number of strata.

Although there is no work directly analogous to the content of this dissertation, there is quite a bit of related work. Brockett (1984) illustrated some of the aspects of the problem of discontinuous or impacting systems, and there is quite a bit written concerning so–called "hybrid systems," (*e.g.*, Branicky (1993), Branicky (1994), Brockett (1993) and Brockett (1994)). However, none of these has exploited the particular geometry of these systems to develop a controllability test or a motion planning algorithm.

Additionally, there is a vast literature on the particular problem of legged robotic locomotion. It is nearly universally true, however, that most legged robotics research has focused on a particular morphology. The following list is just a representative sample:

- *Hopping Monopods:* Raibert (1986), Berkemeier and Fearing (1992), Berkemeier and Fearing (1994), Ostrowski and Burdick (1993) and Gokan, Yamafuji, and Yoshinada (1994);

- *Bipeds:* Kajita and Tani (1991), Mulder, Shaw, and Wagner (1990), McGeer (1993), Hodgins and Raibert (1991) and Alexander (1992);

- *Quadrapeds:* Lee and Song (1991), Todd (1991), Shin and Streit (1993) and Collins and Richmond (1994); or
- *Hexapod:* Song and Waldron (1989), Halme, Hartikainen, and Kärkkäinen (1994), Pfeiffer, Eltze, and Weidemann (1995), Cruse, Bartling, Cymbalyuk, Dean, and Deifert (1995), Beer, Chiel, Quinn, Espenschied, and Larsson (1992) and Collins and Stewart (1993).

While it is true that many of the listed references are based upon general principles, such as symmetry breaking bifurcations (Collins and Stewart (1993)) or conservation of energy (Kajita and Tani (1991)), philosophically, none of the references attempt *general applicability*, *i.e.*, determining provable properties or algorithms which are independent of morphology. In contrast to nonlinear control theory, the question of controllability for legged robots apparently has not been previously considered.

Nonlinear control theory, on the other hand, is typically formulated in complete generality. The starting point is simply a differential equation. See, for example, Isidori (1989), Nijmeijer and der Schaft (1990), Sontag (1990). However, these general nonlinear results require that the equations of motion be smooth. Some recent works have started to uncover some of the fundamental structure specific to locomotion mechanics and control. Kelly and Murray (1995) showed that a number of kinematic locomotive systems can be modeled using connections on principal fiber bundles. They also provide results on controllability, as well as an interpretation of movement in terms of geometric phases. Ostrowski, Burdick, Murray, and Lewis (1995) and Ostrowski (1995) developed analogous results for a class of dynamic nonholonomic locomotion systems. Also, Krishnaprasad and Tsakiris (1994) have used methods from nonlinear control theory to develop motion planning schemes for "G"-snakes, a class of kinematic undulatory mechanisms. All of these approaches, unfortunately, are limited to *smooth* systems, and thus are not directly applicable to stratified systems.

For motion planning, the legged robotics research has, again, focused on a particular morphology (see the above–listed references), and additionally, has, for example, presumed a collection of gaits which enable the robot to locomote as desired

(Orin (1982)). For smooth nonlinear systems there are general motion planning algorithms. For *differentially flat* systems, control and motion planning are trivial because of a special relationship between the control inputs and the *output function* (*e.g.*, Fliess, Lévine, Martin, and Rouchon (1992) and van Nieuwstadt, Rathinam, and Murray (1998)) Unfortunately, this structure is rather special. Another class of systems are the so–called *chained* systems, which can be steered using sinusoidal inputs (Murray and Sastry (1993) and Tilbury, Murray, and Sastry (1995)). Again, this structure is somewhat special. The most general method for steering nonholonomic systems is due to Lafferriere and Sussmann (1993). This method is the basis of our work for two reasons. First, it is the most general method. Secondly, it uses piecewise constant inputs, which has a direct appeal to stratified systems because of the piecewise nature inherent in the equations of motion for the system. Closely related to the piecewise constant inputs are "highly oscillatory inputs" (Sussmann and Liu (1991a), Sussmann and Liu (1991b)).

Outline and Contributions of this Dissertation

The main contribution of this thesis is, for both controllability and motion planning, the extension or modification of general results for smooth systems to stratified system, in particular, legged robotic systems.

Chapter 2 presents background mathematical material which is primarily comprised of topics from differential geometry and algebra. One aspect of the chapter which is new is the definition and geometric description of stratified configuration spaces.

Chapter 3 presents the most basic nonlinear controllability result, Chow's theorem, and two alternative formulations thereof. In particular, Chow's theorem can be restated in terms of tools from exterior differential systems and also in terms of tools related to principal fiber bundles. These results provide the basis for the result presented in Chapter 4. Additionally, stratified configuration spaces are topologically distinct from the standard smooth case. Therefore, a novel focus of a part of this chapter is the impact of such topological concerns on the fundamental notion and definition of controllability for stratified systems. Also presented in this chapter is the general motion planning algorithm which is the basis for the motion planning results in Chapter 5.

Chapter 4 contains controllability results for stratified systems. The contributions in this chapter are several controllability tests; namely, a test based on distributions, a test using tools from exterior differential systems and a test using the special geometry of principal fiber bundles. Each of these tests can be viewed as a new extension of the corresponding smooth test to the stratified case. Additionally, the definition of gait controllability and a related test are also new. The use of these tests are illustrated using the example from Figure 1.1.

Chapter 5 contains a motion planning algorithm for stratified systems. The contribution of this chapter is the extension of the motion planning method of Lafferriere and Sussmann (1993) to the stratified case. The use of the algorithm is illustrated using the example from Figure 1.1. Also in this chapter, a grasping example illustrates the fact that even for completely controllable smooth systems, directly exploiting any stratified structure in the system may yield efficiency gains which make it easier for to satisfy stability or obstacle avoidance criteria.

Chapter 6 contains a controllability test for systems with unilateral inputs. The test itself is new and is essentially a simpler reformulation of some results from Sussmann (1987). The test is simpler because it is formulated in terms of vector fields rather than a more difficult algebraic test.

Finally, Chapter 7 summarizes the results in this dissertation and provides details of potentially fruitful future work.

Chapter 2

Mathematical Preliminaries

This chapter reviews the mathematical concepts upon which our results are based and presents some commentary to indicate the role or interpretation of particularly important concepts. For the most part, the topics are presented in a manner most consistent with establishing an intuitive understanding of the rest of this thesis. This approach is usually consistent with the desire for rigor, but, unfortunately does not lend itself to a basic, fully self–contained presentation. Because of this, we frequently cite appropriate references to which the reader can turn for a more complete explanation of the relevant concept.

Section 2.1 reviews some basic differential geometry. Some of the more basic control results have a nice geometric interpretation and can be proved using geometric tools. In particular, Section 3.2 presents three alternative formulations of controllability results. Correspondingly, three sections of this chapter present the underlying mathematical tools: Section 2.1.1 presents some of the most basic definitions from differential geometry, including vector fields, distributions and foliations, Section 2.2.2 presents some concepts from exterior differential systems and Section 2.1.2 presents the rather special geometry of principal fiber bundles. Section 2.2 presents some algebra underlying the results of Chapter 5 and Chapter 6. Finally, Section 2.3 defines a stratified control system in terms of the particular geometry present in such systems.

2.1 Differential Geometry

This section reviews some basic differential geometry that are most relevant to this thesis. The reader should refer to Abraham, Marsden, and Ratiu (1988) and Boothby (1986) for a complete treatment of this subject.

2.1.1 Vector Fields, Distributions and Foliations

This thesis is concerned with differential equations of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u^i, \tag{2.1}$$

defined on a smooth manifold M, where $x \in M$ represents the state of the control system, $f(x)$ and $g_i(x)$ are *vector fields* on M and the u^i are *control inputs*, which belong to a set of *admissible controls*, $u^i \in \mathcal{U}$. The system in Equation 2.1 is called *driftless* if $f(x)$ is identically zero; otherwise, it is called a system *with drift*, and the vector field $f(x)$ is called the *drift term*.

On a fundamental level, one way to understand the control system expressed in Equation 2.1 is to understand the geometry of the vector fields $f(x)$ and $g_i(x)$. Given a manifold M, denote the tangent bundle by TM. Define a *vector field* as a section of the tangent bundle TM of M. This is a mapping $X : M \longrightarrow TM$, which assigns to each point in $x \in M$ a vector $v \in T_xM$. Let $\mathfrak{X}^r(M)$ denote the C^r vector fields on M, and let $\mathfrak{X}(M) = \mathfrak{X}^\infty(M)$. Central to geometric nonlinear control theory is the *Lie bracket*.

Definition 2.1: (Lie bracket) If $g_1, g_2 \in \mathfrak{X}^r(M)$ and g_1 has a flow $\phi_t^{g_1}$, the vector field $[g_1, g_2] \in \mathfrak{X}^{r-1}(M)$ defined by

$$[g_1, g_2] = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^{g_1 *} g_2) = \left.\frac{d}{dt}\right|_{t=0} ((T\phi_t^{g_1})^{-1} \circ g_2 \circ \phi_t^{g_1})$$

is called the *Lie bracket* of g_1 and g_2. The flow, $\phi_t^g(x_0)$ represents the solution of the differential equation $\dot{x} = g(x)$ at time t starting from x_0. $\square$

In coordinates, the Lie bracket between $g_1(x)$ and $g_2(x)$, is computed as

$$[g_1(x), g_2(x)] = \frac{\partial g_2(x)}{\partial x} g_1(x) - \frac{\partial g_1(x)}{\partial x} g_2(x)$$

and can be interpreted as the leading order term that results from the sequence of flows

$$\phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(x) = \epsilon^2 [g_1, g_2](x) + \mathcal{O}(\epsilon^3). \tag{2.2}$$

Roughly speaking, then, one way to interpret a Lie bracket is that it is a "new direction" in which the system can flow, by executing the sequence of flows in Equation 2.2. An important relationship between flows of vector fields is given by the Campbell–Baker–Hausdorff formula:

$$\phi_1^{g_2} \circ \phi_1^{g_1}(x) = \phi_1^{g_1 + g_2 + \frac{1}{2}[g_1, g_2] + \frac{1}{12}([g_1,[g_1,g_2]] - [g_2,[g_1,g_2]]) + \cdots}(x) \tag{2.3}$$

(a proof can be found in Varadarajan (1984)). Essentially, if given the composition of multiple flows along multiple vector fields, this formula gives the one flow along one vector field which results in the same net flow. This will play a central role in the motion planning results in Section 3.4 and Chapter 5.

Lie brackets satisfy three important properties, namely,

1. Skew–symmetry: $[g_1, g_2] = -[g_2, g_1]$,
2. Jacobi identity: $[g_1, [g_2, g_3]] + [g_3, [g_1, g_2]] + [g_2, [g_3, g_1]] = 0$, and
3. Chain rule: $[\alpha g_1, \beta g_2] = \alpha\beta[g_1, g_2] + \alpha(L_{g_1}\beta)g_2 - \beta(L_{g_2}\alpha)g_1$, where $L_{g_1}\beta$ and $L_{g_2}\alpha$ are the Lie derivatives (directional derivatives) of β and α along the vector fields g_1 and g_2, respectively.

Note that the first two of these properties make the set of smooth vector fields equipped with the Lie bracket a Lie algebra.

In control theory, the set of all possible directions in which the system can move, or the set of all points the system can reach, is of obvious fundamental importance.

Geometrically, this is related to *distributions*.

Definition 2.2: (Distribution) Let M be a manifold. A *distribution* assigns a subspace of the tangent space to each point in M in a smooth way. A distribution Δ is *involutive* if, for any two vector fields $X, Y \in \Delta$, $[X, Y] \in \Delta$. A distribution Δ is *integrable* if, for any $x \in M$, there is a submanifold $N \subset M$ containing x such that the tangent bundle, TN, is exactly Δ restricted to N, *i.e.*, $TN = \Delta|_N$. □

It is natural to consider distributions generated by the vector fields appearing in Equation 2.1. In this case, consider the distribution defined by

$$\Delta = \text{span}\{f, g_1, \ldots g_m\},$$

where the span is taken over the set of smooth real–valued functions. Denote by $\overline{\Delta}$ the *involutive closure* of the distribution Δ, which is the closure of Δ under bracketing. Then, $\overline{\Delta}$ is the smallest subalgebra of $\mathfrak{X}(M)$ which contains $\{f, g_1, \ldots, g_m\}$. We will often need to "add" distributions. Since distributions are, pointwise, vector spaces, define the sum of two distributions,

$$(\Delta_1 + \Delta_2)(x) = \Delta_1(x) + \Delta_2(x).$$

Similarly, define the intersection

$$(\Delta_1 \cap \Delta_2)(x) = \Delta_1(x) \cap \Delta_2(x).$$

Similar addition and intersections follow for codistributions.

Frobenius' theorem asserts that integrability and involutivity are equivalent. Thus, associated with an involutive distribution is a partition of M into disjoint connected immersed submanifolds called *leaves.* This partition is called a *foliation.* In control theory, these leaves are related to the set of points that a control system can reach starting from a given initial condition.

The motion planning results of Section 3.4 and Chapter 5 need a basis for the Lie algebra generated by a set of vector fields. However, because of Jacobi's identity and the fact that a Lie bracket is skew symmetric, it is not easy to select such a basis. A *Philip Hall basis* is a particular way to select a basis.

Definition 2.3: (Philip Hall basis) Given a set of vector fields $\{g_1, \ldots, g_m\}$, define the *length* of a *Lie product* as

$$\begin{aligned} l(g_i) &= 1 \qquad i = 1, \ldots, m \\ l([A, B]) &= l(A) + l(B), \end{aligned}$$

where A and B may be Lie products. A *Philip Hall basis* is an ordered set of Lie products $H = \{B_i\}$ satisfying:

1. $g_i \in H, i = 1, \ldots, m$
2. If $l(B_i) < l(B_j)$, then $B_i < B_j$
3. $[B_i, B_j] \in H$ if and only if
   (a) $B_i, B_j \in H$ and $B_i < B_j$, and
   (b) either $B_j = g_k$ for some k or $B_j = [B_l, B_r]$ with $B_l, B_r \in H$ and $B_l \leq B_i$.

$\square$

For a proof, see Serre (1992). Essentially, the ordering aspect of the Philip Hall basis vectors accounts for skew symmetry and Jacobi's identity to determine a basis.

2.1.2 Principal Fiber Bundles

The controllability results in Sections 3.2.3 and 4.2.3 are based on a particular geometry generic to locomotion systems. The object of ultimate interest is a *principal fiber bundle* and its associated *connection*. There is quite a bit of general theory associated with principal fiber bundles (Kobayashi and Nomizu (1963)); however, this dissertation only requires particularly simple aspects of it. Illustrations of the use of principal fiber bundles in control can be found in Kelly and Murray (1995), which will be the basis for our results in Section 4.2.3. Use of the more general aspects of the theory is by Montgomery (1993) (for control theory) and by Bloch, Krishnaprasad, Marsden, and Murray (1996) (for mechanics).

The central concept in this section is that of a *Lie group*. A partial explanation of its significance is that it is almost universally true that the position and orientation in space of a locomotion system can be represented by a Lie group. Most of the definitions in this section are from Kelly and Murray (1995) or Marsden and Ratiu (1994). Additionally, the reader is referred to Abraham and Marsden (1978), Olver (1993) and Serre (1992) for a complete exposition on Lie groups.

Definition 2.4: (Lie group) A *Lie group* is a finite dimensional smooth manifold, G, that is a group and for which the group operations of multiplication and inversion are smooth. A *Lie subgroup* H of a Lie group G is a subgroup of G for which the inclusion mapping $i : H \to G$ is an immersion, that is, $i(H)$ is an immersed submanifold of G. □

Rigid body motion corresponds to *translation* by elements of the Lie group. For every $g \in G$ define the map $L_g : G \to G : h \mapsto gh$, called *left translation*. Similarly, define $R_g : G \to G : h \mapsto hg$, called *right translation*. Part of the definition of a principal fiber bundle includes the notion of a *left action*.

Definition 2.5: (Left action of a group on a manifold) A *left action* of a Lie group G on a manifold M is a smooth mapping $\Phi : G \times M \to M$ such that

1. $\Phi(e, x) = x \quad \forall x \in M$;
2. $\Phi(g_2, \Phi(g_1, x)) = \Phi(g_2 g_1, x)$ for every $g_1, g_2 \in G$ and $x \in M$.

A left action Φ of G is called *free* if $\Phi(g, x) = x \Rightarrow g = e$ for each $x \in M$. □

Together, a Lie group and left action define a principal fiber bundle.

Definition 2.6: (Globally trivial principal fiber bundle) Let $M = B \times G$, B a manifold and G a Lie group. A *globally trivial principal fiber bundle* with *total space* M, *base space* (or *shape space*) B, and the *structure group*, G, is the manifold M together with a free left action of G on M given by left translation in the group variable. □

As an aside, note that in the more general theory, the manifold M is only locally trivial, and the base space B is a quotient manifold determined by identifying configurations that differ only by rigid body displacements.

Many biological systems and biomimetic robots move about and reorient themselves by coupling a change in their shape to an external constraint. In such instances, desired changes in spatial location directly result from changes in object shape. If the relationship between such shape changes and resulting changes in spatial location can be naturally expressed, the question of controllability can be more efficiently addressed by studying that relationship, which is expressed as a *connection*.

In order to define a connection, we need to consider properties of vector fields. Of particular importance, are *invariant* vector fields. A vector field X on G is *left invariant* if the vector field is invariant with respect to the *push forward* of left translation, $(L_g)_* X = X$, that is,

$$T_h L_g X(h) = X(gh) \qquad \text{for every } h \in G. \tag{2.4}$$

Denote the set of left invariant vector fields on G by $\mathfrak{X}_L$. For each $\xi \in T_e G$, define a vector field X_ξ on G by $X_\xi(g) = T_e L_g \xi$. It is easy to show that X_ξ is left invariant and that $\mathfrak{X}_L$ and $T_e G$ are isomorphic as vector spaces. Also note that defining a Lie bracket in $T_e G$ by

$$[\xi, \eta] = [X_\xi, X_\eta](e) \qquad \text{for } \xi, \eta \in T_e G$$

makes $T_e G$ into a Lie algebra. The vector space $T_e G$ with this Lie algebra structure is called the *Lie algebra* of G and is denoted by $\mathfrak{g}$. A *Lie subalgebra*, of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $\xi, \eta \in \mathfrak{h} \Rightarrow [\xi, \eta] \in \mathfrak{h}$.

Definition 2.7: (Exponential mapping) For every $\xi \in T_e G$ let $\phi_\xi : \mathbb{R} \to G : t \mapsto \exp(t\xi)$ denote the integral curve of X_ξ passing through e at $t = 0$. The function $\exp : T_e G \to G : \xi \mapsto \phi_\xi(1)$ is called the *exponential mapping* of the Lie algebra of G into G. □

The derivative of the exponential map is the *infinitesimal generator*. Let Φ be a left action of G on a manifold M and $\xi \in \mathfrak{g}$. The infinitesimal generator of Φ corresponding to ξ is the vector field on M defined by

$$\xi_M(x) = \frac{d}{dt}(\Phi(exp(\xi t), x))|_{t=0}.$$

A *connection one-form* Γ on $M = B \times G$ is a $\mathfrak{g}$–valued one form on M satisfying

1. $\Gamma(\xi_M) = \xi$ for every $\xi \in \mathfrak{g}$;
2. $\Gamma(T_x\Phi_g v_x) = Ad_g\Gamma(v_x)$ for every $v \in TM$.

The connection defines a splitting of the tangent space, T_xM, into two complementary subspaces. The *vertical subspace*, $V_xM \subset T_xM$ is

$$V_xM = \{v_x \in T_xM : v_x = \xi_M(x) \text{ for some } \xi \in \mathfrak{g}\}. \tag{2.5}$$

The *horizontal subspace* H_xM of T_xM is

$$H_xM = \{v_x \in T_xM : \Gamma(v_x) = 0\}.$$

In the types of systems considered in this dissertation, connections arise from the constraints where the horizontal space is defined to be the the set of velocities which satisfies the constraints on the system. In the more general theory from the references (particularly Bloch, Krishnaprasad, Marsden, and Murray (1996)), this is referred to as the *kinematic case*, and is one of several possible situations which give rise to a connection. Nonholonomic constraints can be expressed as *one–forms* (see Section 2.2). Just as the left action induced an action on tangent vectors (Equation 2.4) it induces an action on one–forms via the *pull back*,

$$(L_g)^*\omega(x)(v) = \omega(L_g x)(T_x L_g v).$$

Group invariance of constraints now follows the same as for vector fields.

The following result is important because it says that a kinematic system with group invariant constraints defines a connection.

Lemma 2.8 *If the constraints are group invariant and $H_xM \oplus V_xM = T_xM$, the constraints define a connection on M.*

For a proof, see Kobayashi and Nomizu (1963). Intuitively, it is natural to define the horizontal subspace as any velocity vector that satisfies the constraints. Having a sufficient number of constraints to render the system fully kinematic gives the necessary splitting of the tangent space. The group invariance of the constraints then allows one to define the vertical subspace as required by Equation 2.5.

It is natural to split any velocity vector into its horizontal and vertical components. Denote the projections onto the horizontal and vertical subspaces by writing $v_x = \mathrm{hor}_x v_x + \mathrm{ver}_x v_x$, where

$$\mathrm{ver}_x v_x = (\Gamma(v_x))_M(x) \qquad \text{and} \qquad \mathrm{hor}_x v_x = v_x - (\Gamma(v_x))_M(x).$$

For a trivial principal fiber bundle, the connection can always be written in local coordinates $x = (g, b)$ as

$$\Gamma(x)v = \mathrm{Ad}_g(\xi + A(b)v_b), \tag{2.6}$$

where $\xi \in \mathfrak{g}$ and $A : TB \to \mathfrak{g}$. A is called the *local connection one–form.* Equation 2.6 follows because any velocity vector can be translated back to the identity (in the group component) and the Ad_g pulls out via the second defining property of a connection (called equivariance). The fact that $A(b)$ is only a function of shape variables also follows from equivariance. The significance of $A(b)$ is that it maps shape velocities into the corresponding rigid body velocities. The controllability result in Sections 3.2.3 and 4.2.3 is actually formulated in terms of the derivative of the local connection one form. The *covariant exterior derivative* of a Lie algebra valued one form is defined by applying the ordinary exterior derivative (defined in

Section 2.2), d to the horizontal parts of vectors:

$$D\Gamma(X,Y) = d\Gamma(\mathrm{hor}X, \mathrm{hor}Y).$$

Finally, if X is a vector field on TB, the *horizontal lift* of X is

$$X^h(x) = \begin{pmatrix} X(b) \\ -gA(b)\cdot X(b) \end{pmatrix}.$$

2.2 Algebra

The controllability results in Sections 3.2.2, 4.2.2 and 6.1 and the motion planning results in Section 3.4 and Chapter 5 require some background algebra.

2.2.1 Basic Tensor Algebra

Let E be a vector space, and E^* the dual space of E. A *tensor*, contravariant of order r and covariant of order s, denoted by $T^r_s(E)$ is a $r+s$ multilinear map from $E_1^* \times \cdots \times E_r^* \times E_1 \times \cdots \times E_s$ to $\mathbb{R}$. To extend this to the tangent bundle, let $T^r_s(M) = T^r_s(TM) = \cup_{x\in M} T^r_s(T_xM)$. Then a tensor field of type (r,s) is a smooth section of $T^r_s(M)$.

Given two tensors, $t_1 \in T^{r_1}_{s_1}(E)$ and $t_2 \in T^{r_2}_{s_2}(E)$, define the tensor product of t_1 and t_2, $t_1 \otimes t_2$, as

$$\begin{aligned} t_1 \otimes t_2(\alpha^1, \ldots, \alpha^{r_1}, \beta^1, \ldots, \beta^{r_2}, a_1, \ldots, a_{s_1}, b_1, \ldots, b_{s_2}) = \\ t_1(\alpha^1, \ldots, \alpha^{r_1}, a_1, \ldots, a_{s_1}) t_2(\beta^1, \ldots, \beta^{r_2}, b_1, \ldots, b_{s_2}), \end{aligned}$$

where $\alpha^i, \beta^i \in E^*$ and $a_i, b_i \in E$.

2.2.2 Exterior Differential Systems

The exterior algebra of a vector space, E (which can be extended to a bundle), is concerned with a specific type of tensor; in particular, tensors of the type $T^0_k(E)$ which are completely skew symmetric. Given a product which preserves the skew symmetry, the collection of these tensors will then form an algebra, called the *exterior algebra*.

Let S_k denote the permutation group on k elements. A transposition is a permutation that swaps two elements. An *even* permutation can be written as the composition of an even number of transpositions; correspondingly, an *odd* permutation is the composition of an odd number of transpositions. Let $\text{sign}(\sigma) = 1$ if $\sigma \in S_k$ is even, and $\text{sign}(\sigma) = -1$ if σ is odd. An element of $t \in T_k^0(E)$ is called *skew symmetric* if

$$t(e_1, \ldots, e_k) = (\text{sign}(\sigma))t\left(e_{\sigma(1)}, \ldots, e_{\sigma(k)}\right),$$

for all $e_1, \ldots, e_k \in E$ and $\sigma \in S_k$. A tensor, $t \in T_k^0(E)$ is an *exterior k form* if it is skew symmetric.

In order to construct a product that preserves skew symmetry, define the *alternation mapping*. The *alternation mapping*, $\mathbf{A} : T_k^0(E) \to T_k^0(E)$ is defined by

$$\mathbf{A}t(e_1, \ldots, e_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (sign(\sigma))t\left(e_{\sigma(1)}, \ldots, e_{\sigma(k)}\right),$$

where the sum is over all $k!$ elements of S_k. Define the *wedge product*, which takes a k form α, and a l form β and returns a $k + l$ form,

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!}\mathbf{A}(\alpha \otimes \beta).$$

Let Ω^k denote the set of smooth exterior k–forms on M, and let Ω be the algebra of exterior differential forms on M, defined as the direct sum of Ω^k, $k = 0, 1, \ldots$.

Definition 2.9: (Exterior derivative) The *exterior derivative* is the unique map $d : \Omega^r \to \Omega^{r+1}$ which satisfies:

1. if $f \in \Omega^0(M) = C^\infty(M)$ then $df = \frac{\partial f}{\partial x_i} dx_i$;
2. if $\theta \in \Omega^r, \sigma \in \Omega^s$ then $d(\theta \wedge \sigma) = d\theta \wedge \sigma + (-1)^r \theta \wedge d\sigma$;
3. $d^2 = 0$.

□

In controls, it is often convenient to work with the *constraints* on a system rather than directly with the equations of motion. In particular, nonholonomic constraints are conveniently expressed as one–forms, and their span generates a codistribution on M. Let I be a codistribution on M spanned by a set of linearly independent one–forms $\{\omega^1, \dots, \omega^m\}$. The exterior derivative induces a mapping

$$\begin{aligned} \delta &: I \to \Omega^2(M)/I \\ \delta &: \lambda \mapsto d\lambda \bmod I. \end{aligned}$$

It follows from linearity that the kernel of δ is a codistribution on M. Call the subspace $I^{(1)}$ the *first derived system* of I. This construction can be continued to generate a nested sequence of codistributions

$$I = I^{(0)} \supset I^{(1)} \supset \cdots \supset I^{(N)}. \tag{2.7}$$

If the dimension of each $I^{(i)}$ is constant, then this construction terminates for some finite integer N. Equation 2.7 defines the *derived flag* of I. In controls, this nested sequence of codistributions plays a role analogous to constructing the involutive closure of a distribution.

2.2.3 Lie Algebra of Indeterminates

While controllability results for driftless systems have a nice geometric interpretation, the same is not true for general nonlinear systems. One approach, which appears frequently in the work of Sussmann, (Sussmann (1986), Sussmann (1983), Sussmann (1987) and Lafferriere and Sussmann (1993)), is associating with the vector fields in the control system a set of *indeterminates*. Developing sufficient algebraic structure for the indeterminates allows one to associate solutions of the control system with solutions of a related differential equation involving the indeterminates.

The majority of this section is from Sussmann (1987), Sussmann (1983) and Lewis (1995). Other algebraic material is from Hungerford (1974) and Serre (1992).

The first task is to develop a Lie algebraic structure for indeterminates analogous to that of vector fields. Let $\mathbf{X} = \{X_0, \ldots, X_m\}$ be a finite sequence of indeterminates. Let M be a smooth manifold. Consider the system

$$\dot{x} = f_0(x) + g_i(x)u_i, \qquad x \in M \tag{2.8}$$

with a control constraint

$$u = (u_1, \ldots, u_m) \in K,$$

where $\mathbf{f} = (f_0, g_1 \ldots, g_m)$ is an m–tuple of smooth vector fields on M, and $K \subseteq \mathbb{R}^m$ such that $\mathrm{Aff}(K) = \mathbb{R}^m$, where $\mathrm{Aff}(K)$ denotes the affine hull of K, *i.e.*, the set of all finite linear combinations $\sum \alpha_i u^i$ where $\sum \alpha_i = 1$. Call the triple $(M, \mathbf{f}, K)$ the *control system*. Note that the control systems here are not limited to driftless systems.

A *magma* is a set M with a map from $M \times M \to M$. If $\mathbf{X}$ is a set, generate the *free magma on* $\mathbf{X}$ as follows. Define $\mathbf{X}_1 = \mathbf{X}$ and inductively define $\mathbf{X}_n = \coprod_{p+q=n} \mathbf{X}_p \times \mathbf{X}_q$ for $n \geq 2$. The free magma on $\mathbf{X}$ is the set

$$M(\mathbf{X}) = \coprod_{n=1}^{\infty} \mathbf{X}_n.$$

Define the *free algebra* associated with the set $\mathbf{X}$, denoted $A(\mathbf{X})$ which consists of all finite linear combinations

$$\sum_{m \in M(\mathbf{X})} a_m m$$

where $a_m \in \mathbb{R}$. Denote the homogeneous components of degree N of $A(\mathbf{X})$ by $A^{N,hom}(\mathbf{X})$.

Let I be the two–sided ideal of $A(\mathbf{X})$ generated by elements of the form $a \cdot a$ and $a \cdot (b \cdot c) + c \cdot (a \cdot b) + b \cdot (c \cdot a)$ for $a, b, c \in A(\mathbf{X})$. The *free Lie algebra* generated by $\mathbf{X}$ is the quotient algebra $L(\mathbf{X}) = A(\mathbf{X})/I$. The product on $L(\mathbf{X})$ is denoted $[\cdot, \cdot]$, and is called the *bracket*. The Lie algebra $L(\mathbf{X})$ is spanned by the *formal brackets* of $X_0, \ldots, X_m$. Define $\mathrm{Br}(\mathbf{X})$ to be the smallest subset of $L(\mathbf{X})$ that contains $X_0, \ldots, X_m$ and is closed under bracketing. Elements of $\mathrm{Br}(\mathbf{X})$ are called *brackets* of X. Similar to homogeneous components of $A(\mathbf{X})$, the homogeneous components of elements in $L(\mathbf{X})$ are defined and denoted

$$L^{N,hom}(\mathbf{X}) = L(\mathbf{X}) \cap A^{N,hom}(\mathbf{X}).$$

Let $\hat{A}(\mathbf{X})$ denote the set of all *formal power series* $\sum_I a_I X_I$, where $X_I = X_{i_1} X_{i_2} \cdots X_{i_k}$ for the multi–index $I = (i_1, i_2, \ldots, i_k)$ and let $\hat{A}_0(\mathbf{X})$ denote the set of formal power series for which $a_\emptyset = 0$. The *exponential map* is the well defined bijection

$$\exp : \hat{A}_0(\mathbf{X}) \to 1 + \hat{A}_0(\mathbf{X})$$

whose inverse is denoted by log, both of which are defined by their usual series definitions. In particular, define the *formal exponential*

$$\phi_t^X(x) := \exp(tX) = (I + tX + \frac{t^2}{2} X^2 + \cdots) = \sum_{k=0}^{\infty} \frac{q}{k!} X^k t^k \tag{2.9}$$

and

$$\log(1 + S) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} S^k.$$

Note that one way to prove the Campbell–Baker–Hausdorff formula (Equation 2.3) is to expand the product of two exponentials and equate terms in the resulting formal power series. Note that for formal exponential calculations, composition is from left to right; conversely, when composition flows of vector fields, composition is usually denoted from right to left. We use the "e" notation for the former and "ϕ" notation for the latter.

Also, let $\hat{L}(\mathbf{X}) \subseteq \hat{A}_0(\mathbf{X})$ be the set of all formal sums $\sum_{N=1}^{\infty} S_N$ such that each S_N is in $L^{N,hom}(\mathbf{X})$, *i.e.*, the set of those elements of $\hat{A}(\mathbf{X})$ whose homogeneous components are Lie. Note that the exponential map is well defined on $\hat{L}(\mathbf{X})$. The elements of $\hat{A}(\mathbf{X})$ that are of the form $\exp(S)$ for some $S \in \hat{L}(\mathbf{X})$ are the *exponential Lie series in* $X_0, \ldots, X_m$. The set of all such series is denoted by $\hat{G}(\mathbf{X})$, which is a group. The exponential map, restricted to $\hat{L}(\mathbf{X})$ is a bijection from $\hat{L}(\mathbf{X})$ to $\hat{G}(\mathbf{X})$.

Given this algebraic structure, the task now is to develop a differential equation analogous to that of the control system. Consider the differential equation

$$\dot{S} = S\left(X_0 + \sum_{i=1}^{m} u_i X_i\right) \tag{2.10}$$

for an $\hat{A}(\mathbf{X})$–valued function $t \to S(t)$, with the initial condition $S(0) = 1$. This is the *formal differential equation*, solutions to which are related to solutions of the differential equation describing the control system evolving on M. Sussmann (1983) notes that the solution to this differential equation exists and is unique, and is given by

$$S(t) = \sum_I \left(\int_0^t u_I\right) X_I \tag{2.11}$$

where

$$\int_0^t u_I = \int_0^t \int_0^{\tau_k} \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} u_{i_k}(\tau_k) u_{i_{k-1}}(\tau_{k-1}) \cdots u_{i_1}(\tau_1) d\tau_1 \cdots d\tau_k.$$

To see this, note that $S(t)$ is a solution of Equation 2.11 if and only if, for each I,

$$\begin{aligned} S_I(0) &= 1 \\ \dot{S}_I(t) &= S_J(t) u_i(t) \end{aligned}$$

where $I = J * \{i\}$, (where $*$ is concatenation). Then

$$S_{J*\{i\}} = \int_0^t u_i(\tau) S_J(\tau) d\tau.$$

The series $S(T(u(\cdot)))$, with $t \to S(t)$ as above is the *formal power series associated with the control* $u(\cdot)$, and will be denoted by $\mathrm{Ser}(u(\cdot))$, where $T(u(\cdot))$ is the terminal time of $u(\cdot)$.

Let $\mathcal{U}$ be the set of all functions $u(\cdot)$ whose domain is a compact interval of the form $[0, T]$ such that $u(\cdot)$ takes values in $\mathbb{R}^m$ and is Lebesgue integrable on $[0, T]$. If K is an arbitrary subset of $\mathbb{R}^m$, consider $\mathcal{U}_m(K)$, the subsemigroup of $\mathcal{U}_m$ whose elements are the K–valued controls. The image of $\mathcal{U}_m(K)$ under Ser will be denoted by $\hat{S}(\mathbf{X}, K)$.

Relationship between Formal Exponential Lie Series and Solutions of the Control System

This section explores the relationship between solutions of Equation 2.10 (the differential equation on $\hat{A}(\mathbf{X})$) and solutions of Equation 2.8 (the control system expressed as vector fields).

The basic mechanism by which indeterminates are related to vector fields is via the *evaluation map*. Each g_j and f is a member of $D(M)$, the algebra of all partial differential operators. Define the evaluation map obtained by "plugging in f for X_0 and the g_j for the X_j"

$$\begin{aligned} \mathrm{Ev}(\mathbf{f}) : A(\mathbf{X}) &\rightarrow D(M) &(2.12)\\ \mathrm{Ev}(\mathbf{f})\left(\sum_I a_I X_I\right) &\mapsto \sum_I a_I f_I. &(2.13) \end{aligned}$$

Also, define the evaluation map at a point p

$$\mathrm{Ev}_p(\mathbf{f})(S) = (\mathrm{Ev}(\mathbf{f})(S))(p).$$

The evaluation map can be restricted to $L(\mathbf{X})$. Denote the image $\mathrm{Ev}(\mathbf{f})(L(\mathbf{X}))$ by

$L(\mathbf{f})$.

A system satisfies the *Lie algebra rank condition* ("LARC") if $L(\mathbf{f})(p) = T_p(M)$. Also note that, attempting to apply the evaluation maps to series in $\hat{A}(\mathbf{X})$ raises technical difficulties (because $\hat{A}(\mathbf{X})$ contains infinite series), thus motivating the use of the nilpotent approximation.

Now we can explicitly relate solutions of Equation 2.10 to solutions of Equation 2.8. Let $\pi(\mathbf{f}, u, t, x_0)$ denote the trajectory of Equation 2.8 (vector field system) corresponding to the control input u, starting at point x_0. To make the equations less cluttered, we will suppress the explicit dependence on $\mathbf{f}, u$, and x_0, and write $\pi(t)$.

Let $\phi \in C^\infty(M)$. Then $\frac{d}{dt}\phi(\pi(t))$ is the Lie derivative of $\phi(\pi(t))$ along the vector field $\sum_{i=0}^m u_i(t) f_i(\pi(t))$. Therefore,

$$\phi(\pi(t)) = \phi(x_0) + \sum_{i=0}^{m} \int_0^t u_i(s)(f_i\phi)(\pi(s))ds. \tag{2.14}$$

From this, the Lie derivative of ϕ along f_i is

$$(f_i\phi)(\pi(s)) = (f_i\phi)(x_0) + \sum_{j=0}^{m} \int_0^s u_j(\sigma)(f_i f_j\phi)(\pi(\sigma))d\sigma. \tag{2.15}$$

Therefore, substituting from Equation 2.15 into Equation 2.14,

$$\begin{aligned} \phi(\pi(t)) = \phi(x_0) \quad &+ \quad \sum_{i=0}^{m} \left[\int_0^t u_i(s)ds\right](f_i\phi)(x_0) \\ &+ \quad \sum_{i,j} \int_0^t \int_0^s u_i(s)u_j(\sigma)(f_i f_j\phi)(\pi(\sigma))d\sigma ds. \end{aligned}$$

Continuing this procedure,

$$(f_i f_j\phi)(\pi(\sigma)) = (f_i f_j\phi)(\pi(x_0)) + \sum_{k=0}^{m} \int_0^\sigma (f_i f_j f_k\phi)(\pi(\tau))d\tau, \tag{2.16}$$

and substituting as before,

$$\begin{aligned}\phi(\pi(t)) &= \phi(\pi(x_0)) + \sum_{i=0}^{m}\left[\int_0^t u_i(s)ds\right](f_i\phi)(\pi(x_0)) \\ &+ \sum_{i=0}^{m}\sum_{j=0}^{m}\left[\int_0^t\int_0^s u_j(\sigma)u_i(s)\right](f_if_j\phi)(\pi(x_0)) \\ &+ \sum_{k=0}^{m}\sum_{j=0}^{m}\sum_{i=0}^{m}\int_0^t\int_0^s\int_0^\sigma u_k(\tau)u_j(\sigma)u_i(s)(f_if_jf_k\phi)(\pi(\tau))d\tau d\sigma ds.\end{aligned}$$

Iterating this procedure yields

$$\phi(\pi(t)) = \mathrm{Ev}(\mathbf{f})(\mathrm{Ser}_N(u(t))(\phi(x_0)) + R_N(\phi)(x_0),$$

where the $R_N(\phi)(x_0)$ is the remainder term and the Ser term comes from the terms in the square brackets.

Let $\pi_p(u(\cdot))$ be the point to which $u(\cdot)$ steers p. Proposition 4.1 of Sussmann (1983) proves that the remainder term is appropriately bounded so that the series $\mathrm{Ev}_p(\mathbf{f})(\mathrm{Ser}(u(\cdot))$ gives an asymptotic expansion for $\pi_p(u(\cdot))$, in the sense that

$$\|\phi(\pi_p(u(\cdot))) - \mathrm{Ev}_p(\mathbf{f})(\mathrm{Ser}_\nu(u(\cdot)))\phi\| < \beta_\nu T(u(\cdot))^{\nu+1},$$

for all ν and all controls $u(\cdot)$ such that $T(u(\cdot)) \leq \tau_\nu$, (recall that $T(u(\cdot))$ is the terminal time of the control $u(\cdot)$). Taking ϕ as the coordinate functions finally gives the desired result.

Dilations and Input Symmetries

The two main concepts appearing in the general controllability theorems are dilations and input symmetries. Basically, a dilation allows for scaling the input vector fields. Input symmetries are in recognition of the fact that some control inputs can be interchanged or reflected.

If V is a linear space over $\mathbb{R}$, a *group of dilations* of V is a mapping $\rho \to \Delta(\rho)$ that assigns to every real $\rho > 0$ a linear endomorphism $\Delta(\rho) : V \to V$, in such a way that

1. $\Delta(1) = \text{identity}$,
2. $\Delta(\rho_1)\Delta(\rho_2) = \Delta(\rho_1\rho_2) \quad \forall \rho_1, \rho_2$,
3. V has a direct sum decomposition $V = \oplus V_j$ such that the subspaces V_j are invariant under the $\Delta(\rho)$, and the action of $\Delta(\rho)$ on each V_j is given by multiplication by ρ^{α_j} for some $\alpha_j \geq 0$.

The decomposition is unique if $\alpha_j \neq \alpha_k$ whenever $j \neq k$. In that case, the V_j are referred to as *the homogeneous components of* V *with respect to* Δ. Note that any $v \in V$ can be expressed in a unique way as a sum $\sum_j v_j$, $v_j \in V_j$. The Δ*–degree of* v is the largest α_j such that $v_j \neq 0$. A group of dilations is called *strict* if it has no component of degree zero. If Δ is a strict group of dilation of $L^{1,hom}(x)$, call Δ as an *admissible group of dilations*. Say that Δ is *compatible with* $\hat{S}(\mathbf{X}, K)$ if $\Delta(\rho)\left(X_0 + \sum_{i=1}^m u_i X_i\right)$ is of the form $T\left(X_0 + \sum_{i=1}^m v_i X_i\right)$ for some $T > 0$, where $(v_1, \ldots, v_m) \in K$, whenever $0 < \rho \leq 1$ and $(u_1, \ldots, u_m) \in K$. If $Z \in L(\mathbf{X})$ is Δ–homogeneous, Z is called Δ*–neutralized* for $\mathbf{f}$ at p if $\mathrm{Ev}_p(\mathbf{f})(Z)$ can be expressed as a sum of vectors $\mathrm{Ev}_p(\mathbf{f})(Q_i)$, where the Q_i are elements of $L(\mathbf{X})$ of lower Δ–degree than Z.

The class of controls is embedded as a subsemigroup $\hat{S}(\mathbf{X}, K)$ of the group $\hat{G}(\mathbf{X}) = \{\exp(Z) : Z \in \hat{L}(\mathbf{X})\}$. An automorphism λ of $L(\mathbf{X})$ gives rise to a mapping $\hat{\lambda}$, where, if $Z = \sum_{i=1}^\infty P_i$, where P_i is homogeneous of degree i, then $\hat{\lambda}(Z) = \sum_{i=1}^\infty \lambda(P_i)$. Also, define $\lambda^\#$ from $\hat{G}(\mathbf{X})$ to $\hat{G}(\mathbf{X})$ by letting $\lambda^\#(\exp(Z)) = \exp(\hat{\lambda}(Z))$ for $Z \in \hat{L}(\mathbf{X})$. An *input symmetry* is an automorphism λ of $L(\mathbf{X})$ such that the corresponding map $\lambda^\#$ maps $\hat{S}(\mathbf{X}, K)$ to $\hat{S}(\mathbf{X}, K)$. A linear map $\lambda : L(\mathbf{X}) \to L(\mathbf{X})$ is *graded* if λ maps $L^{j,hom}(\mathbf{X})$ into $L^{j,hom}(\mathbf{X})$ for each j.

Finally, an element of $L(\mathbf{X})$ is called *totally odd* if all its homogeneous components have odd degree.

2.3 Stratified Control Systems

This section defines a stratified configuration space and addresses the issue of existence and uniqueness of solutions of a differential equation defined on a stratified configuration space.

2.3.1 Geometry of Stratified Systems

We will motivate our definition of a stratified configuration space with a simple example.

Example 2.10: (Biped Robot) Consider a biped robot as shown in Figure 2.1. The configuration manifold for the robot describes the spatial position and orientation of a reference frame rigidly attached to the robot as well as variables such as joint angles which describe its internal geometry. The robot's motion will be subjected to constraints if one or more of its feet is in contact with the ground. The set of configurations corresponding to one of the feet in contact with the ground is a codimension one submanifold of the configuration space. The same is true when the other foot contacts the ground. The fact that these sets are submanifolds is clearly the true, since the set of points corresponding to a foot in contact with the ground can be described by the preimage of a function describing the foot's height. Similarly, when both feet are in contact with the ground, the system is on a codimension 2 submanifold of the configuration space formed by the intersection of the single contact submanifolds. The structure of the configuration manifold for such a biped is abstractly illustrated in Figure 2.2. Our approach is to exploit this type of geometric structure of such configuration spaces.

Also note that when a foot contacts the ground, the robot is subjected to additional constraints. In particular, the velocity of the foot relative to the ground must be zero. Also, except for when the robot transitions from a state where a foot is off of the ground to one where a foot contacts the ground, the equations of motion for the system are smooth. In other words, restricted to each stratum, the equations of motion are smooth. □

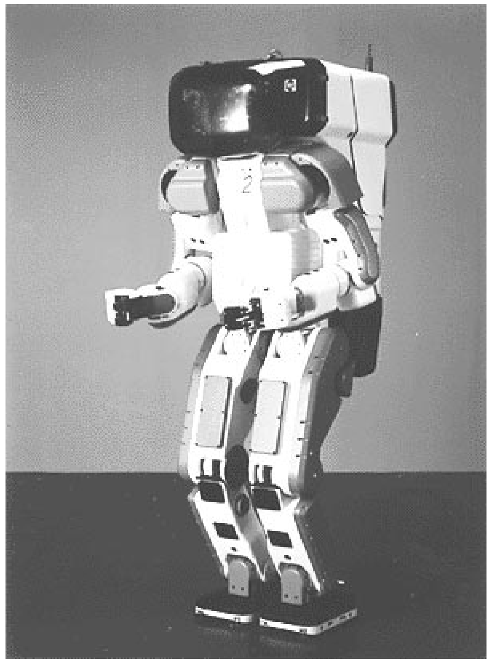


Figure 2.1. Honda biped robot.

We will refer to the configuration space for the biped robot in Example 2.10 as *stratified*. Classically, a *regularly stratified* set $\mathcal{X}$ is a set $\mathcal{X} \subset \mathbb{R}^m$ decomposed into a finite union of disjoint smooth manifolds, called *strata*, satisfying the Whitney condition. The dimension of the strata varies between zero, which are isolated point manifolds, and m, which are open subsets of $\mathbb{R}^m$. The Whitney condition requires that the tangent spaces of two neighboring strata "meet nicely," and for our purposes it suffices to say that this condition is generically satisfied. See Goresky and Macpherson (1980) for details on such stratifications. Note that the terms "stratification" and "strata" are also used in a different context; namely, describing the topology of orbit spaces of Lie group actions, and are a slight generalization of the notion of a foliation (Abraham, Marsden, and Ratiu (1988)).

By considering legged robot systems more general than the biped in Example 2.10, we can develop a general definition of stratified configuration spaces. Let M denote the legged robot's entire configuration manifold (it will often be con-

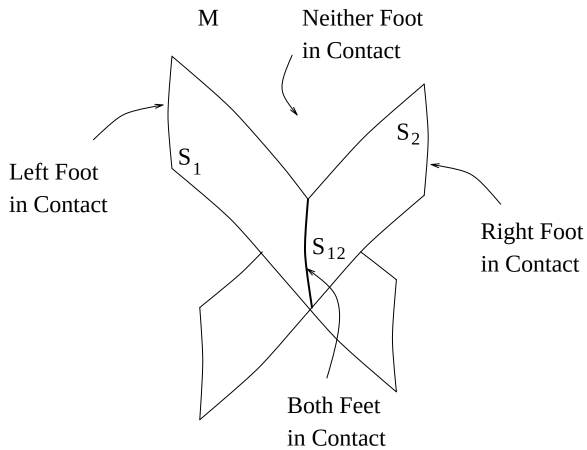


Figure 2.2. Schematic view of the configuration manifold structure of a biped robot.

venient to denote this space as S_0). Let $S_i \subset M$ denote the codimension one submanifold of M that corresponds to all configurations where only the ith foot contacts the terrain. Denote, the intersection of S_i and S_j, by $S_{ij} = S_i \cap S_j$. The set S_{ij} physically corresponds to states where both the ith and jth feet are on the ground. Further intersections can be similarly defined in a recursive fashion: $S_{ijk} = S_i \cap S_j \cap S_k = S_i \cap S_{jk}$, etc. Note that the ordering of the indices is irrelevant, *i.e.*, $S_{ij} = S_{ji}$. In the classical definition of a stratification, stratum $\mathcal{X}_i$ consists of the submanifold S_i with all lower dimensional strata (that arise from intersections of S_i with other submanifolds) removed. However, in our case, we will refer to the submanifolds S_i, as well as their recursive intersections S_{ij}, S_{ijk}, etc, as strata. We will call the stratum with the lowest dimension containing the point x as the *bottom stratum*, and any other submanifolds containing x as *higher strata*. When making relative comparisons among different strata, we will refer to lower dimensional (*i.e.* higher codimension) strata as *lower strata*, and higher dimensional (*i.e.* lower codimension) strata as *higher strata*.

Furthermore, assume that on each stratum, S_i, the system may be subjected to constraints in addition to those present on M. Denote the set of constraints on $M =$

S_0 by $\{\omega^{0,1}, \dots, \omega^{0,m_0}\}$. On a stratum, S_i, denote the *additional* constraints with a superscript i. Thus, the set of constraints on S_i is $\{\omega^{0,1}, \dots, \omega^{0,s}, \omega^{i,1}, \dots, \omega^{i,m_i}\}$. Note that, on any particular stratum, the system is subjected to, at a minimum, all the constraints present on all the higher strata whose intersection defines that stratum. For example, on the stratum $S_{ij} = S_i \cap S_j$, the system is subjected to the all the constraints in M, S_i and S_j, as well as any additional constraints that may be present on S_{ij}.

A codimension one submanifold, S_i, is locally defined by a level set of a function $\Phi_i(x) : M \to \mathbb{R}$. When the system transitions from M to S_i, if the system is going to evolve on the stratum S_i for some finite time, the system must not only satisfy all the constraints that are present on the stratum, but also the constraint $d\Phi_i(x)\dot{x} = 0$, *i.e.*, it must satisfy the set of constraints $\{\omega^{0,1}, \dots, \omega^{0,m_0}, d\Phi_i, \omega^{i,1}, \dots, \omega^{i,m_i}\}$. Note that whether the intersection, $S_i \cap S_j = S_{ij}$ is a submanifold depends upon the functional independence of the functions, Φ_i and Φ_j, respectively defining, S_i and S_j. A basic assumption throughout this paper is that for the multi–index, $I = i_1 i_2 \dots i_k$, the set, $S_I = S_{i_1 i_2 \dots i_k} = S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}$ is a regular submanifold of $M = S_0$. If the strata $S_{i_1}, S_{i_2}, \dots, S_{i_k}$ are locally described by the functions $\Phi_{i_1}, \Phi_{i_2}, \dots, \Phi_{i_k}$, respectively, then S_I will be a submanifold of M if the functions $\Phi_{i_1}, \Phi_{i_2}, \dots, \Phi_{i_k}$ are functionally independent.

We will write the equations of motion for the system at $x \in M = S_0$ as

$$\dot{x} = g_{0,1}(x)u^{0,1} + \cdots g_{0,n_0}(x)u^{0,n_0}, \tag{2.17}$$

and the equations of motion for the system in one of the strata at $x \in S_i$ as

$$\dot{x} = g_{i,1}(x)u^{i,1} + \cdots g_{i,n_i}(x)u^{i,n_i}, \tag{2.18}$$

where n_i depends upon the codimension of S_i and the nature of the additional constraints imposed on the system in S_i. For an arbitrary stratum, S_I,

$$\dot{x} = g_{I,1}(x)u^{I,1} + \cdots + g_{I,n_I}(x)u^{I,n_I}.$$

Assume that the vector fields in the equations of motion for any given stratum are well defined at all points in that stratum, including points contained in any substrata of that stratum. For example, the vector fields $g_{0,i}(x)$ are well defined for $x \in S_i$. Note, however, that they do *not* represent the equations of motion for the system in the substrata, but, nonetheless, are still well defined as vector fields.

Because it is necessary to consider objects defined on different spaces, we need a rigorous way to "include" objects in a subspace into higher dimensional spaces. This is done by way of the inclusion map. When proving our stratified controllability results, we will construct the reachable set for the control system restricted to one stratum, and then extend the construction to the higher strata. In order to rigorously extend a construction from a lower to higher stratum, we need the inclusion map:

$$\begin{aligned} i \quad &: \quad S_I \longrightarrow S_J \qquad |I| > |J| \\ i(x|_{S_I}) \quad &= \quad x \in S_J. \end{aligned}$$

In words, the inclusion map i takes points in a submanifold, and returns the same point in the higher dimensional manifold. Throughout this dissertation we will often intentionally skip the step of including a set defined on a lower stratum into the higher strata, since it will often be clear from the context when a set needs to be included from a lower stratum into a higher stratum.

In order to control whether the system stays on the stratum S_i, we require that we can algebraically solve the equation

$$d\Phi_i \left(g_{i,1}(x)u^{i,1} + \cdots + g_{i,n_i}(x)u^{i,n_i} \right) = 0 \qquad (2.19)$$

for one of the control inputs, which we can always do unless all of the $g_{0,j}(x)$'s are contained in the submanifold described by $\Phi_i(x)$. However, this would be a degenerate case because if all the vector fields $g_{0,j}$ are contained in the tangent space to the stratum described by $\Phi_i(x)$, it would not be possible to move from the ambient manifold, M, onto the stratum S_i in the first place. In other words, in such a case, the configuration manifold is foliated by the involutive closure of the collection of control vector fields $\{g_{0,1}, \ldots, g_{0,n}\}$ in a manner such that the stratum S_i is either a leaf of the foliation, or the leaves of the foliation are submanifolds of S_i.

When the system encounters a stratum with additional constraints, we can choose, by constraining our control inputs to satisfy Equation 2.19, for the system to stay on the stratum. Arguably, for a real system, it is impossible to control a system to stay on a submanifold, since the submanifold will have measure zero. However, in physical systems where the submanifold corresponds to a physical boundary, it will be relatively easy to control the system in such a manner. Note that as long as $d\Phi_i$ is an independent constraint, which we can choose to violate, the system can move off of the submanifold at any time. Throughout this paper, we assume that this is the case. Note that so far we have only discussed whether it is possible to move *off* of a stratum. The converse situation, whether it is possible to move *onto* a stratum from a higher stratum, is a more difficult question and will be briefly discussed in Section 4.4.

Remark 2.11: The equations of motion on S_i, (Equation 2.18), have fewer control inputs because the inputs are constrained according to equations of the form of Equation 2.19, codimension–S_i times. The codimension of S_i and the number $(n_0 - n_i)$ (recall that n_i is the number of inputs on S_i) may not be equal because constraints other than of the type in Equation 2.19, *i.e.*, the $\omega^{i,j}$, may also constrain the inputs. Note also that $g_{i,j}$ is not necessarily the same as $g_{0,j}$, because the additional constraints imposed on the system in S_i may modify the form of each of the $g_{0,j}$'s. □

Finally, assume that the only discontinuities present in the equations of motion are due to transitions on and off of the strata S_i or their intersections. We also make a similar assumption regarding the control vector fields restricted to any stratum, *i.e.*, the control vector fields restricted to any stratum are smooth away from points contained in intersections with other strata. When a configuration manifold is consistent with the above description, we will refer to it as a *stratified configuration* *manifold.*

Definition 2.12: (Stratified configuration manifold) Let M be a mainifold, and n functions $\Phi_i : M \mapsto \mathbb{R}$, $i = 1, \ldots n$ be such that the level sets $S_i = \Phi_i^{-1}(0) \subset M$ are regular submanifolds of M, for each i, and the intesection of any number of the level sets, $S_{i_1 i_2 \cdots i_m} = \Phi_{i_1}^{-1}(0) \cap \Phi_{i_2}^{-1}(0) \cap \cdots \cap \Phi_{i_m}^{-1}(0)$, $m \leq n$, is also a regular submanifold of M. Then M and the functions Φ_n define a *stratified configuration space.* □

2.3.2 Existence and Uniqueness of Solutions

Since, for the problems in this dissertation, the right–hand side of the differential equation

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u^i(x,t) = f(x,t) \tag{2.20}$$

is not continuous everywhere, we must generalize the notion of a solution of a differential equation.

Definition 2.13: (Solution of a differential equation) If there is a continuous function, $\phi(t)$ which satisfies the initial conditions for Equation 2.20 and

$$\dot{\phi}(t) = f(\phi(t), t) \quad \text{almost everywhere,} \tag{2.21}$$

then $\phi(t)$ is called a *solution* of Equation 2.20. □

In order to assure existence and uniqueness of solutions, we make the following assumptions regarding the flow of the control system.

Assumption 2.14: Except for the points where the right–hand side of Equation 2.20 is discontinuous, the differential equation has a unique solution. Since we have assumed that the only discontinuities in the equations of motion are due to transitions between strata, this assumption holds for points in M which have a neighborhood that do not contain points $x \in S_i$. Also, restricted to any stratum, assume that equations of motion have a unique solution for all points in that stratum which have neighborhoods that do not contain any substrata. $\square$

Assumption 2.15: If the flow of the system encounters a stratum of the configuration space in which the right–hand side of the differential equation which describes the motion of the system is discontinuous, then the system evolves on the stratum for a finite time before leaving it. $\square$

These assumptions eliminate the possibility, for instance, when, in a neighborhood of a stratum, all solution curves intersect the stratum, but on the stratum, the vector fields are directed away from the stratum towards the region from which the solutions came. This is basically the "chattering" problem in sliding mode control. In this case, a solution of the differential equation as defined by Equation 2.21 will not exist. Since the purpose of this analysis is to investigate how to exploit the differences between the equations of motion in the various strata of which the configuration space is comprised, and this is a pathological situation in the sense that a choice of a control law generally would be necessary to achieve such a situation, we will specifically exclude its possibility. Filippov (1964) has generalized the notion of a solution to a differential equation to address this situation.

The above assumptions now guarantee existence and uniqueness of solutions of Equation 2.20 because of piecewise existence and uniqueness. Points of discontinuity of the right–hand side of Equation 2.20 are encountered only when a trajectory encounters a substratum of the stratum on which it is evolving, or when the trajectory leaves the stratum on which it is evolving. In either case, since the point of discontinuity uniquely establishes initial conditions for the system, and we have assumed existence and uniqueness for the equations of motion restricted to any strata away from substrata, we have existence and uniqueness of solutions in the sense of Equation 2.21.

Chapter 3

Review of Nonlinear Control Theory

This chapter presents the concepts in nonlinear control upon which the results of this dissertation are based. Note that except where explicitly recognized as relating to stratified systems, the results in this chapter are limited to smooth systems. Chapters 4 and 5 extend these results to the stratified case.

First, Section 3.1, defines the concept of controllability, both for smooth systems as well as stratified systems. Certain topological issues which arise with respect to controllability for stratified systems are also addressed. Section 3.2 reviews three alternative formulations of Chow's theorem: Section 3.2.1 presents Chow's theorem itself, Section 3.2.2 presents an alternative formulation using the tools from exterior differential systems, and Section 3.2.3 presents yet another formulation in the special case where the configuration space is a principal fiber bundle. Then, Section 3.3 presents the most general known controllability test, due to Sussmann (1987), which is the basis for the results in Chapter 6. Finally, Section 3.4, reviews the motion planning algorithm due to Lafferriere and Sussmann (1993).

3.1 Controllability

This section reviews various approaches to nonlinear controllability because our controllability tests extend these approaches to the case where the configuration space is stratified and also address some basic topological properties of stratified configuration spaces. The review of controllability and Chow's theorem is primarily from

Isidori (1989), Murray, Li, and Sastry (1994), Nijmeijer and der Schaft (1990) and Sontag (1990), the material relating to controllability using methods from exterior differential system is from Abraham, Marsden, and Ratiu (1988), Boothby (1986), Flanders (1989) and Murray (1994) and the material relating to principal fiber bundles is primarily from Abraham and Marsden (1978), Boothby (1986), Kelly and Murray (1995), Kobayashi and Nomizu (1963) and Murray, Li, and Sastry (1994).

This section is concerned with *kinematic* control systems. A control system is called kinematic if it is of the form

$$\dot{x} = g_1(x)u^1 + \cdots + g_k(x)u^k. \tag{3.1}$$

Such systems are alternatively called *driftless* or *control linear*. A stratified system is kinematic if the equations of motion are of the this form when restricted to each stratum. In the legged locomotion context, this assumption means that we limit our attention to legged robotic systems which walk in a quasi–static manner.

First, it is necessary define the term "controllable." Given an open set $V \subseteq M$, define $R^V(x_0, T)$ to be the set of states x such that there exists $u : [0,T] \to \mathcal{U}$ that steers the control system from $x(0) = x_0$ to $x(T) = x_f$ and satisfies $x(t) \in V$ for $0 \leq t \leq T$, where $\mathcal{U}$ is the set of admissible controls. Assume that the input space $\mathcal{U}$ is such that the linear span of the set

$$\{\sum_i g_i(x)u : u \in \mathcal{U}\}$$

contains all the vector fields, g_i. Define

$$R^V(x_0, \leq T) = \bigcup_{0<\tau\leq T} R^V(x_0, \tau). \tag{3.2}$$

We will refer to $R^V(x_0, \leq T)$ as the set of states reachable up to time T.

Definition 3.1: (Small time local controllability) A system is *small time locally controllable* ("STLC," or simply "controllable") if $R^V(x_0, \leq T)$ contains a neighborhood of x_0 for all neighborhoods V of x_0 and $T > 0$. □

For stratified systems, Definition 3.1 must be modified for two topological reasons. First, in terms of controllability, it may not be desirable or possible to reach an open neighborhood in the entire configuration space, but rather an open set on a collection of the strata within the whole configuration space. For example, for the biped, it may be desirable that the robot always has at least one foot in contact with the ground, *i.e.*, it is walking, as opposed to running. In such a case it is most natural to consider controllability in terms of reaching an open neighborhood defined in the union of the two strata S_1 and S_2 (corresponding to each foot in contact with the ground) as illustrated in Figure 3.1. The following definition is from basic topology (Abraham, Marsden, and Ratiu (1988)).

Definition 3.2: (Relative topology) If A is a subset of a topological space S with topology O (the collection of open sets), the *relative topology* on A is defined by $O_A = \{U \cap A : U \in O\}$. □

Thus, in the biped example, as illustrated in Figure 3.1, the dotted regions illustrate an open set in the union $S_1 \cup S_2$. The dotted regions represent the intersection of an open ball in S_0 with $S_1 \cup S_2$. Stratified controllability now is defined as in Definition 3.1, where the reachable set contains a neighborhood of the starting point, where the neighborhood is open in $S_1 \cup S_2$. Clearly this notion extends to the case where there is the union of more than two strata.

The second modification is a result of the fact that, until now, we have considered a stratum to be a submanifold of the configuration space for a stratified system. In fact, it will often be the case that the strata defining the stratification are *boundaries* of the configuration space because the submanifolds upon which the system is subjected to additional constraints will often be a physical boundaries. In such a case, it is necessary to redefine a neighborhood of a point x_0 contained in the boundary to be the union of the portion of the standard neighborhood on the "allowable" side of the manifold with the intersection of the standard neighborhood with the boundary.

As a simple planar example, Figure 3.2 illustrates the standard open neighborhood; whereas, Figure 3.3 illustrates the open set when there is a boundary present.

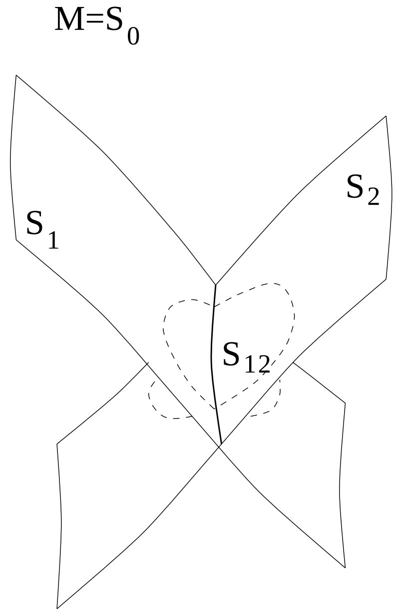


Figure 3.1. Stratified open neighborhood.

Note that this again is a relative topology in accordance with Definition 3.2. Now, for the biped example, only one of the four "quadrants" defined by the intersecting strata is "allowable" (the other three correspond to one or both feet penetrating the ground). Figure 3.4 illustrates an open set for such a stratified configuration space with boundary. Stratified controllability amounts to reaching an open neighborhood of the starting point, where an open set is determined by the natural topology of the problem. We will typically only consider strata as regular submanifolds and not as boundaries. Where appropriate, we will comment on the effect that a boundary would have as opposed to a regular submanifold.

Definition 3.3: (Stratified controllability) Given a stratified configuration manifold and a collection of strata, $\{S_{I_1}, S_{I_2}, \ldots, S_{I_m}\}$, a system is *small time locally stratified controllable* if $R^V(x_0, \leq T)$ contains a neighborhood of x_0 in $S_{I_1} \cup S_{I_2} \cup \cdots \cup S_{I_m}$, for all neighborhoods V of x_0 in $S_{I_1} \cup S_{I_2} \cup \cdots \cup S_{I_m}$ and $T > 0$. □

Note that this definition includes the definition of controllability in Definition 3.1 as a special case when $S_{I_1} \cup S_{I_2} \cup \cdots \cup S_{I_m} = M$.

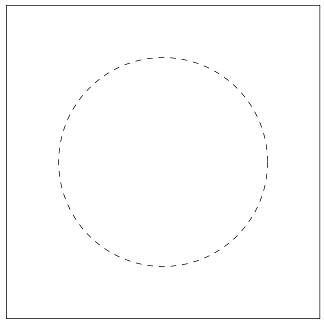


Figure 3.2. Standard open set.

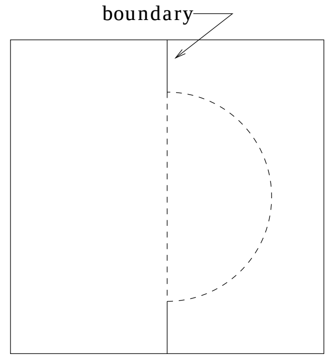


Figure 3.3. Open set with boundary.

3.2 Three Versions of Chow's Theorem

Now we present three versions of the same controllability theorem.

3.2.1 Controllability Based on Distributions

The first controllability result is based upon a construction involving distributions. Let $\mathcal{C}$ denote the smallest subalgebra of $V^{\infty}(M)$ (the Lie algebra of smooth vector fields on a manifold M whose product is the Lie bracket, $[\cdot,\cdot]$) that contains $g_1, \ldots, g_m$. If $\dim(\mathcal{C}) = m$ at a point x, then the system described by Equation 3.1 satisfies the *Lie Algebra Rank Condition* ("LARC") at x. The following is well known as "Chow's Theorem."

Theorem 3.4 *If the system described by Equation 3.1 satisfies the LARC at a point* x_0 *then it is STLC from* x_0.

Since the proof is similar to the construction of our stratified controllability proof in Section 4.2, we will present it here.

Proof: Let $n = \dim(\overline{\Delta})$ and let W be a neighborhood of x_0. Choose $X_1 \in \Delta$ such that $X_1(x_0) \neq 0$. For ϵ_1 sufficiently small, $N_1 = \{\phi_{t_1}^{X_1}(x_0) : t_1 \in (0, \epsilon_1)\}$ is a submanifold of M of dimension one.

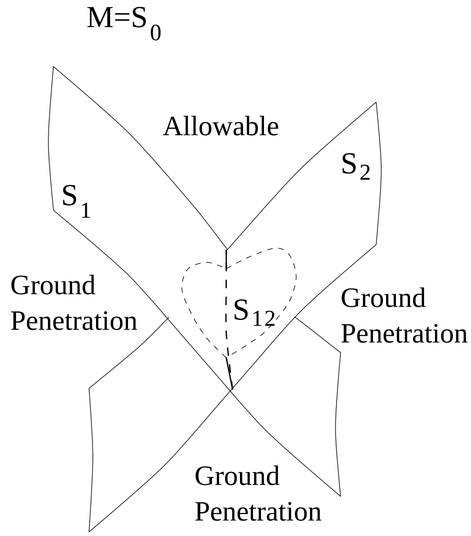


Figure 3.4. Stratified open set.

Now proceed by induction. Assume that

$$N_{j-1} = \{\phi_{t_{j-1}}^{X_{j-1}} \circ \phi_{t_{j-2}}^{X_{j-2}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0) : t_i \in (0, \epsilon_i)\}$$

is a $(j-1)$–dimensional submanifold of M. If $j-1 < n$, then there exists a $X_j \in \Delta$ and a $q \in W$ such that $X_j(q) \notin T_q N_{j-1}$. If this were not true, then $X(q) \in W$ $\forall X \in \Delta$, which would imply that $\dim(\overline{\Delta}) < n$, which is a contradiction. Therefore,

$$N_j = \{\phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0) : t_i \in (0, \epsilon_i)\}$$

is a j–dimensional submanifold of M. Therefore, N_n is an open in M.

Thus, the set

$$N = \{\phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{j-1}}^{-X_{j-1}} \circ \phi_{s_j}^{-X_j} \circ \phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0) : s_i, t_i \in (0, \epsilon_i)\}$$

is an open set of M containing x_0. Thus, the reachable set contains an open neighborhood of x_0, and hence the system is controllable. ■

See any one of the references Isidori (1989), Nijmeijer and der Schaft (1990), Jurdjevic (1997) or Murray, Li, and Sastry (1994) for a more complete exposition.

3.2.2 Controllability Based on Exterior Differential Systems

This section presents Chow's theorem in its "dual" version. The reason that tools from exterior differential systems are useful is because, for nonholonomic systems, the constraints on the system can be expressed as one forms. A *Pfaffian* system is an exterior differential system generated by a set of linearly independent one–forms. One way to consider a control problem is to consider the collection of constraints on the system (one forms spanning a codistribution) rather than the equations of motion (vector fields spanning a distribution).

Aside from controllability considerations, note that this approach has proved useful for the trajectory generation problem (Tilbury, Murray, and Sastry (1995) and Murray (1994)). Also, the notion of *differential flatness* has received quite a bit of attention (Fliess, Lévine, Martin, and Rouchon (1992) and van Nieuwstadt, Rathinam, and Murray (1998)).

The following is the "dual" of Theorem 3.4.

Theorem 3.5 *Let $I = \{\omega^1, \ldots, \omega^m\}$ represent a set of constraints and assume that the derived flag of the system exists. Then, there exists a path $x(t)$ between any two points satisfying $\omega^i(x)\dot{x} = 0 \ \forall i$ if there exists an N such that $I^{(N)} = 0$.*

Rather than prove this directly, we will relate it distributions. If we define a nested sequence of distributions, $E_0 \subset E_1 \subset \cdots \subset E_N$ as

$$\begin{aligned} E_0 &= \Delta \\ E_i &= E_{i-1} + [E_{i-1}, E_{i-1}], \end{aligned} \tag{3.3}$$

then, $I^{(i)} = E_i^{\perp}$. (See Murray (1994), Tilbury, Murray, and Sastry (1993b) or Tilbury, Murray, and Sastry (1993a)). The increasing dimension of the distributions in the filtration in Equation 3.3 correspond exactly to the decreasing dimension of the derived systems in Equation 2.7, and if $I^{(N)} = 0$, then $\dim(E_N) = \dim(M)$ and vice versa. So, this theorem is equivalent, from the point of view of controllability, to Theorem 3.4.

3.2.3 Controllability Based on Principal Fiber Bundles

Because of the global split between shape and group variable for systems on principal fiber bundles, it is natural to consider different types of controllability. Kelly and Murray (1995) distinguish the between the following two definitions of controllability.

Definition 3.6: (Strong controllability) A system on a principal fiber bundle is said to be *strongly controllable* if, for any $x_o = (b_0, g_0)$ and $x_f = (b_f, g_f)$, there exists a time $T > 0$ and a curve $c(\cdot)$ connecting b_0 to b_f in the base space such that the horizontal lift of b passing through x_0 satisfies $b^h(0) = x_0$ and $b^h(T) = x_f$. □

This definition is the standard definition of controllability. However, it is called strong controllability to contrast it with the following notion of weak (or fiber) controllability.

Definition 3.7: (Fiber controllability) A system on a principal fiber bundle is said to be *fiber controllable* if, for any initial position $g_0 \in G$, final position $g_f \in G$, and initial shape $b_0 \in B$, there exists a time $T > 0$ and a base space curve $b(\cdot)$ satisfying $b(0) = b_0$ such that the horizontal lift of b passing through (b_0, g_0) satisfies $b^h(T) = (b(T), g_f)$. □

The definition of weak controllability encapsulates the notion that, for locomotion systems, final shape of the mechanism is often irrelevant. Of primary importance is the fact that the locomotor can reach a desired location in G.

The following is the main result in Kelly and Murray (1995). This result is proved by showing that these conditions are essentially equivalent to Chow's Theorem.

Theorem 3.8 *Define the following sequence of subspaces of the Lie algebra* $\mathfrak{g}$ *at a* *fixed point $b \in B$:*

$$\begin{aligned}
\mathfrak{h}_1 &= \text{span}\{A(X) : X \in T_b B\} \\
\mathfrak{h}_2 &= \text{span}\{DA(X,Y) : X, Y \in T_b B\} \\
\mathfrak{h}_3 &= \text{span}\{L_Z DA(X,Y) - [A(Z), DA(X,Y)], [DA(X,Y), DA(W,Z)] : \\
&\quad W, X, Y, Z \in T_b B\} \\
&\vdots \\
\mathfrak{h}_k &= \text{span}\{L_X \xi - [A(X), \xi], [\eta, \xi] : X \in T_b B, \xi \in \mathfrak{h}_{k-1}, \eta \in \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{k-1}\}.
\end{aligned}$$

A system defined on a trivial principal fiber bundle M over B with structure group G and local connection $A(b)$ is locally weakly controllable near $x \in M$ if and only if

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots .$$

The system is locally strongly controllable if and only if

$$\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_4 \oplus \cdots .$$

Note that in the expression for $\mathfrak{h}_k$, the elements ξ and η are not fixed elements of the Lie subalgebras $\mathfrak{h}_i$, but rather, result from the connection, its curvatures and higher order derivatives of the connection (otherwise, if they were fixed elements of a Lie subalgebra, their Lie derivatives would automatically be zero).

3.3 General Nonlinear Controllability

The following is the most general nonlinear controllability result. The result in Chapter 6 will follow from the following theorem, due to Sussmann, which is actually a corollary to the main result in Sussmann (1987).

Theorem 3.9 *Let $(M, \mathbf{f}, K)$ be a control system. Assume that $\mathbf{f}$ satisfies the LARC at p, and that there exists (a) an admissible group of dilations Δ of $L^{1,hom}(\mathbf{X})$ which is compatible with $\hat{S}(X, K)$, (b) a finite group Λ_0 of graded linear maps from $L(\mathbf{X})$*

to $L(\mathbf{X})$ that are input symmetries, such that every totally odd Λ_0–fixed element of $L(\mathbf{X})$ is Δ–neutralized for $\mathbf{f}$ at p. Then $(M, \mathbf{f}, K)$ is STLC from p.

Note that the difficulty with this very general result is that one must work with indeterminates rather than vector fields (or one forms) and this requires all the machinery associated with “translating” the problem into indeterminates.

Although extremely abstract, this theorem does allow for a (rough) intuitive interpretation. The underlying method is to construct a “group action” on the configuration manifold corresponding to the action of the controls. If the group action is transitive, *i.e.*, “full rank”, and the isotropy subgroup, *i.e.*, group actions corresponding to control inputs that leave the initial condition fixed, is in the interior of action of the group on the initial condition, then the system is controllable. The dilations simply recognize that individual control vector fields can be scaled by scaling the associated input. A vector field that is fixed under the action of the group of input symmetries is one which cannot be compensated by any other input. Finally, the “totally odd” requirement is a reflection of the fact that under time reversal, even order brackets change sign and odd order brackets do not, and thus even order brackets cannot be fixed under the “time reversal” input symmetry.

3.4 Nonlinear Motion Planning

To set the context for our approach in Chapter 5, this section outlines the method in Lafferriere and Sussmann (1993) for generating trajectories for smooth, kinematic nonholonomic systems of the form

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \qquad x \in M. \tag{3.4}$$

Since the method is limited to kinematic systems, so too will our method. The practical implication of this is that our method applies only to quasi–static locomotion.

A system with equations of motion having the form of Equation 3.4 is to be *nilpotent of order k* if all the Lie brackets between control vector fields of order greater than k are 0. (More precisely, the Lie algebra generated by the vector fields is nilpotent). The method presented in this section works exactly for nilpotent systems, and approximately for systems which are not nilpotent. For non-nilpotent systems, arbitrary precision can be obtained by iterating the algorithm.

A nonholonomic control system typically does not have enough controls to directly drive each state variable along a given trajectory, *i.e.*, the number m in Equation 3.4 is less than the dimension of the configuration space. The motion planning problem for systems with such a deficit can be managed by using an "extended system," where "fictitious controls," corresponding to higher order Lie bracket motions, are added. If enough Lie brackets are added to the system to span all possible motion directions, then the motion planning problem becomes trivial for the extended system.

The extended system is constructed by simply "adding on" Lie bracket directions to the original system from Equation 3.4,

$$\dot{x} = b_1 v^1 + \cdots b_m v^m + b_{m+1} v^{m+1} + \cdots + b_s v^s \tag{3.5}$$

where $b_i = g_i$ for $i = 1, \dots, m$, and the $b_{m+1}, \dots, b_s$ correspond to higher order Lie brackets of the g_i, chosen so that $\dim(\text{span}\{b_1, \dots, b_s\}) = \dim(M)$. The v^i's are called *fictitious inputs* since they may not correspond with the actual system inputs. The higher order Lie brackets must belong to the Philip Hall basis for the Lie algebra. The control inputs v^i which steer the extended system can be found as follows. To go from a point p to a point q, define a curve, $\gamma(t)$ connecting p and q (a straight line would work, but is not necessary). After determining γ, simply solve

$$\dot{\gamma}(t) = g_1(\gamma(t))v^1 + \cdots + g_s(\gamma(t))v^s \tag{3.6}$$

for the fictitious controls v_i. This will involve inverting a square matrix or determining a pseudo–inverse, depending on whether or not there are more b_i's than the dimension of the configuration space.

The actual control inputs can be found as follows. Determine the Philip Hall basis for the Lie algebra generated by $g_1, \dots, g_m$, and denote it by $B_1, B_2, \dots, B_s$.

It is possible to represent all flows of Equation 3.4 in the form

$$S(t) = e^{h_s(t)B_s} e^{h_{s-1}(t)B_{s-1}} \cdots e^{h_2(t)B_2} e^{h_1(t)B_1} \tag{3.7}$$

for some functions $h_1, h_2, \ldots, h_s$, called the backward Philip Hall coordinates. Furthermore, $S_t(x)$ satisfies the *formal differential equation*

$$\dot{S}(t) = S(t)(B_1 v_1 + \cdots + B_s v_s); \quad S(0) = 1. \tag{3.8}$$

Define the *adjoint mapping*

$$\mathrm{Ad}_{e^{-h_i B_i}} B_j = e^{-h_i B_i} B_j e^{h_i B_i}.$$

Now it is straight–forward to show that

$$\mathrm{Ad}_{e^{-h_i B_i} \cdots e^{-h_{j-1} B_{j-1}}} B_j \dot{h}_j = \left(\sum_{k=1}^{s} p_{j,k}(h) B_k \right) \dot{h}_j, \tag{3.9}$$

for some polynomials $p_{j,k}(h)$. (For a complete derivation, see Murray, Li, and Sastry (1994)). Equating coefficients yields the differential equations

$$\dot{h} = A(h)v, \qquad h(0) = 0. \tag{3.10}$$

These equations specify the evolution of the Philip Hall coordinates in response to the fictitious inputs, which were found via Equation 3.6. Next determine the actual inputs using the Philip Hall coordinates.

It is easier to determine the real inputs using the forward rather than backward Philip Hall coordinates. The transformation from the backward to forward coordinates is a "simple algebraic transformation" (see Lafferriere and Sussmann (1993)), and this transformation results in an equation of the form

$$S = e^{\tilde{h}_1 B_1} e^{\tilde{h}_2 B_2} \cdots e^{\tilde{h}_{s-1} B_{s-1}} e^{\tilde{h}_s B_s}.$$

One way to compute this transformation, is to take

$$\begin{aligned} S &= e^{h_s B_s} e^{h_{s-1} B_{s-1}} \cdots e^{h_2 B_2} e^{h_1 B_1} \\ &= e^{\tilde{h}_1 B_1} e^{\tilde{h}_2 B_2} \cdots e^{\tilde{h}_{s-1} B_{s-1}} e^{\tilde{h}_s B_s}, \end{aligned}$$

where the $\tilde{h}_i$ represent the forward Philip Hall coordinates, and equate coefficients of the basis elements, B_i. We will illustrate this by way of example.

Example 3.10: (Two–Input Nilpotent System) Consider a two input system that is nilpotent of order three,

$$\dot{x} = g_1(x)u^1 + g_2(x)u^2,$$

with an extended system,

$$\dot{x} = b_1(x)v^1 + b_2(x)v^2 + b_3(x)v^3 + b_4(x)v^4 + b_5(x)v^5,$$

where

$$\begin{aligned} b_1 &= g_1 \\ b_2 &= g_2 \\ b_3 &= [g_1, g_2] \\ b_4 &= [g_1, [g_1, g_2]] \\ b_5 &= [g_2, [g_1, g_2]]. \end{aligned}$$

We need to equate the basis elements in the expansions for

$$e^{h_5 B_5} e^{h_4 B_4} e^{h_3 B_3} e^{h_2 B_2} e^{h_1 B_1} = e^{\tilde{h}_1 B_1} e^{\tilde{h}_2 B_2} e^{\tilde{h}_3 B_3} e^{\tilde{h}_4 B_4} e^{\tilde{h}_5 B_5}.$$

Using the Campbell–Baker–Hausdorff formula (Equation 2.3), the formal exponential on the left–hand side becomes

$$
\begin{aligned}
1 &+ h_5B_5 + h_4B_4 + h_3B_3 + h_2B_2 + h_1B_1 + \frac{1}{2}h_3h_2[B_3, B_2] + \frac{1}{2}h_3h_1[B_3, B_1] \\
&+ \frac{1}{2}h_2h_1[B_2, B_1] + \frac{1}{12}(h_2^2h_1[B_2, [B_2, B_1]] - h_2h_1^2[B_1, [B_2, B_1]]) = \\
1 &+ h_5B_5 + h_4B_4 + h_3B_3 + h_2B_2 + h_1B_1 + \frac{1}{2}h_2h_3B_5 - \frac{1}{2}h_1h_3B_4 \\
&- \frac{1}{2}h_1h_2B_3 - \frac{1}{12}(h_1h_2^2B_5 - h_1^2h_2B_4) = \\
1 &+ h_1B_1 + h_2B_2 + (h_3 - \frac{1}{2}h_1h_2)B_3 + (h_4 - \frac{1}{2}h_1h_3 + \frac{1}{12}h_1^2h_2)B_4 \\
&+ (h_5 + \frac{1}{2}h_2h_3 - \frac{1}{12}h_1h_2^2)B_5,
\end{aligned}
$$

(recall that the system is nilpotent of degree three). Similarly, for the right–hand side,

$$
\begin{aligned}
1 &+ \tilde{h}_1B_1 + \tilde{h}_2B_2 + \tilde{h}_3B_3 + \tilde{h}_4B_4 + \tilde{h}_5B_5 + \frac{1}{2}\tilde{h}_1\tilde{h}_2[B_1, B_2] \\
&+ \frac{1}{12}(\tilde{h}_1^2\tilde{h}_2[B_1, [B_1, B_2]] - \tilde{h}_1\tilde{h}_2^2[B_2, [B_1, B_2]]) + \frac{1}{2}\tilde{h}_1\tilde{h}_3[B_1, B_3] + \frac{1}{2}\tilde{h}_2\tilde{h}_3[B_2, B_3] = \\
1 &+ \tilde{h}_1B_1 + \tilde{h}_2B_2 + \tilde{h}_3B_3 + \tilde{h}_4B_4 + \tilde{h}_5B_5 + \frac{1}{2}\tilde{h}_1\tilde{h}_2B_3 \\
&+ \frac{1}{12}(\tilde{h}_1^2\tilde{h}_2B_4 - \tilde{h}_1\tilde{h}_2^2B_5) + \frac{1}{2}\tilde{h}_1\tilde{h}_3B_4 + \frac{1}{2}\tilde{h}_2\tilde{h}_3B_5 = \\
1 &+ \tilde{h}_1B_1 + \tilde{h}_2B_2 + (\tilde{h}_3 + \frac{1}{2}\tilde{h}_1\tilde{h}_2)B_3 + (\tilde{h}_4 + \frac{1}{12}\tilde{h}_1^2\tilde{h}_2 + \frac{1}{2}\tilde{h}_1\tilde{h}_3)B_4 \\
&+ (\tilde{h}_5 - \frac{1}{12}\tilde{h}_1\tilde{h}_2^2 + \frac{1}{2}\tilde{h}_2\tilde{h}_3)B_5.
\end{aligned}
$$

Equating the coefficients of the basis elements, gives the $\tilde{h}_i$'s in terms of the h_i's,

$$
\begin{aligned}
\tilde{h}_1 &= h_1 \\
\tilde{h}_2 &= h_2 \\
\tilde{h}_3 &= h_3 - h_1h_2 \\
\tilde{h}_4 &= h_4 - h_1h_3 + \frac{1}{2}h_1^2h_2 \\
\tilde{h}_5 &= h_5 + h_2h_3 - \frac{1}{2}h_1h_2^2.
\end{aligned}
$$

Note that this transformation bears a striking resemblance to the transformation between *chained form* system (see Section 5.4) and *power form* (see Teel, Murray, and Walsh (1992)), but is not precisely the same (neither is its inverse). □

From the example, it is clear that, regardless of the number of inputs and degree of nilpotency, $\tilde{h}_i = h_i$ for vector fields in the non–extended system; however, in general, the degree of nilpotency and number of inputs will affect the structure of the higher order terms in the transformation.

Now, to determine the real inputs, simply approximate any Lie bracket by its first order piecewise approximation given in Equation 2.2, as illustrated by the following example.

Example 3.11: (From Lafferriere and Sussmann (1993)). Consider the same system as in Example 3.10. Assume further that we have followed the construction and solved the differential equations in Equation 3.10, which gives the values for the forward Philip Hall coordinates, $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4, \tilde{h}_5$. We know that the fictitious inputs steer the system to the desired final value. The task now is to determine the real control inputs, u^i. Given the forward Philip Hall coordinates, the final position is given by

$$S(t) = e^{\tilde{h}_1 b_1} e^{\tilde{h}_2 b_2} e^{\tilde{h}_3 b_3} e^{\tilde{h}_4 b_4} e^{\tilde{h}_5 b_5}.$$

Now, since $b_1 = g_1$ the first flow, $e^{\tilde{h}_1 b_1}$ is just along g_1, so the control input $u_1 = \operatorname{sgn}(\tilde{h}_1)$ for time $0 < t \leq |\tilde{h}_1|$ will result in the flow $e^{\tilde{h}_1 g_1} = e^{\tilde{h}_1 b_1}$ at time $t = \tilde{h}_1$ as desired. Similarly, for time $|\tilde{h}_1| < t \leq |\tilde{h}_2|$ the control input $u_2 = \operatorname{sgn}(\tilde{h}_2)$ will result in the flow $e^{\tilde{h}_1 g_2} = e^{\tilde{h}_2 b_2}$, so the composition, $u_1 = \operatorname{sgn}(h_1)$ for $0 < t \leq |\tilde{h}_1|$ followed by $u_2 = \operatorname{sgn}(h_2)$, $|\tilde{h}_1| < t \leq |\tilde{h}_2|$ results in the net flow $e^{\tilde{h}_1 b_1} e^{\tilde{h}_2 b_2}$.

Now, we need to flow in the Lie bracket direction $e^{\tilde{h}_3 b_3} = e^{\tilde{h}_3 [g_1, g_2]}$. To leading order, the sequence

$$\begin{array}{rcll} u_1 & = & \mathrm{sgn}(\tilde{h}_3), & |\tilde{h}_1| + |\tilde{h}_2| < t \leq |\tilde{h}_1| + |\tilde{h}_2| + \sqrt{\tilde{h}_3} \\ u_2 & = & 1, & |\tilde{h}_1| + |\tilde{h}_2| + \sqrt{\tilde{h}_3} < t \leq |\tilde{h}_1| + |\tilde{h}_2| + 2\sqrt{\tilde{h}_3} \\ u_1 & = & -\mathrm{sgn}(\tilde{h}_3), & |\tilde{h}_1| + |\tilde{h}_2| + 2\sqrt{\tilde{h}_3} < t \leq |\tilde{h}_1| + |\tilde{h}_2| + 3\sqrt{\tilde{h}_3} \\ u_2 & = & -1 & |\tilde{h}_1| + |\tilde{h}_2| + 3\sqrt{\tilde{h}_3} < t \leq |\tilde{h}_1| + |\tilde{h}_2| + 4\sqrt{\tilde{h}_3} \end{array}$$

gives rise to $e^{\tilde{h}_3[g_1,g_2]}$. However, using the Campbell–Baker–Hausdorff formula, and assuming a nilpotency of degree three, an easy, but tedious calculation shows that it actually gives rise to

$$e^{\tilde{h}_3[g_1,g_2]} e^{\frac{1}{2}\tilde{h}_3^{3/2}[g_1,[g_1,g_2]]} e^{\frac{1}{2}\tilde{h}_3^{3/2}[g_2,[g_1,g_2]]} = e^{\tilde{h}_3 b_3} e^{\frac{1}{2}\tilde{h}_3^{3/2} b_4} e^{\frac{1}{2}\tilde{h}_3^{3/2} b_5}.$$

We now must find a control that gives rise to $e^{(\tilde{h}_4 - \frac{1}{2}\tilde{h}_3^{3/2})b_4} e^{(\tilde{h}_5 - \frac{1}{2}\tilde{h}_3^{3/2})b_5}$, to cancel out the third–order error from above and give the correct third–order exponentials. If $\rho = \left(\delta - \frac{1}{2}\gamma^{\frac{3}{2}}\right)^{\frac{1}{3}}$, and $\sigma = \left(\epsilon - \frac{1}{2}\gamma^{\frac{3}{2}}\right)^{\frac{1}{3}}$, the 20 sequence move that accomplishes this is

$$\begin{array}{l} \rho g_1 \circ \rho g_1 \circ \rho g_2 \circ (-r g_1) \circ (-\rho g_2) \circ (-\rho g_1) \circ \rho g_2 \circ \rho g_1 \circ (-\rho g_2) \circ (\rho g_1) \circ \\ \sigma g_2 \circ \sigma g_1 \circ \sigma g_2 \circ (-r g_1) \circ (-\sigma g_2) \circ (-\sigma g_2) \circ \sigma g_2 \circ \sigma g_1 \circ (-\sigma g_2) \circ \in \sigma g_1), \end{array}$$

where the notation, ρg_1 means flow along g_1 for time ρ, *i.e.*, turn on control input $u_1 = \alpha$ for ρ length of time, where $\alpha = \pm 1$, as is necessary for any given bracket. The basic idea is to construct the control inputs that gives the appropriate first–order flow, and then take care of the higher order error when constructing the control input sequence to execute the Lie bracket directions. $\square$

This method generates the actual control inputs necessary to follow the desired trajectory. If the system is nilpotent this method exactly steers the system to the desired final state. If the system is not nilpotent, it steers it to a point that is, at worst, only half the distance to the desired configuration. The algorithm can thus be iterated to generate arbitrary precision. This iterated method also includes the notion of a "critical" step length. It is possible to analytically determine the critical step length. Alternatively, note that an appropriate step length can be determined by simulation or experiment, and the simulation results in Lafferriere and Sussmann (1993) show that the actual critical length can be significantly larger than the estimated bound.

Chapter 4

Stratified Controllability

This chapter considers controllability of stratified systems. Section 4.1 presents a simple example which illustrates the issue of fundamental importance to our result, which is transversality of foliations. Section 4.2 presents three alternative controllability tests (corresponding to Chow's theorem, and the exterior differential systems and principal fiber bundle reformulations thereof) for a nested sequence of strata. Section 4.3, presents a robotic example to illustrate the application of the theory. Particularly useful for and motivated by robotics applications is the notion of gait controllability, which is the subject of in Section 4.4.

4.1 Stratified Controllability: Introductory Example

In order to clarify the presentation and provide an intuitive understanding of our approach, we first consider a subset of all possible stratified systems for which the controllability analysis is greatly simplified. In particular, this section focuses on the case where the configuration manifold contains only one submanifold (or stratum) upon which the system is subjected to additional constraints, so that the only stratum is also the bottom stratum. By focusing on this situation, it is rather straightforward to motivate and obtain a basic controllability result. Also, as will become clear, this simple result is easily generalized.

The following example, while extremely simple, nevertheless exemplifies the fundamental nature of a stratified control system. As we develop our results, we will repeatedly return to this problem.

Example 4.1: (Kinematic Leg) Consider the kinematic leg illustrated in Figure 4.1. The configuration space, M, for the leg is parameterized by the variables $q = (x, l, \theta)$, corresponding to the lateral position of the body, the length of the leg and the angular displacement of the leg, respectively. Assume that the height of the body off of the ground remains fixed, so when the leg is lifted off of the ground, the body does not fall down. While this assumption is clearly unrealistic for, say, a hopping robot, it may be realistic, for example if the leg under consideration is only one leg of a multi–legged robot, where the focus is solely on the effect that this one leg has on the system. The two inputs for the system are the joint velocities $u_1 = \dot{l}$ and $u_2 = \dot{\theta}$.

In this case, the bottom stratum (or boundary) is the set of points

$$S = \{q \in M : l \cos\theta = h\},$$

where h is some fixed height. The equations of motion are given by

$$\frac{d}{dt}\begin{pmatrix} x \\ l \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2, \tag{4.1}$$

when the foot is off the ground, and

$$\frac{d}{dt}\begin{pmatrix} x \\ l \\ \theta \end{pmatrix} = \begin{pmatrix} -\frac{l}{\cos\theta} \\ l\tan\theta \\ 1 \end{pmatrix} u_2, \tag{4.2}$$

when the foot is in contact with the ground (on the bottom stratum, *i.e.*, $q \in S$).

Note that, consistent with Remark 2.11, when the leg is off the ground, there are two control inputs; conversely, when the leg is in contact with the ground, there is only one input because the inputs are constrained when the foot contacts the ground. Note also that even though Equation 4.1 is not the equation of motion for

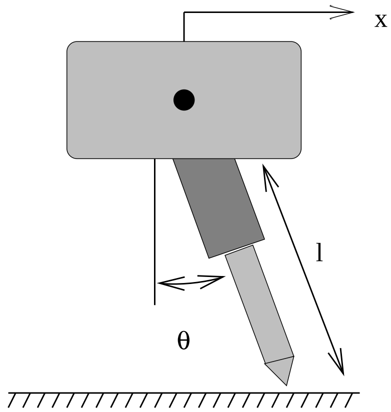


Figure 4.1. Kinematic leg.

the system when $q \in S$, the two vector fields in that equation are still well defined for all points in S. $\square$

It is clear that if the leg needs to move laterally (in the x–direction) while still retaining control over the joint variables, it must cyclically move the leg in and out of contact with the ground. This observation motivates the need to formulate a controllability test which incorporates the equations of motion for the system both in and out of contact with the ground. Figure 4.2 schematically illustrates the configuration space for the simple kinematic leg example. It is comprised of the "ambient" space, M, where the leg is off of the ground, and the submanifold (or boundary), S, which represents the set of points where the leg is in contact with the ground.

Since we know the equations of motion in each strata, we can calculate the associated involutive closures of the distributions associated with M and also with S, denoted $\overline{\Delta}_M$ and $\overline{\Delta}_S$, respectively. Note that in Figure 4.2, the symbols for the involutive distributions are pointing to the manifolds to which they are the tangent space.

It follows from Chow's theorem that, if the system starts at a point in S, then the set of points it can reach in S is the leaf of the foliation of S defined by $\overline{\Delta}_S$ which contains that point. In Figure 4.2, such a leaf is represented by the lines in

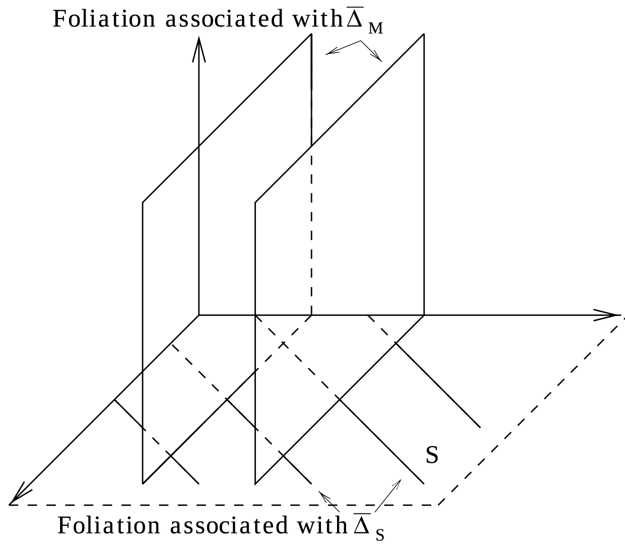


Figure 4.2. Controllability of a stratified system.

S. Similarly, if the system starts at a point in M, then the set of points that can be reached in M is represented in Figure 4.2 by the vertical sheets in M, which represent the foliation defined by $\overline{\Delta}_M$. Any arbitrary point in S is contained in one leaf of the foliation of M defined by $\overline{\Delta}_M$ and one leaf of the foliation of S defined by $\overline{\Delta}_S$. By the proof of Chow's Theorem (Theorem 3.4), $\overline{\Delta}_M$ and $\overline{\Delta}_S$ are the directions in which the system can flow on M and S, respectively. Since any point in S is also contained in M, and by assumption, the system can move from S to M arbitrarily, then the vector space sum of $\overline{\Delta}_M|_{x_0}$ and $\overline{\Delta}_S|_{x_0}$ represents all the directions in which the system can flow. Thus, if $\overline{\Delta}_M$ and $\overline{\Delta}_S$ intersect *transversely*, *i.e.*

$$\overline{\Delta}_M|_{x_0} + \overline{\Delta}_S|_{x_0} = T_{x_0}M,$$

then the system can flow in any direction in M. This argument suggests the following Proposition.

Proposition 4.2 *If*

$$\overline{\Delta}_M|_{x_0} + \overline{\Delta}_S|_{x_0} = T_{x_0}M,$$

then the system is STLC from x_0.

Since this proposition will follow trivially as a corollary of a following more general result (Proposition 4.4), we will not provide the proof.

Example 4.3: (Kinematic leg — continued) To show that the kinematic leg is controllable, we must show that its equations of motion satisfy the requirement of Proposition 4.2. Since the vector fields in the equations of motion when the foot is not in contact with the ground are constant, then

$$\overline{\Delta}_M = \operatorname{span}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

When the foot is in contact with the ground, there is only one vector field, so

$$\overline{\Delta}_S = \operatorname{span}\begin{pmatrix} -\frac{l}{\cos\theta} \\ l\tan\theta \\ 1 \end{pmatrix}.$$

Clearly, away from singularities at $\pm\frac{\pi}{2}$,

$$\overline{\Delta}_M + \overline{\Delta}_S = TM.$$

Thus the kinematic leg is controllable. □

4.2 Controllability for a Nested Sequence of Strata

The central aspect of the above controllability discussion was the transversal relationship between the foliations defined by the control vector fields on M and S. This notion is easy to generalize to a nested sequence of submanifolds:

$$S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0 = M.$$

In this sequence of submanifolds, the subscript is the codimension of the submanifold. Note that there may be multiple submanifolds with the same codimension at a point. If there are multiple submanifolds with the same codimension, this sequence contains only *one* of them. Also, denote the distribution defined by the control vector fields defined on a stratum S_i by Δ_i, and its associated involutive closure by $\overline{\Delta}_i$.

Clearly, the fact that we limit our attention to the equations of motion in a nested sequence is rather limiting. Section 4.4 presents results which eliminate this restriction. However, this comes at the expense of a more complicated test. Also, later in this section we will show that this limitation actually encompasses quite a broad class of legged locomotion problems.

4.2.1 The Distribution Approach

Here we present a test using distributions, which may be considered an extension of Chow's theorem. This result is central to this chapter because the results that follow are based upon it.

Proposition 4.4 *If there exists a nested sequence of submanifolds*

$$x_0 \in S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0,$$

such that the associated involutive distributions satisfy

$$\sum_{j=0}^{p} \overline{\Delta}_{S_j}|_{x_0} = T_{x_0}M$$

then the system is stratified controllabile from x_0.

Proof: Let V_p be a neighborhood of the point x_0 in the submanifold S_p, which is the bottom stratum, *i.e.*, the manifold of smallest dimension in the nested sequence at x_0. Choose $X_1 \in \Delta_{S_p}$. For ϵ_1 sufficiently small,

$$N_p^1 = \{\phi_{t_1}^{X_1}(x_0) : 0 < t_1 < \epsilon_1\}$$

is a smooth manifold of dimension one. This follows from, for example, the Flow-Box theorem (Theorem 2.26 of Nijmeijer and der Schaft (1990)), the Straightening Out Theorem (Theorem 4.1.14 of Abraham, Marsden, and Ratiu (1988)) or the Orbit Theorem (Theorem 1, Chapter 2 of Jurdjevic (1997)).

Now, construct N_p^j by induction. Assume that the collection of vector fields, $\{X_1, \ldots, X_{j-1}\}$, $X_i \in \Delta_{S_p}$ is such that the mapping

$$(t_1, \ldots, t_{j-1}) \mapsto \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0), \qquad X_i \in \Delta_{S_p} \quad 0 < t_i < \epsilon_i, \tag{4.3}$$

has rank $j-1$. Thus, by the immersion theorem (see, for example, Theorem 3.5.7 of Abraham, Marsden, and Ratiu (1988), Theorem 2.19 of Nijmeijer and der Schaft (1990)), the set

$$N_p^{j-1} = \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0) \qquad 0 < t_i < \epsilon_i$$

is a $(j-1)$–dimensional manifold. Also, for the ϵ_i sufficiently small, $N_p^{j-1} \subset V_p$.

If $(j-1) < \dim\left(\overline{\Delta}_{S_p}\right)$, then there exists $x \in N_p^{j-1}$ and $X_j \in \Delta_{S_p}$ such that $X_j(x) \notin T_x N_p^{j-1}$. If this were not so, then $\overline{\Delta}_{S_p} \subset T_x N_p^{j-1}$ for any x in some open set $W \subset V_p$. This cannot be true since $\dim\left(\overline{\Delta}_{S_p}\right) > \dim\left(N_p^{j-1}\right)$. Thus, for ϵ_j sufficiently small the mapping

$$(t_1, \ldots, t_j) \mapsto \phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0), \qquad X_i \in \Delta_{S_p}, \quad 0 < t_i < \epsilon_i \tag{4.4}$$

has rank j. To see this, consider the tangent mapping

$$\begin{aligned} &T\left(\phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)\right) = \\ &\quad \left[X_j \Phi_j(x_0) \quad \left(\phi_{t_j}^{X_j}\right)_* X_{j-1}\Phi_j(x_0) \quad \cdots \quad \left(\phi_{t_j}^{X_j} \circ \cdots \circ \phi_{t_2}^{X_2}\right)_* X_1 \Phi_j(x_0)\right], \end{aligned}$$

where $\Phi_j(x_0) = \phi_{t_j}^{X_j} \circ \phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0)$. For a diffeomorphism, ϕ, and a vector field, X, $\phi_* X = T\phi \circ X \circ \phi^{-1}$. If the rank of this mapping is not j, then

$$X_j(\Phi_j(x_0)) = \sum_{i=1}^{j-1} \alpha_i \left(\phi_{t_j}^{X_j} \circ \cdots \circ \phi_{t_{i+1}}^{X_{i+1}}\right)_* X_i(\Phi_j(x_0)),$$

for some coefficients, α_i. However, if we pull this back along the flow of X_j, then

$$\begin{aligned} \left(\phi_{t_j}^{-X_j}\right)_* X_j(\Phi_j(x_0)) &= \sum_{i=1}^{j-1} \alpha_i \left(\phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_{i+1}}^{X_{i+1}}\right)_* X_i(\Phi_{j-1}(x_0)) \\ \implies \quad X_j\left(\Phi_{j-1}(x_o)\right) &= \sum_{i=1}^{j-1} \alpha_i \left(\phi_{t_{j-1}}^{X_{j-1}} \circ \cdots \circ \phi_{t_{i+1}}^{X_{i+1}}\right)_* X_i \Phi_{j-1}(x_0), \end{aligned}$$

which contradicts the fact that $X_j \notin TN_p^{j-1}$. Thus,

$$N_p^j = \{\phi_{t_j}^{X_j} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0) : 0 < t_i < \epsilon_i, i = 1, \ldots, j\}$$

is a j dimensional manifold. Since ϵ can be made arbitrarily small, $N_p^j \subset V_p$. Now, if $k = n_p$, $N_p^k \subset V_p$ is an n_p–dimensional manifold.

Now, let $(s_1, \ldots, s_n)$ satisfy $0 < s_1 < \epsilon_i$ and consider the map

$$(t_1, \ldots, t_{n_p}) \mapsto \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}, \qquad 0 < t_i < \epsilon_i. \tag{4.5}$$

Since $\phi_s^{-X} = \phi_{-s}^X$, it follows that the image of this map is an open set of N_n containing the point x_0. Hence $\mathcal{R}^V(x_0, \epsilon)$ contains x_0 and an open set in the manifold whose tangent space is $\overline{\Delta}_{s_p}$. By restricting each $\epsilon \leq T/(2n_p)$, there is such an open set for any $T > 0$.

So far, we have constructed the reachable set for the system restricted to the bottom stratum, S_p. The process is to extend the reachable set by using vector fields defined on the next higher stratum, S_{p-1}, and then proceed to each higher stratum in order. (Note that we are following the indices in *reverse* order). Proceed by induction. Assume that we have constructed the reachable set up to and including stratum S_{k+1}, and denote this reachable set by N_{k+1}. Without loss of generality, assume that

$$\dim\left(\sum_{j=k}^{p}\overline{\Delta}_{S_j}|_{x_0}\right) > \dim\left(\sum_{j=(k+1)}^{p}\overline{\Delta}_{S_j}|_{x_0}\right),$$

(otherwise, the control distribution, $\overline{\Delta}_{S_k}$ would not contribute any "new directions" to the reachable set, in which case we can proceed to the next higher stratum, S_{k-1}).

Now, let $n_i = \sum_{j=i}^{p} \dim\left(\overline{\Delta}_{S_i}\right)$, let the vector fields $X_1, \ldots, X_{n_p}$ be defined on S_p, let the vector fields $X_{n_p+1}, \ldots X_{n_{p-1}}$ be defined on S_{p-1}, etc. We will be considering compositions of flows of the following type:

$$\underbrace{\phi_{t_{n_k}}^{X_{n_k}} \circ \cdots \circ \phi_{t_{n_{k+1}+1}}^{X_{n_{k+1}+1}}}_{\mathrm{on} S_k} \circ \underbrace{\cdots}_{\mathrm{on} S_{k+1},\ldots,S_{p-1}} \circ \underbrace{\phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}}_{\mathrm{on} S_p}(x_0),$$

where the construction starts on the bottom stratum, S_p, using vector fields defined there, and proceeds to the higher strata in order.

Also assume (as part of the induction hypothesis) that the mapping

$$(t_1, \ldots, t_{n_p}, \ldots, t_{n_{k+1}}) \mapsto \phi_{t_{n_{k+1}}}^{X_{n_{k+1}}} \circ \cdots \circ \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0),$$

has rank n_{k+1}, so the set

$$N_{k+1}^{n_{k+1}} = \phi_{t_{n_{k+1}}}^{X_{n_{k+1}}} \circ \cdots \circ \phi_{s_1}^{-X_1} \circ \cdots \circ \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0),$$

is a (n_{k+1})–dimensional manifold.

By continuity, there exists a neighborhood, $V_k \subset S_k$ in which $\dim\left(\overline{\Delta}_{S_k}\right)$ is constant. Since $\dim(N_{k+1}) < \dim\left(\sum_{j=k}^{p}\overline{\Delta}_{S_j}|_{x_0}\right)$, there exits a vector field, $X \in \Delta_{S_k}$, and a point, $x \in V_k$, such that $X(x) \notin T_x N_k$. If this were not possible, then $X(x) \in T_x N_{k+1} \quad \forall X \in \Delta_{S_k}$ and $x \in V_k$. But this implies that $\overline{\Delta}_{S_k} \subset TN_{k+1}$. Since, by construction, $\overline{\Delta}_{S_i} \subset TN_i \subset TN_{k+1}$ for $i = (k+1), \ldots, p$,

$$\left(\sum_{j=k}^{p}\overline{\Delta}_{S_j}|_{x_0}\right) \subset T_{x_0}N_{k+1},$$

which implies that

$$\dim\left(\sum_{j=k}^{p}\overline{\Delta}_{S_j}|_{x_0}\right) \leq \dim\left(N_{k+1}\right)$$

which is a contradiction.

By exactly the same argument as before, then, the set

$$\begin{aligned} N_k^{n_k} = \phi_{s_{n_{k+1}+1}}^{-X_{n_{k+1}+1}} \circ \cdots \circ \phi_{s_{n_k}}^{-X_{n_k}} \circ \phi_{t_{n_k}}^{X_{n_k}} \quad &\circ \quad \cdots \circ \phi_{s_1}^{-X_1} \circ \cdots \\ &\circ \quad \phi_{s_{n_p}}^{-X_{n_p}} \circ \phi_{t_{n_p}}^{X_{n_p}} \circ \cdots \circ \phi_{t_1}^{X_1}(x_0), \end{aligned}$$

is an n_k–dimensional manifold containing the point x_0, and by construction $N_p^k \subset \mathcal{R}^V(x_0, \leq \epsilon_1 + \cdots + \epsilon_{n_p})$. Hence $\mathcal{R}^V(x_0, \epsilon)$ contains x_0 and an open set in the manifold whose tangent space is $\overline{\Delta}_{s_p}$. By restricting each $\epsilon \leq T/(2n_p)$, there is such an open set for any $T > 0$. ■

The proof of Proposition 4.2 now follows trivially by considering a nested sequence containing only one submanifold.

Note that it is not necessary that the nested sequence actually include the full configuration space M. It may, in fact, terminate at some stratum, S_p. In such a case, however, controllability amounts to reaching an open neighborhood of the starting point in the relative topology of the highest stratum, S_p.

Also note that if the configuration space has a boundary, Proposition 4.4 still applies with a simple modification of the proof. In a manner similar to that in the proof, when extending the reachable set from the submanifold boundary into the manifold in which it is contained, we can always choose the first vector field along which the system flows to be the one that violates the constraint $d\Phi_i(x)\dot{x} = 0$, in the "allowable" direction (Section 4.4 elaborates more on this notion). However, in the constructed "reversed" flow (Equation 4.5), do not include this reversed flow corresponding to this vector field which moves the system off of the boundary. In this case, the reachable set will be open in the interior of the manifold and contain points arbitrarily close to points in the boundary. By assumption, it is possible to move from the interior of the manifold to points in the boundary. In this manner, then, the final constructed manifold contains x_0 and will be an open neighborhood of x_0 defined in the appropriate relative topology, *i.e.*, the topology of a manifold with boundary.

Digression: Manifolds with Boundary

At this point, a natural question is whether it would be beneficial to explicitly incorporate into our analysis machinery associated with manifolds with boundary. This seems appealing for at least two reasons. First, it is the case physically that the configuration space for many stratified systems are actually manifolds with boundary. Secondly, the basic manifold structure (including the tangent bundle, the notion of diffeomorphisms and so on) is well developed (see, for example, Boothby (1986) and Abraham, Marsden, and Ratiu (1988)). One would hope, then, that the manifold with boundary structure, and associated machinery, lends itself to the direct application of Chow's theorem.

Mathematically, the primary purpose of manifolds with boundary seems to be in connection with integration on manifolds, with the ultimate purpose of formulating Stokes' theorem. (In particular, the integral of an $(n-1)$–form on the boundary an n–dimensional orientable manifold is equal to the integral of its exterior derivative over the manifold itself).

The problem with stratified systems that prevents the direct application of Chow's theorem (even recognizing the boundary structure) is the fact that the equations of motion on the boundary are not the same as the equations of motion in the interior of the manifold. This is in contrast with Stokes' theorem, where the form on the boundary and the derivative of the *same* form in the interior are the objects of interest. This leads to the necessity of treating the equations of motion on a stratum by stratum basis, as was necessary in the proof of Proposition 4.4. Additionally, as the equations of motion for the kinematic leg in Example 4.1 illustrates, the "natural" equations of motion for the system provide no indication that there is a boundary present in the problem. For example, one cannot determine that there is a boundary by inspecting the vector fields that make up the equations of motion for the leg when the foot is off the ground. In fact, the vector fields in the equations of motion are well defined everywhere, including points in the boundary and points corresponding to penetrating the ground. The presence of the boundary is only indicated by the fact that we provide an additional set of equations that we specify as the equations of motion on the boundary. Since the focus of the analysis is on the vector fields in the equations of motion which do not naturally indicate the boundary structure and since, unlike Stokes' theorem, there is no natural relationship between the equation of motion in the interior and the boundary, we choose not to incorporate explicitly any manifold with boundary machinery, and treat possible boundaries as simple submanifolds.

Combinations of Nested Sequences of Strata

Proposition 4.4 only directly applies to a single nested sequence of strata; however, repeatedly applying the test to multiple sequences is possible. The usefulness of this approach is that if the top stratum in each sequence is different, then the test determines controllability for the *union* of the top strata. For example, for the configuration space shown in Figure 4.3, Proposition 4.4 applied to the sequence $S_{12} \subset S_1$ will tell if the system can reach an open set in S_1 and applied to $S_{12} \subset S_2$ will tell if it can reach an open set in S_2, and taken together, gives controllability in the relative topology of the union $S_1 \cup S_2$. This is useful because, for problems like the biped from Example 2.10, reaching open sets in the relative topology of the union of strata is often the most natural way to define controllability.

4.2.2 The Exterior Differential Systems Approach

This section and the following section offer two alternative approaches to the formulation of Proposition 4.4. As mentioned in Section 2.2.2, in some cases these two alternative formulations simplify the calculations necessary to determine controllability.

Recall the definition of the derived flag in Section 2.2.2. The derived flag de-

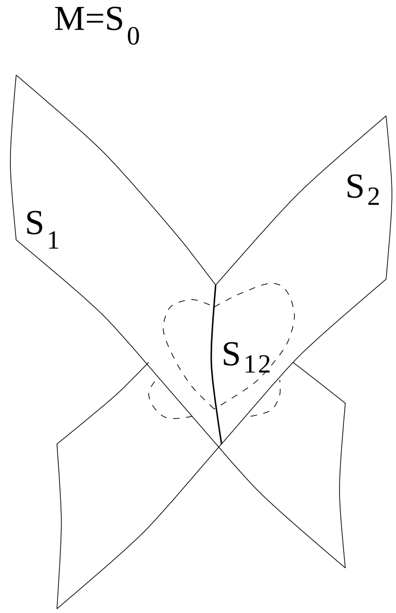


Figure 4.3. Stratified open neighborhood.

scribes the integrability properties of the ideal generated by I. It follows from Frobenius' Theorem that $I^{(N)}$ is the largest integrable subsystem contained in I, therefore, if $I^{(N)}$ is not empty, there exists functions $h_1, \ldots, h_r$ such that $\{dh_i\} \subset \{I\}$. Thus, if the bottom derived flag is not empty, there exists functions which describe a foliation of the state space.

Recall that on M, there is collection of constraints

$$\{\omega^{0,1}, \ldots, \omega^{0,m_0}\},$$

and on a codimension one submanifold S_{1_i} there is the collection of constraints

$$\{\omega^{0,1}, \ldots, \omega^{0,s}, d\Phi_{1_i}\omega^{1,1}, \ldots, \omega^{1,m_1}\}.$$

On any stratum, S_I, we will have all the regular constraints in the codistribution along with the derivatives of the functions, Φ_I which locally defines the stratum. Let $I_M^{(N_0)}$ be the bottom derived flag for the constraints on M. Let $I_{S_i}^{(N_{S_i})}$ be the bottom derived flag for the set of constraints on S_i. Similarly to the manner of construction in the previous section, we can calculate derived flags associated with each submanifold S_i.

Proposition 4.5 *If there exists a nested sequence of submanifolds*

$$S_{p_i} \subseteq S_{(p-1)_i} \subseteq \cdots \subseteq S_{1_i} \subseteq S_0 = M,$$

such that the associated derived flags satisfy

$$I_{S_{p_i}}^{(N_{S_{p_i}})} \cap \cdots \cap I_{S_{1_i}}^{(N_{S_{1_i}})} \cap I_M^{(N_M)} = 0,$$

at x_0 then the system is STLC from x_0.

Proof: Note that the subspace of T_xM such that $\langle v, \omega \rangle = 0, v \in T_xM, \omega \in \operatorname{span}(I_M^{(N_M)})$ is $\overline{\Delta}_M|_x$. Similarly, the subspace of T_xM such that $\langle v, \omega \rangle = 0, v \in T_xM, \omega \in \operatorname{span}(I_{S_i}^{(N_{S_i})})$ is and $\overline{\Delta}_{S_i}|_x$. A complete explanation on the relationship between the levels of the derived flags and distributions (in a filtration) can be found in Murray (1994).

Now, the collection of tangent vectors that satisfies $\langle v, \omega \rangle = 0, v \in T_xM$,

$$\omega \in \operatorname{span}(I_M^{(N_M)}) \cap \operatorname{span}(I_{S_{1_i}}^{(N_{S_{1_i}})}) \cap \cdots \cap \operatorname{span}(I_{S_{p_i}}^{(N_{S_{p_i}})})$$

is the whole tangent space at x, T_xM. Therefore

$$\overline{\Delta}_M|_x + \sum_i \overline{\Delta}_{S_i}|_x = T_xM.$$

Thus, by Proposition 4.4, the system is STLC from x_0. ■

Example 4.6: (Kinematic leg — revisited) We now return to the simple kinematic leg example to illustrate the application of the exterior differential systems approach.

In this case, the application of Proposition 4.5 is simple because the constraints are integrable in all the strata. In particular, note that when the leg is not in contact with the ground, there is the constraint

$$dx = 0.$$

This constraint is integrable, so it is also the bottom derived system.

When the leg is in contact, there are the constraints

$$\cos\theta dl - l\sin\theta d\theta = 0,$$
$$dx + \sin\theta dl + l\cos\theta d\theta = 0.$$

Again, these constraints are integrable, so they comprise the bottom derived system. Clearly,

$$\text{span}\,\{dx\} \cap \text{span}\,\{\cos\theta dl - l\sin\theta d\theta, dx + \sin\theta dl + l\cos\theta d\theta\} = 0,$$

so the system is STLC. □

Again, as in the construction using distributions, there is no necessity that the highest stratum in the nested sequence be the entire configuration manifold. It could be a stratum, S_I. In this case, however, the intersections of the derived flags will not be empty. This results from the fact that the functions defining the strata are included in the codistribution describing the constraints, with the ultimate result that for controllability on a stratum which is not S_0 the bottom derived system will contain the derivatives of the functions which describe that stratum. In practice, this will be easy to recognize since these functions will typically be the height of the feet off of the terrain. Section 4.3.2 contains an example illustrating this.

4.2.3 Principal Fiber Bundle Approach

A question naturally arises regarding whether there are any circumstances in which the above analyses can be simplified. In many instances, particularly with robotic systems, it is clear that the configuration manifold of the system is naturally considered in two parts, namely, the shape of the robot and its location in space. In such a context, the question of controllability is more naturally (and frequently more simply) addressed by considering directly the relationship between changes in shape and the resulting change in position in space.

The interpretation of the system's configuration space as a principal fiber bundle arises naturally in the study of robotic locomotion and many rigid body mechanical systems. The configuration space associated to a such mechanisms can naturally be decomposed into two sets of variables: those that describe the location of a frame rigidly affixed to one of the system's bodies, and those that describe the internal shape of the system. The set of all possible location of the body fixed frame is the set rigid body displacements, which is a Lie group: $SE(2)$ for systems restricted to motion in the plane, or $SE(3)$ in general. Let G denote a Lie group and B the set of internal shape variables.

Obviously, formulating controllability in terms of connections, and their curvature is problematic when the connection is not continuous or sufficiently differentiable. This section essentially combines the previous results in this dissertation to extend the results of 3.2.3 to the stratified case. We emphasize the fundamental assumptions in Kelly and Murray (1995); namely, that the constraints must be invariant under the action of the structure group and that there is full shape space actuation, *i.e.*, all the shape variables are directly controlled via an independent control input. In a nutshell, in a manner similar to formulating controllability in terms of the transversality of submanifolds of the configuration space of the controlled system (as we did in the previous sections of this paper), we will formulate controllability here in terms of direct sums of Lie subalgebras of the Lie algebra associated with the Lie group giving rise to the principal fiber bundle structure.

As before, consider a nested sequence of strata

$$S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0 = M.$$

Assume, furthermore, that each strata, S_i is a principal fiber bundle and so can be expressed as $S_i = B_i \times G$, where the structure group, G is the *same group* for each stratum. Again, as before, if the equations of motion for the system on each stratum are kinematic, then there is a connection defined on each stratum, Γ_i. Now, in accordance with the construction illustrated in subsection 2.1.2, define the sequence of Lie algebra subspaces $\mathfrak{h}_1^i, \mathfrak{h}_2^i, \ldots, \mathfrak{h}_k^i, \ldots$, associated with each stratum, S_i.

Proposition 4.7 *If there exists a nested sequence of strata*

$$S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0 = M,$$

such that

$$\mathfrak{g} = \sum_{i=0}^{p} \sum_{j} \mathfrak{h}_j^i,$$

then the system is weakly or fiber controllable. If

$$\mathfrak{g} = \sum_{i=0}^{p} \sum_{j \neq 1} \mathfrak{h}_j^i, \tag{4.6}$$

then the system is strongly controllable (STLC).

Proof: The proof of Proposition 3.8 (Kelly and Murray (1995)) shows the equivalence of Proposition 3.8 and Chow's theorem. In particular, it is assumed that there is complete shape space controllability in the sense that the equations of motion for the system can be expressed as the horizontal lift of m linearly independent vector fields defined on the shape space, B, where m is the dimension of B. Furthermore, it is assumed that the Lie bracket between any of these vector fields is zero. Thus, any "new directions" with result from bracketing contribute only in the fiber or group directions.

In the case of strong controllability, then, there is a direct correspondence between the series $\mathfrak{h}_2^i \oplus \mathfrak{h}_3^i \oplus \cdots$ and the distributions $\overline{\Delta}_{S_i}$ defined in association with

Proposition 4.4. In particular,

$$\overline{\Delta}_{S_i} = \text{span}\ \{g_{i,1}, \dots, g_{i,n_i}\} \oplus \mathfrak{h}_2^i \oplus \mathfrak{h}_3^i \oplus \cdots, \tag{4.7}$$

where the vector fields $\{g_{i,1}, \dots, g_{i,n_i}\}$ are the shape or base components of the control vector fields on stratum S. For this sum to make sense, the base component must be evaluated at the identity and each Lie algebra subspace, $\mathfrak{h}_j^i$ must be included via an inclusion into TM. Now, the result follows from Proposition 4.4.

In the case of weak controllability, Proposition 3.8 is just Chow's Theorem restricted to the structure group, G. In this case, there is a correspondence with the $\overline{\Delta}_{S_i}$ in Proposition 4.4 in that

$$\overline{\Delta}_{S_i} = \mathfrak{h}_1^i \oplus \mathfrak{h}_2^i \oplus \cdots.$$

Now, the result follows directly from Proposition 4.4. ■

Note that unlike Propositions 4.4 and 4.5, for strong controllability, this result requires that the last stratum in the sequence be the top stratum, because otherwise, complete shape controllability is not guaranteed. Thus, if the sequence is truncated at a lower stratum, the constraints in the shape space corresponding to various strata may not allow complete shape space controllability. Alternatively, the sequence can be truncated at a lower stratum if the sequence is such that, in combination, the system has complete shape space actuation controllability. When considering strong controllability in the union of a set of strata, one must consider the effect of constraints on the shape of the robot on each sequence of strata. The example in Section 4.3 will illustrate this concept. We emphasize that this limitation applies only to strong controllability, not weak controllability, because, in that case, the shape is irrelevant.

In contrast to this limitation, however, one aspect of this result is, in fact, too restrictive. In Proposition 3.8, the result for strong controllability is the sum of Lie algebra subspaces not including $\mathfrak{h}_1$ because these are the terms that give complete controllability over the shape space. The Lie algebra subspaces, $\mathfrak{h}_i$, $i > 1$, are the

Lie bracket terms that give rise to controllability in the group directions. In the above Proposition 4.7, in condition for strong controllability where none of the $\mathfrak{h}_1^j$ are considered in Equation 4.7 is in fact too strong. As long as *some* of the $\mathfrak{h}_1^j$ span all the shape directions, the other ones may be considered to give controllability in the group directions; however, which of the $\mathfrak{h}_i^k$'s that can be included in the sum in Equation 4.6 is problem specific, and thus cannot be set forth in a general form.

4.3 An Example

Because the kinematic leg example was so simple, it is instructive to include a more complicated example. The following is adapted from Kelly and Murray (1995). Consider the six–legged robot shown in Figure 4.4. It will be clear from the equations of motion for the system that each leg has only two degrees of freedom. In particular, the leg can move "up and down" and "forward and backward," but *not* "side to side" (in a direction outward from the body). In such a case it is not obvious how the robot can move in any direction.

Assume that the robot walks with a tripod gait, alternating movements of legs 1–4–5 with movements of legs 2–3–6. Hence, we are considering motions in only a subset of all possible strata. Suppose that

$$
\begin{aligned}
\dot{x} &= \cos\theta\,(\alpha(h_1)u_1 + \beta(h_2)u_2) \\
\dot{y} &= \sin\theta\,(\alpha(h_1)u_1 + \beta(h_2)u_2) \\
\dot{\theta} &= l\alpha(h_1)u_1 - l\beta(h_2)u_2 \\
\dot{\phi}_1 &= u_1 \\
\dot{\phi}_2 &= u_2 \\
\dot{h}_1 &= u_3 \\
\dot{h}_2 &= u_4
\end{aligned}
$$

where (x, y, θ) represents the planar position of the center of mass, ϕ_i is the front to back angular deflection of the legs and h_i is the height of the legs off the ground.

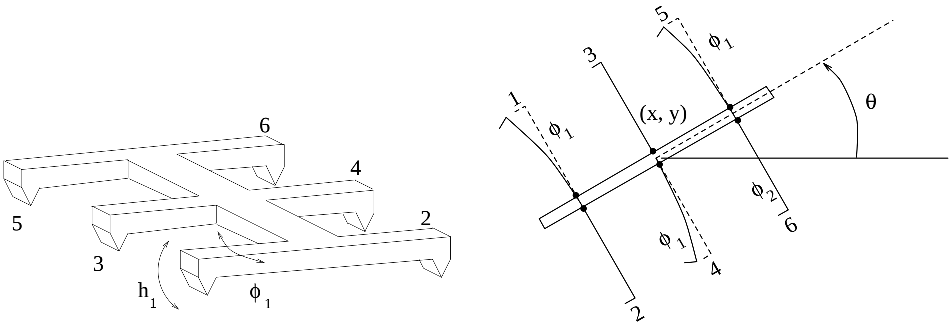


Figure 4.4. Six legged robot.

The tripod gait assumption requires that all the legs in a tripod move with the same angle $\dot{\phi}_i$. The inputs u_1 and u_2 control the leg swing velocities, while the inputs u_3 and u_4 control the leg lifting velocities.

The functions $\alpha(h_1)$ and $\beta(h_2)$ are defined by

$$\alpha(h_1) = \begin{cases} 1 & \text{if } h_1 = 0 \\ 0 & \text{if } h_1 > 0 \end{cases} \qquad \beta(h_2) = \begin{cases} 1 & \text{if } h_2 = 0 \\ 0 & \text{if } h_2 > 0 \end{cases}$$

(recall the tripod gait assumption: legs 1–4–5 move in unison as do legs 2–3–6). Denote the stratum when all the feet are in contact ($\alpha = \beta = 1$) by S_{12} (short for S_{123456}), the stratum when leg one is in contact ($\alpha = 1, \beta = 0$), by S_1 (short for S_{145}), the stratum when leg two is in contact ($\alpha = 0, \beta = 1$), by S_2 (short for S_{236}). and the stratum when no legs are in contact ($\alpha = \beta = 0$), by S_0.

This is a very simple model. In fact, it would not be possible to actuate both control inputs u_1 and u_2 independently without the feet slipping on the ground. However, if we allow the feet to slip as required by the equations of motion, then these equations roughly model the effect of the net frictional force on the body of the robot if the unactuated legs are completely passive. The "twisting" in the θ direction accounts for the fact that two feet are pushing on one side of the body; whereas, only one foot is pushing on the other side. This is a somewhat unsatisfying model in that it requires an "unnatural" consideration of forces when considering a purely kinematic model. However, we adopt it for clarity of presentation. A slightly more sophisticated model of this same robot is considered in Section 5.3.

4.3.1 The Distribution Approach

If all legs are in contact with the ground, the equations of motion are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta & 0 & 0 \\ \sin\theta & \sin\theta & 0 & 0 \\ l & -l & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4.8}$$

where u_3 and u_4 are constrained to be 0. Note that if f represents the first column, and g represents the second column, then

$$[f,g] = \begin{pmatrix} -2l\sin\theta \\ 2l\cos\theta \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad [[f,g],f] = \begin{pmatrix} 2l^2\cos\theta \\ 2l^2\sin\theta \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.9}$$

Clearly, with all the legs in contact with the ground, these vector fields span the (x, y, θ) directions. However, at this point we have not generated enough directions to span all the shape variables as well.

If legs 1, 4 and 5 are in contact with the ground, but legs 2, 3 and 6 are not in contact, the equations of motion are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 & 0 \\ \sin\theta & 0 & 0 & 0 \\ l & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4.10}$$

where u_3 is constrained to be 0.

If legs 2, 3 and 6 are in contact with the ground and legs 1, 4 and 5 are not, then the equations of motion are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{h}_1 \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta & 0 & 0 \\ 0 & \sin\theta & 0 & 0 \\ 0 & -l & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{4.11}$$

where u_4 is constrained to be 0.

If none of the legs are in contact with the ground,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{4.12}$$

If we consider either the distributions associated with the sequence $S_{12} \subset S_1 \subset$

S_0 or $S_{12} \subset S_2 \subset S_0$, the distributions spanned by the vector fields comprising Equations 4.8, 4.9, 4.10 and 4.12, or the distributions spanned by the vector fields comprising Equations 4.8, 4.9, 4.11 and 4.12, respectively, the hypotheses of Proposition 4.4 are satisfied. Note that this example has the somewhat unrealistic requirement of considering the equations of motion when none of the feet are in contact with the ground. In fact, this is *required* for controllability in the entire configuration space since both leg heights are variables.

Since it is undesirable to lift all the feet of the robot out of contact with the ground at once, a better notion of controllability may be to ask that the system reach an open set in the union $S_1 \cup S_2$. Thus, we need to consider the nested sequences $S_{12} \subset S_1$ and $S_{12} \subset S_2$ simultaneously. From Equations 4.8, 4.9 and 4.10, the sum of the associated distributions is six dimensional, as is the sum from Equations 4.8, 4.9 and 4.11. Thus, the system is controllable because it can reach an open neighborhood of a starting point in the bottom strata defined in the relative topology of the union $S_1 \cup S_2$.

Note also, that for this particular model, *all* gaits are controllable because the Lie brackets in Equation 4.9 span the (x, y, θ) directions. Since the robot is kinematic, whenever a foot is not in contact with the ground, the motion of that leg will be decoupled and independent of the other degrees of freedom of the robot. Thus, with the (x, y, θ) directions spanned with all the feet in contact with the ground, control over the leg variables is obtained whenever any of the feet are not in contact with the ground, thus giving controllability for any gait which allows each foot out of contact with the ground at some point during the gait.

4.3.2 The Exterior Differential Systems Approach

When all legs are in contact with the ground, the constraints are

$$\begin{aligned}
\omega^{12,1} &= dx - \cos\theta(d\phi_1 + d\phi_2) = 0 \\
\omega^{12,2} &= dy - \sin\theta(d\phi_1 + d\phi_2) = 0 \\
\omega^{12,3} &= d\theta - ld\phi_1 + ld\phi_2 = 0 \\
\omega^{12,4} &= dh_1 = 0 \\
\omega^{12,5} &= dh_2 = 0.
\end{aligned}$$

Computing the derived system for this system,

$$\begin{aligned}
I_{S_{12}}^{(1)} = \{-\csc\theta d\phi_1 - \csc\theta d\phi_2 + \cot\theta dx &- \cot^2\theta dy + \csc^2\theta dy, \\
& ld\phi_2 - ld\phi_1 + d\theta, dh_1, dh_2\},
\end{aligned}$$

and

$$I_{S_{12}}^{(2)} = \{-ld\phi_1 + ld\phi_2 + d\theta, dh_1, dh_2\},$$

which is clearly the bottom derived system.

When legs 1, 4 and 5 are in contact with the ground, but legs 2, 3 and 6 are not in contact, the constraints are

$$\begin{aligned}
\omega^{1,1} &= dx - \cos\theta d\phi_1 = 0 \\
\omega^{1,2} &= dy - \sin\theta d\phi_1 = 0 \\
\omega^{1,3} &= d\theta - ld\phi_1 = 0 \\
\omega^{1,4} &= dh_1 = 0.
\end{aligned}$$

Computing the derived flag for this system,

$$I_{S_1}^{(1)} = \{dy - d\phi_1 \sin\theta, -\cos\theta d\phi_1 + dx, -ld\phi_1 + d\theta, dh_1\},$$

which is the bottom derived system on S_1.

When legs 2, 3 and 6 are in contact with the ground, but legs 1, 4 and 5 are not in contact, the constraints are

$$\begin{aligned} \omega^{2,1} &= dx - \cos\theta d\phi_2 = 0 \\ \omega^{2,2} &= dy - \sin\theta d\phi_2 = 0 \\ \omega^{2,3} &= d\theta - l d\phi_2 = 0 \\ \omega^{2,4} &= dh_2 = 0. \end{aligned}$$

Computing the derived flag for this system,

$$I_{S_2}^{(1)} = \{dy - d\phi_2 \sin\theta, -\cos\theta d\phi_2 + dx, -l d\phi_2 + d\theta, dh_2\},$$

which is the bottom derived system on S_2.

When none of the legs are in contact with the ground,

$$\begin{aligned} \omega^{0,1} &= dx = 0 \\ \omega^{0,2} &= dy = 0 \\ \omega^{0,3} &= d\theta = 0. \end{aligned}$$

Clearly, the bottom derived system is

$$I_{S_0}^{(1)} = \{dx, dy, d\theta\}.$$

Now, the hypotheses of Proposition 4.5 are satisfied by either the nested sequence $S_{12} \subset S_1 \subset S_0$, or $S_{12} \subset S_2 \subset S_0$. As before, we can also consider controllability relative to the topology of the union $S_1 \cup S_2$. The intersection of the bottom derived systems associated with the nested sequence $S_{12} \subset S_1$ contains only $\{dh_1\}$, so the system can reach an open set in S_1. Similarly, intersection of the bottom derived systems associated with the nested sequence $S_{12} \subset S_2$ contains only $\{dh_2\}$, so the system can reach an open set in S_2. Thus, the robot is controllable in $S_1 \cup S_2$.

4.3.3 The Principal Fiber Bundle Approach

The configuration space splits globally into shape and group variables. The variables (x, y, θ) can locally parameterize the group SE(2), and the remaining variables, $(\phi_1, \phi_2, h_1, h_2)$, describe the shape. From Kelly and Murray (1995), the local connection one form for the hexapod is given by

$$A(x) = \begin{pmatrix} -\alpha(h_1)d\phi_1 - \beta(h_2)d\phi_2 \\ 0 \\ -l\alpha(h_1)d\phi_1 + l\beta(h_2)d\phi_2 \end{pmatrix}$$

and that the curvature is given by

$$DA = \begin{pmatrix} -d\alpha \wedge d\phi_1 - d\beta \wedge d\phi_2 \\ 0 \\ -ld\alpha \wedge d\phi_1 + ld\beta \wedge d\phi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2l\alpha\beta d\phi_1 \wedge d\phi_2 \\ 0 \end{pmatrix}.$$

For the upcoming computations, define the two vector fields $X_1 = \frac{\partial}{\partial \phi_1}$ and $X_2 = \frac{\partial}{\partial \phi_2}$.

When both feet are in contact with the ground,

$$A_{12}(x) = \begin{pmatrix} -d\phi_1 - d\phi_2 \\ 0 \\ -ld\phi_1 + ld\phi_2 \end{pmatrix} \qquad \text{and} \qquad DA_{12} = \begin{pmatrix} 0 \\ 2ld\phi_1 \wedge d\phi_2 \\ 0 \end{pmatrix}.$$

The vectors

$$A_{12} \cdot X_1 = \begin{pmatrix} -1 \\ 0 \\ -l \end{pmatrix}, \quad \text{and} \quad A_{12} \cdot X_2 = \begin{pmatrix} -1 \\ 0 \\ l \end{pmatrix}, \tag{4.13}$$

span $\mathfrak{h}_1^{12}$, and the vector

$$DA_{12} \cdot (X_1, X_2) = \begin{pmatrix} 0 \\ 2l \\ 0 \end{pmatrix}, \tag{4.14}$$

spans $\mathfrak{h}_2^{12}$. Clearly, at this point, the group directions are spanned, and so the robot is weakly controllable from Proposition 4.7 without even considering the higher strata.

For strong controllability, we must consider the higher strata. When leg 1 is in contact with the ground, and leg 2 is not in contact,

$$A_1(x) = \begin{pmatrix} -d\phi_1 \\ 0 \\ -ld\phi_1 \end{pmatrix} \qquad \text{and} \qquad DA_1 = 0.$$

When leg 2 is in contact with the ground, and leg 1 is not in contact,

$$A_2(x) = \begin{pmatrix} -d\phi_2 \\ 0 \\ ld\phi_2 \end{pmatrix} \qquad \text{and} \qquad DA_2 = 0.$$

The vectors

$$A_1 \cdot X_1 = \begin{pmatrix} -1 \\ 0 \\ -l \end{pmatrix} \qquad \text{and} \qquad A_2 \cdot X_2 = \begin{pmatrix} -1 \\ 0 \\ l \end{pmatrix}.$$

Constructing the Lie algebra subspaces in accordance with Proposition 4.7,

$$\begin{aligned}
\mathfrak{h}_1^{12} &= \mathrm{span} A_{12}(x) \\
\mathfrak{h}_2^{12} &= \mathrm{span} DA_{12}(x) \\
\mathfrak{h}_1^{1} &= \mathrm{span} A_{1}(x) \\
\mathfrak{h}_1^{2} &= \mathrm{span} A_{2}(x) \\
\mathfrak{h}_1^{0} &= 0.
\end{aligned}$$

As before, considering $S_{12} \subset S_1 \subset S_0$, we have $\mathfrak{g} = \mathfrak{h}_1^{12} \oplus \mathfrak{h}_2^{12}$ and excluding $\mathfrak{h}_1^0$ from the sum gives strong controllability. The same is true for $S_{12} \subset S_1 \subset S_0$. Considering $S_{12} \subset S_1$ and $S_{12} \subset S_1$ simultaneously, $\mathfrak{g}$ is spanned by the connection and its curvature on S_{12}, and *in combination* S_1 and S_2 allow for complete shape controllability (each allows control over a different leg height).

4.4 General Stratified Systems

This Section extends the previous results to overcome the limitation in Propositions 4.4 and 4.5 which considered only the geometry of a nested sequence of submanifolds, thus possibly excluding the effect of multiple submanifolds with the same codimension.

First, we must consider the structure of a general stratified system in more detail. Recall the definitions of the bottom and higher strata from Section 2.3. At a point, x, the lowest dimension stratum containing the point x is the bottom stratum, and any other submanifolds containing x are higher strata. Denote an arbitrary stratum by $S_I = S_{i_1 i_2 \cdots i_n}$, $I = \{i_1 i_2 \cdots i_n\}$, and note that its codimension is n, the length of the multi–index subscript.

Assume that at point, x_0, the stratum $S_B = S_{i_1 i_2 \cdots i_n}$ is the bottom stratum. We will refer to the *level* of the stratum as its codimension. Thus, the bottom stratum is on the n^{th} level, the $(n-1)^{th}$ level contains all the strata with codimension $(n-1)$, and so forth. It is easy to verify that if every stratum is accessible, then k^{th} level

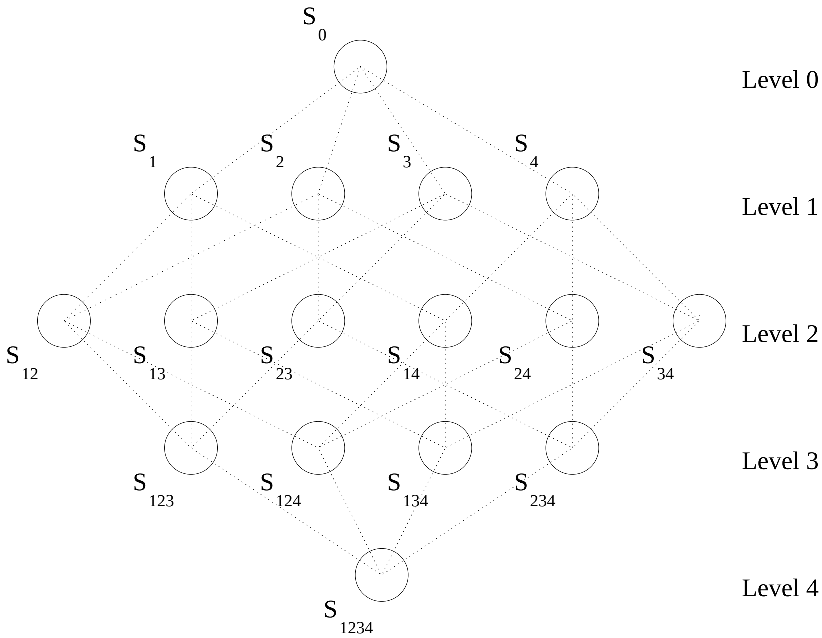


Figure 4.5. Four level stratification.

contains

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{(n-k)!k!}$$

strata. Figure 4.5 illustrates the combinatorial structure of a stratification with four levels. In Figure 4.5, the nodes of the graph correspond to the different strata. The edges connecting the nodes indicate whether it is possible for the system to move from one stratum to another, *i.e.*, if the nodes are connected by an edge, then the system can move between the strata, if there is no edge, then the system cannot move between the strata. Note that while the figure simply illustrates edges between nodes only one level apart, it may be the case that multi–level jumps are possible, in which case there would be an edge connecting strata of two levels that are more than one level apart.

If there are n codimension one strata, then the total number of strata is

$$\sum_{k=1}^{n} \begin{pmatrix} n \\ k \end{pmatrix} = 2^n - 1,$$

which clearly increases quickly with n. The corresponding graph structure also grows similarly in complexity. Even with this simplistic pictorial view, it is evident that the a general stratified configuration space is characterized by an interesting algebraic structure. Specifically, as illustrated by the dotted lines connecting the strata, there is an naturally defined graph structure in which to consider the problem. Note that one way to consider a gait is simply a choice of a cyclic path through this graph structure. Specify a gait as an ordered sequence of strata,

$$\mathcal{G} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}, S_{I_{n+1}} = S_{I_1}\}. \tag{4.15}$$

In this ordered sequence, the first and last element are identical, indicating that the gait is a closed loop. Clearly, in order for the gait to be meaningful, it must be possible for the system to switch from stratum S_{I_i} to $S_{I_{i+1}}$ for each i. In Figure 4.5, this corresponds to each stratum S_{I_i} in the sequence being connected to $S_{I_{i+1}}$ and S_{I_n} being connected to S_{I_1}. Limitations on gaits, such as stability requirements, could be expressed as limitations (possibly as a function of configuration) on the cyclic gait paths.

Here assume that we know the physical constraints on the system and the manner by which these constraints are manifested as constraints in its graph representation. In other words, assume that there is a collection of strata (or nodes), $\mathcal{S} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}\}$ which are deemed "permissible," and similarly a collection of "permissible" edges connecting the nodes, denoted by

$$\mathcal{C} = \{(S_{I_1}, S_{J_1}), (S_{I_2}, S_{J_2}), \ldots, (S_{I_n}, S_{J_n})\}.$$

Which strata and edges are permissible may, of course, be a function of the configuration of the system.

Whether a stratum is permissible depends upon whether the equations of motion for the system can be expressed as a kinematic system (recall Equation 3.1), in a neighborhood of the point of interest. For example, for a biped robot, clearly if it lifts both feet off of the ground, it is not a kinematic system because the fact that gravity will make it fall back to the ground.

Whether or not edges between nodes are permissible is a more complicated issue. We have assumed that the system can always move off of a stratum into a higher stratum (recall Equation 2.19). Clearly, a system can always return to a stratum from whence it came. The more difficult problem is whether the system can leave a stratum into a higher stratum, and then move to a *different* substratum of the higher stratum. The answer to this question is different depending upon whether the strata are defined by boundaries or simple submanifolds.

Consider the situation illustrated in Figure 4.6. Assume that starting from S_{12} the reachable sets in S_1 and S_2 are open in their respective topologies. Further assume that the foliation associated with the control system on S_0 are one–dimensional lines as shown. For this system, it is not possible to leave either S_1 or S_2 and move to the other stratum if S_1 and S_2 are boundaries. If S_1 and S_2 were not boundaries, then it would be possible to flow "through" S_1 to S_2 and vice–versa.

If the strata are defined by simple submanifolds, then locally, the reachable set on the higher stratum, S_H must intersect each of the substrata. If the substrata are codimension one submanifolds of the higher stratum, this will always be true since we have assumed that there is a vector field that moves the system off of any stratum. Because it is a codimension one submanifold, the stratum and reachable set will then intersect transversely. Recall that two submanifolds intersect *transversally* if

$$T_x S_1 + T_x S_2 = T_x M, \tag{4.16}$$

where S_1 and S_2 are submanifolds of M. If S_1 and S_2 are transversal, then following theorem (Corollary 3.5.13 of Abraham, Marsden, and Ratiu (1988)) is useful.

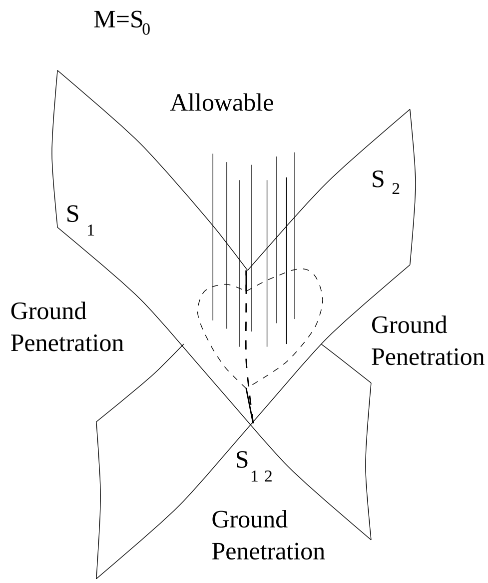


Figure 4.6. Stratification that prohibits a gait.

Lemma 4.8 *If S_1 and S_2 are transversal and have finite codimension in M, then*

$$\operatorname{codim}(S_1 \cap S_2) = \operatorname{codim}(S_1) + \operatorname{codim}(S_2).$$

If the strata are actually boundaries of the manifold, then the reachable set in the higher stratum, S_H, must intersect the substrata with a particular orientation. We refer the reader to Abraham, Marsden, and Ratiu (1988) for a complete exposition on orientations of manifolds and adopt a rather simplistic approach here. Note that for a codimension one stratum, S_i, with corresponding level set function Φ_i, the exterior derivative of Φ_i, $d\Phi$, in a local sense defines an "orientation" in the following manner. If $d\Phi \neq 0$ in a neighborhood of a point, x_0, then, the set of tangent vectors, $v \in T_{x_0}$ oriented "into" S_i are those that satisfy $\langle v, d\Phi_i \rangle < 0$. Thus, if there exists a vector field, g defined on S_0 such that $\langle g, d\Phi_1 \rangle < 0$ and $\langle g, d\Phi_2 \rangle > 0$, the system will be able to locally "hop" from S_1 to S_2. Intuitively, the way to interpret this is that the vector points into one stratum and out of the other.

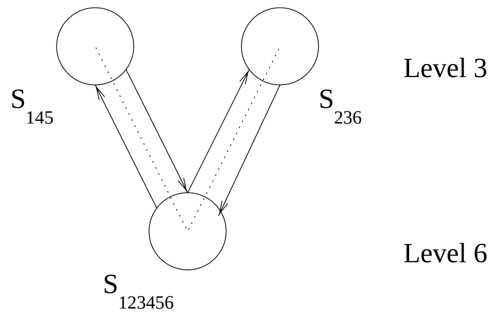


Figure 4.7. The simplified hexapod graph.

Note that for applications such as robotic systems, whether or not it is possible to move from a higher stratum onto a lower stratum naturally will be obvious from the kinematics of the problem.

Example 4.9: (Hexapod — revisited) Recall that the hexapod example in Section 4.3, assumed that the hexapod walked with a tripod gait. That assumption reduced the high dimensional and complex graph structure of the system to a very low dimensional and simple one, as illustrated in Figure 4.7. The arrows in the figure represent the tripod gait.

Note that, for this problem, it will always be possible for the system to move from a higher stratum onto the bottom stratum. This is manifested in the fact that the robot can always put its feet on the ground regardless of its configuration. □

4.4.1 Gait Controllability

This section considers the problem of whether a particular gait is controllable. Recall that a gait is defined as a cyclic path through the graph structure discussed in Section 4.4 and illustrated in Figure 4.5. In this section, we will limit our attention to a particular form of controllability; namely, *gait controllability*. Assume that if $S_{I_{i+1}} \subset S_{I_i}$, then the reachable set is transversal to the substratum, $S_{I_{i+1}}$. As noted previously, this is natural if dim $\left(S_{I_{i+1}}\right) = \dim\ (S_{I_i}) - 1$. Switches between strata with dimensions which vary by more than one are allowable as long as this transversality assumption is satisfied.

In the complete stratified structure, there is one bottom stratum, defined by the intersection of all the codimension one strata in the configuration space. In Figure 4.5, this corresponds to stratum S_{1234}. For a locomotion system, such as a legged robot, this bottom stratum corresponds to the set of points in the configuration space where all the feet are in contact with the ground. Now, for gait controllability, the reachable set, $R^V(x_0, \leq T)$, is defined as before, but is restricted to control inputs consistent with the gait, *i.e.*, the reachable set must be constructed consistent with the ordering of the strata that define the gait.

Definition 4.10: (Gait controllability) A gait, $\mathcal{G} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}, S_{I_1}\}$ is *gait controllable* from the point x_0 if the reachable set $R^V(x_0, \leq T)$ (defined in Equation 3.2 and consistent with the gait) contains a neighborhood of x_0 for all neighborhoods V of x_0 and $T > 0$, where the neighborhood is defined by the topology of the lowest stratum, S_{I_1}. □

Example 4.11: (Kinematic leg — revisited) In the simple kinematic leg example, Example 4.1, illustrated in Figure 4.1, the bottom stratum is the set of points $q = (x, l, \theta)$ such that

$$l \cos \theta = h, \tag{4.17}$$

for some fixed height, h. This is most naturally parameterized by the variables x and θ, and so an open set in S corresponds to reaching an open neighborhood of x and θ, where l is subject to the constraint expressed by Equation 4.17. □

Let $\overline{\Delta}_I$ denote the involutive closure of the control distribution on S_I, where the subscripted index for $\overline{\Delta}_I$ corresponds to the subscripted index for the stratum S_I to which it is associated. Now construct the *gait distribution.* Given a gait, $\mathcal{G}$, the gait distribution is constructed as follows. First, let $\mathcal{D}_1 = \overline{\Delta}_{S_{I_1}}$. If $S_{I_1} \subset S_{I_2}$, then let $\mathcal{D}_2 = \mathcal{D}_1 + \overline{\Delta}_{S_2}$ (implicitly assuming the appropriate inclusion of $\mathcal{D}_1$ into S_2); else, if $S_{I_2} \subset S_{I_1}$, then let $\mathcal{D}_2 = (\mathcal{D}_1 \cap S_2) + \overline{\Delta}_{S_2}$. In general, then, $\mathcal{D}_i = \mathcal{D}_{i-1} + \overline{\Delta}_{S_i}$ if $S_{I_{i-1}} \subset S_{I_i}$, and $\mathcal{D}_i = (\mathcal{D}_{i-1} \cap S_i) + \overline{\Delta}_{S_i}$ if $S_{I_i} \subset S_{I_{i-1}}$.

In fact, the term "gait distribution" is a slight misnomer because the sum of the distributions in the construction of the gait distribution only make sense for points in the bottom stratum. Therefore, the gait distribution is not actually a section of the tangent bundle of the full configuration space. However, for our purposes, evaluating the sums at the point x_0, so it is just a vector space, will suffice.

Proposition 4.12 *If*

$$dim\,(\mathcal{D}_n) = dim\; T_{x_0} S_{I_1},$$

then the system is gait controllable from x_0.

Proof: Our proof relies on one corollary to Proposition 4.4 and one lemma.

Corollary 4.13 *In the construction of the gait distribution, if $S_{I_i} \subset S_{I_{i+1}}$, then the dimension of the reachable set increases by the same amount as the increase in dimension between $\mathcal{D}_{I_i}$ and $\mathcal{D}_{I_{i+1}}$ and contains the point x_0.*

Lemma 4.14 *In the construction of the gait distribution, if $S_{I_{i+1}} \subset S_{I_i}$, then the dimension of the reachable set increases by the same amount as the increase in dimension between $\mathcal{D}_{I_i}$ and $\mathcal{D}_{I_{i+1}}$ minus the difference between the dimensions of $S_{I_{i+1}}$ and S_{I_i}.*

Proof: This follows from the transversality assumption and the codimension result of Theorem 4.8. ▼

It follows that in the construction of the gait distribution that the dimension of the reachable set will be the dimension of $\mathcal{D}_n$. If the first and last strata in the gait $\mathcal{G}$ is the bottom stratum, then the result follows since the reachable set it contained in S_{I_1} and has dimension equal to the dimension of S_{I_1}. ■

4.4.2 Gait Controllability of the Hexapod Robot Example

This section returns to the hexapod robot example considered in Section 4.3. Here, however, we consider gait controllability, as opposed to regular controllability.

The first step is to construct the gait distribution. Take as the gait the following sequence of strata:

$$\mathcal{G} = \{S_{123456}, S_{145}, S_{123456}, S_{236}, S_{123456}\},$$

as illustrated in Figure 4.7. To simplify notation, let $S_{12} = S_{123456}$, $S_1 = S_{145}$ and $S_2 = S_{236}$. The equations of motion for the system restricted to the bottom stratum, S_{12} are given in Equation 4.8. Also, a Lie bracket is necessary to construct $\overline{\Delta}_{12}$, as given in Equation 2.2. By inspection, $\overline{\Delta}_{12} = \mathcal{D}_1$ has a dimension of three.

Next extend the construction to S_1. Since $S_{12} \subset S_1$, $\mathcal{D}_2 = \mathcal{D}_1 + \overline{\Delta}_1$, where $\overline{\Delta}_1$ is determined from Equation 4.10. By inspection, then, $\dim(\mathcal{D}_2) = 5$.

The construction next returns to the bottom stratum, S_{12}. Note that S_{12} is a codimension 1 submanifold of S_1. Also, since $\mathcal{D}_2$ contains the basis vector $\frac{\partial}{\partial h_1}$, it is clear that the transversality assumption in Equation 4.16 is satisfied. Therefore, $\dim(\mathcal{D}_3) = \dim(\mathcal{D}_2) - 1 = 4$.

The construction is next extended to stratum S_2. As with S_1, S_2 increases the dimension of $\mathcal{D}_4$ by two, so that $\dim(\mathcal{D}_4) = 6$. "Projecting" this back down to S_{12} as before gives the dimension of the reachable set to be 5, which is the dimension of S_{12}. Therefore, the hexapod example is gait controllable.

4.4.3 Gaits and Connections

Now consider the case when the system has the properties described in Section 4.2.3, namely, each stratum in the gait is a principal fiber bundle with the same structure group. As before, assume that each stratum is a globally trivial principal fiber bundle, which can be written as $S_i = B_i \times G_i$. A consequence of this assumption is that for the entire configuration manifold, $S_0 = M = B_0 \times G$, the stratification is of the base space, B_0 and *not* G. Since gait controllability was defined in terms of controllability in the bottom stratum only, we will consider only weak or fiber controllability. (The notion of strong controllability requires us to consider the entire configuration space).

Given a gait, $\mathcal{G} = \{S_{I_1}, S_{I_2}, \ldots, S_{I_n}, S_{I_1}\}$, associated with each stratum, S_I is a connection, Γ_I, and a corresponding sequence of Lie algebra subspaces, $\mathfrak{h}_i^I$ constructed in accordance with Proposition 3.8. Observe that since only the base space is stratified, restricting the dimension of the reachable set as in Lemma 4.14 only restricts the reachable set in the shape or base space, *i.e.*, the reachable set in the group or fiber space is unaffected. Thus, we have the following Proposition.

Proposition 4.15 *If*

$$\sum_I \sum_i \mathfrak{h}_i^I = \mathfrak{g},$$

then the system is weakly controllable.

Note that one nice aspect of this proposition is that there is no need to keep a dimension count as in Proposition 4.12.

Example 4.16: (Hexapod Robot — Revisited) The Lie algebra subspaces $\mathfrak{h}_1^{12}$ and $\mathfrak{h}_2^{12}$ span $\mathfrak{g}$, so any gait which contains the bottom stratum will be weakly controllable. □

4.5 Conclusions

This chapter presented controllability tests for stratified systems. In particular, three controllability tests from nonlinear control which are applicable to smooth systems were extended to encompass the stratified case. The first test was based on calculations involving distributions, the second test used tools from exterior differential systems and the third test used results specific to systems on principal fiber bundles. An additional contribution of this chapter was the controllability test for gaits, which provides sufficient conditions for a specified gait to be controllable. Note that the result for gait controllability is not necessarily limited to legged locomotion problems. For example, as long as the kinematic assumption is valid on each strata and the grasp has the force–closure property (see Murray, Li, and Sastry

(1994) for a definition and discussion of force–closure), it applies to multi–fingered robotic grasping problems as well. The main limitation of these results is that they are limited to kinematic system. In the legged locomotion context, this limits their applicability to quasi–static locomotion.

Chapter 5

Stratified Motion Planning

In contrast to controllability, which is an analysis tool, the motion planning problem is constructive, such that the ultimate goal is to determine control inputs so that the system will behave in a particular manner. Controllability is not completely unrelated, however. It will be clear subsequently that a form of gait controllability is a necessary condition for the motion planning method presented in this chapter. This chapter extends the procedure outlined in Section 3.4, which only works for smooth systems, to kinematic legged systems with a stratified configuration space.

5.1 Legged Trajectory Generation

The main difficulty with stratified systems is that the various sets of equations of motion are defined on different spaces. Since we are ultimately required to consider the vector fields associated with each stratum in one common space, vector fields on different strata must have a particular relationship. In this case, that common space will be the bottom stratum and the particular relationship will involve Lie bracket. A few examples will help motivate this.

Example 5.1: Consider the simple biped configuration space as shown in Figure 2.2. Assume that on stratum S_{12}, the vector field $g_{1,1}$ moves the system off of S_{12} and onto S_1, and correspondingly, $g_{2,1}$ moves the system off of S_{12} onto S_2. Also, consider the vector fields $g_{1,2}$ and $g_{2,2}$, defined on S_1 and S_2 respectively. Consider

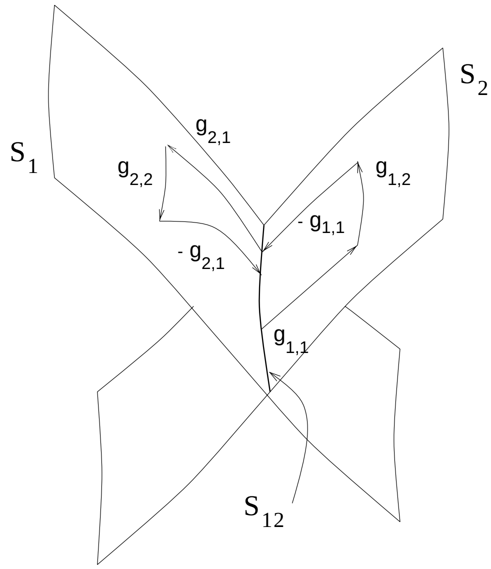


Figure 5.1. Sequence of flows.

the following sequence of flows, starting from the point $x_0 \in S_{12}$

$$x_f = \underbrace{\phi^{t_6}_{-g_{2,1}}}_{S_{12}\leftarrow S_2} \circ \underbrace{\phi^{t_5}_{g_{2,2}}}_{\text{on } S_2} \circ \underbrace{\phi^{t_4}_{g_{2,1}}}_{S_2\leftarrow S_{12}} \circ \underbrace{\phi^{t_3}_{-g_{1,1}}}_{S_{12}\leftarrow S_1} \circ \underbrace{\phi^{t_2}_{g_{1,2}}}_{\text{on } S_1} \circ \underbrace{\phi^{t_1}_{g_{1,1}}}_{S_1\leftarrow S_{12}} (x_0). \tag{5.1}$$

The notation under each flow indicates what the flow is doing, *e.g.*, "$S_{12} \leftarrow S_1$" means that the flow takes the system from S_1 to S_{12} and "on S_1" means that the flow was entirely on S_1. This sequence of flows is illustrated in Figure 5.1. In this sequence of flows, the system first moved off of the bottom stratum into S_1, flowed along the vector field $g_{1,2}$, flowed back onto the bottom stratum, off of the bottom stratum onto S_2, along vector field $g_{2,2}$ and back to the bottom stratum.

It is clear from the Campbell–Baker–Hausdorff formula (Equation 2.3) that if the Lie bracket between two vector fields is zero, then their flows commute. Thus, if

$$[g_{1,1}, g_{1,2}] = 0 \quad \text{and} \quad [g_{2,1}, g_{2,2}] = 0, \tag{5.2}$$

we can reorder the above sequence of flows, by interchanging the flow along $g_{1,1}$ and $g_{1,2}$ and the flows along $g_{2,1}$ and $g_{2,2}$ as follows

$$x_f = \underbrace{\phi^{t_5}_{g_{2,2}} \circ \phi^{t_6}_{-g_{2,1}}}_{\text{interchanged}} \circ \phi^{t_4}_{g_{2,1}} \circ \underbrace{\phi^{t_2}_{g_{1,2}} \circ \phi^{t_3}_{-g_{1,1}}}_{\text{interchanged}} \circ \phi^{t_1}_{g_{1,1}}(x_0). \tag{5.3}$$

If $t_1 = t_3$ and $t_4 = t_6$, this reduces to

$$x_f = \underbrace{\phi^{t_4}_{g_{2,2}} \circ \phi^{t_2}_{g_{1,2}}}_{\text{on } S_{12}}(x_0). \tag{5.4}$$

Note that that $g_{1,2}$ and $g_{2,2}$ are vector fields in the equations of motion for the system on S_1 and S_2, respectively, but *not* on S_{12}. However, the sequence of flows in Equation 5.1, where each flow occurs on a stratum where the associated vector field is in the equations of motion results in the same flow as in Equation 5.3, where the vector fields are evaluated on the bottom stratum, even though they are not part of the equations of motion there. Furthermore, note that if the vector fields $g_{1,2}$ and $g_{2,2}$ are tangent to the substratum S_{12}, then the resulting flow given in 5.3 will remain in S_{12}. In fact, it is implicitly required in the above argument that at least $g_{1,2}$ is tangent to S_{12}.

As a concrete example, consider a biped robot. The above sequence of flows corresponds to lifting one foot out of contact with the ground, moving it parallel to the ground, replacing the foot back in contact with the ground and then doing exactly the same motion with the other foot. Equation 5.4 represents the fact that the final net motion is exactly equivalent to sliding each foot along the ground without ever lifting each foot out of contact. In fact, this sliding motion is not allowed by the equations of motion, but it does give the same net flow. The significance of this fact is that, for the purposes of motion planning, one can consider such sliding motions as part of the equations of motion on the bottom stratum, thus increasing the number of vector fields available for the motion planning method outlined in Section 3.4.

If the bottom stratum is described by the level set of a function, Φ_B, and if a vector field, $g_{1,2}$ is not tangent to the bottom stratum, then, $\langle d\Phi_B, g_{1,2}\rangle = f_1(x) \neq 0$. Also, since the vector field $g_{1,1}$ moves the foot out of contact, we similarly have $\langle d\Phi_B, g_{1,1}\rangle = f_2(x) \neq 0$. Then, the vector field, $\tilde{g}_{1,2} = g_{1,2} - \frac{f_1(x)}{f_2(x)} g_{1,1}$, is tangent to S_B because

$$\langle d\Phi_B, \tilde{g}_{1,2}\rangle = \langle d\Phi_B, g_{1,2}\rangle - \frac{f_1(x)}{f_2(x)}\langle d\Phi_B, g_{1,1}\rangle = 0. \tag{5.5}$$

□

Henceforth, we will just assume that the vector field on the higher stratum is tangent to the lower stratum, and note that if it is not tangent, we can modify it to be so in the above manner.

Example 5.2: Now consider a slightly more complicated sequence of flows using the same stratification and vector fields as in Example 5.1. Consider

$$\begin{aligned} x_f = {} & \phi^{t_{12}}_{-g_{2,1}} \circ \phi^{t_{11}}_{-g_{2,2}} \circ \phi^{t_{10}}_{g_{2,1}} \circ \phi^{t_9}_{-g_{1,1}} \circ \phi^{t_8}_{-g_{1,2}} \circ \phi^{t_7}_{g_{1,1}} \\ & \circ \phi^{t_6}_{-g_{2,1}} \circ \phi^{t_5}_{g_{2,2}} \circ \phi^{t_4}_{g_{2,1}} \circ \phi^{t_3}_{-g_{1,1}} \circ \phi^{t_2}_{g_{1,2}} \circ \phi^{t_1}_{g_{1,1}}(x_0). \end{aligned}$$

The first six flows in this example are the same as in Example 5.1. However, following the first six flows are six more flows wherein the flows that are entirely on S_1, *i.e.*, the flow along $g_{1,2}$, and entirely on S_2, *i.e.*, the flow along $g_{2,2}$, are in the negative direction. If the Lie brackets are zero as in in Equation 5.2, and $t_i = t_{i+2}$, $i = 1, 4, 7, 10$ the flows on and off of S_{12} can be rearranged as

$$x_f = \phi^{t_{11}}_{-g_{2,2}} \circ \phi^{t_8}_{-g_{1,2}} \circ \phi^{t_5}_{g_{2,2}} \circ \phi^{t_2}_{g_{1,2}}(x_0).$$

Now, if $t_2 = t_5 = t_8 = t_{11}$,

$$x_f = \phi^{t_{11}}_{-g_{2,2}} \circ \phi^{t_8}_{-g_{1,2}} \circ \phi^{t_5}_{g_{2,2}} \circ \phi^{t_2}_{g_{1,2}}(x_0) = \phi^{t^2}_{[g_{1,2}, g_{2,2}]} + \mathcal{O}(t^3)(x_0),$$

where $t = t_2 = 5_5 = t_8 = t_{11} \ll 1$, thus, providing a net flow in S_{12} in the direction of the Lie bracket between vector fields which are in the equations of motion for the

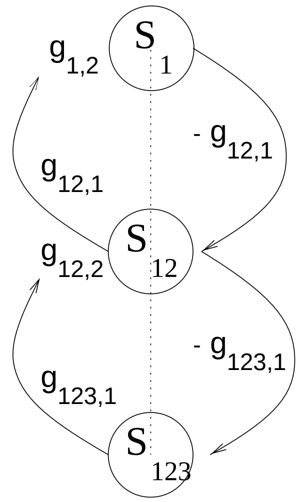


Figure 5.2. Sequence of flows for example 5.3.

system on different strata, S_1 and S_2. □

Example 5.3: Now, consider a third example. Consider the strata, $S_{123} \subset S_{12} \subset S_1$, and let $g_{123,1}$ move the system from S_{123} into S_{12}. Let $g_{12,1}$ move the system from S_{12} into S_1 and let $g_{12,2}$ be a vector field defined on S_{12}. Finally, let $g_{1,2}$ be a vector field defined on S_1. Consider the following sequence of flows

$$x_f = \underbrace{\phi^{t_6}_{-g_{123,1}}}_{S_{123} \leftarrow S_{12}} \circ \underbrace{\phi^{t_5}_{-g_{12,1}}}_{S_{12} \leftarrow S_1} \circ \underbrace{\phi^{t_4}_{g_{1,2}}}_{\text{on} S_1} \circ \underbrace{\phi^{t_3}_{g_{12,1}}}_{S_1 \leftarrow S_{12}} \circ \underbrace{\phi^{t_2}_{g_{12,2}}}_{\text{on} S_{12}} \circ \underbrace{\phi^{t_1}_{g_{123,1}}}_{S_{12} \leftarrow S_{123}} (x_0).$$

These flows are illustrated in the graph representation of the stratification in Figure 5.2. Now, if $[g_{12,1}, g_{1,2}] = 0$ and $t_3 = t_5$, this reduces to

$$x_f = \phi^{t_6}_{-g_{123,1}} \circ \phi^{t_4}_{g_{1,2}} \circ \phi^{t_2}_{g_{12,2}} \circ \phi^{t_1}_{g_{123,1}}(x_0), \tag{5.6}$$

and if $[g_{123,1}, g_{1,2}] = 0$ and $[g_{123,1}, g_{12,2}] = 0$, (which implicitly requires that $g_{1,2}$ be tangent to S_{12}) then Equation 5.6 can be reduced to

$$x_f = \phi^{t_4}_{g_{1,2}} \circ \phi^{t_2}_{g_{12,2}}(x_0),$$

as long as $g_{1,2}$ and $g_{12,2}$ are tangent to S_{123}.

This example illustrates that multi–level "jumps" between higher and lower strata are possible if all the vector fields that move the system off of a substratum are decoupled from all vector fields defined on that and higher level strata in the sense that their Lie bracket is zero. Again, if the tangency requirements are not met, the vector fields can be "parallelized" as in Equation 5.5 of Example 5.1. $\square$

Example 5.4: As a final example, consider the following sequence of flows, which are illustrated in Figure 5.3

$$x_f = \underbrace{\phi^{t_7}_{g_{13,1}}}_{S_{123} \leftarrow S_{13}} \circ \underbrace{\phi^{t_6}_{g_{1a3,2}}}_{\text{on } S_{13}} \circ \underbrace{\phi^{t_5}_{g_{1,3}}}_{S_{13} \leftarrow S_1} \circ \underbrace{\phi^{t_4}_{g_{1,2}}}_{\text{on } S_1} \circ \underbrace{\phi^{t_3}_{g_{1,1}}}_{S_1 \leftarrow S_{12}} \circ \underbrace{\phi^{t_2}_{g_{12,2}}}_{\text{on } S_{12}} \circ \underbrace{\phi^{t_1}_{g_{12,1}}}_{S_{12} \leftarrow S_{123}} (x_0). \tag{5.7}$$

As illustrated in Figure 5.3, this can represent a gait of a three legged robot comprising the following steps:

1. lift foot 3 out of contact with the ground, $(S_{12} \leftarrow S_{123})$;
2. lift foot 2 out of contact with the ground, $(S_1 \leftarrow S_{12})$;
3. replace foot 3 on the ground, $(S_{13} \leftarrow S_1)$; and,
4. replace foot 2 on the ground, $(S_{123} \leftarrow S_{13})$.

Interestingly, note that lifting a foot and placing it down do not necessarily occur on the same stratum. For example, foot 3 is lifted off of the ground by a vector field defined in stratum S_{12}, and replaced in contact by flowing along a vector field defined in S_1. Obviously, our desire is to get these two flows to cancel each other as illustrated in the previous examples, and the only way for this to be possible is for them to be defined on the same manifold. The obvious choice for this manifold is the lower stratum since it is more natural to consider a restriction of a vector field to a submanifold than to construct an (artificial) extension of a vector field into a higher stratum.

Returning to the example illustrated in Figure 5.3, if we have

$$[g_{1,3}, g_{1,2}] = 0, \quad [g_{1,2}, g_{1,1}] = 0, \quad \text{and} \quad [g_{1,1}, g_{1,3}] = 0,$$

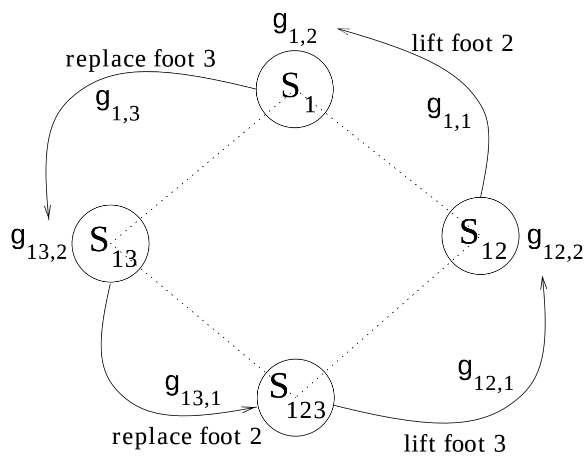


Figure 5.3. Loop gait.

then their corresponding flows (on S_1) commute. Thus, the flow in Equation 5.7 becomes

$$x_f = \phi^{t_7}_{g_{13,1}} \circ \phi^{t_6}_{g_{13,2}} \circ \underbrace{\phi^{t_3}_{g_{1,1}} \circ \phi^{t_4}_{g_{1,2}} \circ \phi^{t_5}_{g_{1,3}}}_{\text{switched}} \circ \phi^{t_2}_{g_{12,2}} \circ \phi^{t_1}_{g_{12,1}}(x_0).$$

The goal is to have the flow along $g_{1,3}$ to commute with the flow along $g_{12,2}$ so that we can switch the flows to cancel the flow along $g_{12,1}$, (lifting foot 3), and the flow along $g_{1,3}$, (replacing foot 3). Unfortunately, $g_{1,3}$ is defined on S_1 and $g_{12,1}$ is defined on S_{12}. However, since $S_{12} \subset S_1$, if $g_{1,3}$ is tangent to S_{12}, then the flow of $g_{1,3}$ restricted to S_{12} will be the same as the flow of $g_{1,3}$ not restricted to S_{12}.

Therefore, if $g_{1,3}$ is tangent to S_{12}, and using the same argument for foot 2, and if $[g_{12,1}, g_{1,3}|_{S_{12}}] = 0$ and $[g_{13,1}, g_{1,1}|_{S_{13}}] = 0$, then

$$\begin{aligned} x_f &= \phi^{t_7}_{g_{13,1}} \circ \phi^{t_3}_{g_{1,1}}|_{S_{13}} \circ \phi^{t_6}_{g_{13,2}} \circ \phi^{t_4}_{g_{1,2}} \circ \phi^{t_2}_{g_{12,2}} \circ \phi^{t_5}_{g_{1,3}}|_{S_{12}} \circ \phi^{t_1}_{g_{12,1}}(x_0) \\ &= \phi^{t_6}_{g_{13,2}} \circ \phi^{t_4}_{g_{1,2}} \circ \phi^{t_2}_{g_{12,2}}(x_0), \end{aligned}$$

which is a flow that can be treated as entirely flowing on S_{123}. $\square$

Examples 5.1, 5.2, 5.3 and 5.4 required that either certain Lie brackets be zero, and Example 5.4, required that foot lifting or replacing vector fields be tangent to other strata. Although the most general approach is to simply check that the necessary conditions are met in a given situation, to simplify the presentation, make the following assumption regarding the kinematics of the problems under consideration.

If it is necessary for the robot to lift a foot off of the ground in a gait cycle, we will assume that the robot can directly control, (either via a single control input, or a combination of control inputs), the height of that foot relative to the ground. Furthermore, for each stratum comprising the gait under consideration, assume that the equations of motion for the system are independent of the foot height (except for whether or not the foot is in contact with the ground). In other words, the motion of the robot is independent of whether a particular foot is very close to the ground, or very far from the ground, but may be dependent upon whether or not a foot is in contact or out of contact with the ground.

Assumption 5.5: Assume that there exists a control input, or a linear combination of control inputs, such that their sum is $\left\{\frac{\partial}{\partial h_i}\right\}$, where h_i is the height of the foot in interest. Furthermore, assume that the equations of motion are otherwise independent of the foot height. □

If this assumption is satisfied, then the Lie bracket of the vector field controlling the height of the foot and any other vector field is zero, so that flows corresponding to lifting a foot and flows corresponding to motions with the foot out of contact with the ground will commute. Additionally, since the same vector field raises and lowers the foot, the tangency requirements for cancelling raising and lowering the foot will automatically be satisfied.

This is arguably a strict assumption; however, for kinematic, legged robots this assumption will often naturally be satisfied. This is because the vector fields in the equations of motion for the system typically will be independent of the leg height. The example we present in Section 5.3 is such a system. Note that one appropriate characterization of the work is that it is a "modification" or "extension" of the results in Lafferriere and Sussmann (1993), and thus carries with it the fundamental limitation that the system considered be kinematic. Until similar results exist for dynamic systems, which hopefully can be similarly extended, we will defer considering such a case. Such an extension to dynamic systems would be difficult because, even if the kinematic restriction were overcome, for dynamic legged systems, the Assumption 5.5 will *not* be naturally satisfied because changing a leg height will affect the inertial properties of the body.

We have assumed that the foot lifting and replacing vector fields are of the form $g_i = \frac{\partial}{\partial h_i}$. Additionally, note that it is impossible for this flow to occur on the stratum S_i since, by definition, S_i corresponds to a constant value of the foot height, h_i. Specifically, in this example, $g_{1,3} = \frac{\partial}{\partial h_3}$, and S_{12} corresponds to all configurations where $h_1 = h_2 = 0$. It will be generically true, then, that $g_i = \frac{\partial}{\partial h_i}$ will be tangent to all strata S_I, where the multi–index I does not contain the index i. Thus, if Assumption 5.5 is satisfied, then the tangency requirement illustrated in Example 5.4 will naturally be satisfied.

In general, then, these Examples 5.1, 5.2, 5.3 and 5.4 show that given a stratified system we can consider the vector fields on any stratum (other than vector fields corresponding to lifting or replacing feet) as part of the equations of motion in the bottom stratum if either certain Lie bracket and tangency conditions are met, or if the more restrictive Assumption 5.5 is satisfied. If the vector fields are not tangent to the bottom stratum, then they must be modified as in Example 5.1.

Now, consider the motion planning planning problem. The above examples show that it is possible to consider vector fields in higher strata as part of the equations of motion for the system on the bottom stratum. At this point, then, we have essentially increased the class of vector fields that we may use in using the motion planning algorithm presented in Section 3.3. Thus, the method presented in Section 3.3 could be used with the modification that whenever the system must flow along a vector field in a higher stratum, it switches to that stratum by lifting the appropriate foot or feet, flows along the vector field, and then replaces the appropriate foot or feet, as in Example 5.1.

In particular, construct the extended system on the bottom stratum using the extended class of vector fields which contain vector fields from higher strata. This effectively takes multiple sets of equations of motion for the system on different strata and combines them into one system on the bottom stratum called the *bottom stratified system*. If Assumption 5.5 is satisfied, then this system will have *all* the control vector fields from every stratum, except the vector fields which correspond to raising and lowering the feet. If Assumption 5.5 is not satisfied, then it will contain all the vector fields which commute with the vector fields that correspond to raising and lowering a foot and that are tangent to the bottom stratum. Then, the bottom *stratified extended system* is constructed by computing Lie brackets among all the vector fields in the bottom stratified system. As in Section 3.4, the brackets must belong to a Philip Hall basis. By including vector fields from higher strata, as well as their brackets, the number of vector fields available for motion planning increases with the number of strata. The construction involving the equations of motion are illustrated as follows.

Multiple Stratified Equations — Bottom Stratified System

$$\left.\begin{array}{lcl} S_0 & : & \dot{x} = g_{0,1}u^{0,1} + \cdots + g_{0,n_0}u^{0,n_0} \\ S_1 & : & \dot{x} = g_{1,1}u^{1,1} + \cdots + g_{1,n_1}u^{1,n_1} \\ S_2 & : & \dot{x} = g_{2,1}u^{2,1} + \cdots + g_{2,n_2}u^{2,n_2} \\ & & \qquad\qquad\vdots \\ S_I & : & \dot{x} = g_{I,1}u^{I,1} + \cdots + g_{I,n_I}u^{I,n_I} \end{array}\right\} \longrightarrow \underbrace{\begin{array}{rcl} \dot{x} & = & g_{0,1}u^{0,1} + \cdots + g_{0,n_0}u^{0,n_0} \\ & + & g_{1,1}|_{S_0}u^{1,1} + \cdots + g_{1,n_1}|_{S_0}u^{1,n_1} \\ & + & g_{2,1}|_{S_0}u^{2,1} + \cdots + g_{2,n_2}|_{S_0}u^{2,n_2} \\ & \vdots & \\ & + & g_{I,1}|_{S_0}u^{I,1} + \cdots + g_{I,n_I}|_{S_0}u^{I,n_I} \end{array}}$$

Stratified Extended System

$$\left.\begin{array}{rcl} \dot{x} & = & g_{0,1}v^{0,1} + \cdots + g_{0,n_0}v^{0,n_0} \\ & + & g_{1,1}|_{S_0}v^{1,1} + \cdots + g_{1,n_1}|_{S_0}v^{1,n_1} \\ & + & g_{2,1}|_{S_0}v^{2,1} + \cdots + g_{2,n_2}|_{S_0}v^{2,n_2} \\ & \vdots & \\ & + & g_{I,1}|_{S_0}v^{I,1} + \cdots + g_{I,n_I}|_{S_0}v^{I,n_I} \end{array}\right\} + \text{Lie brackets}$$

Now proceed just like in Section 3.4. Pick a nominal trajectory $\gamma(t)$ connecting the starting point with the desired final point. Denoting the stratified extended system by $\dot{x} = b_1(x)v_1 + \cdots + b_k(x)v^k$, set

$$\dot{\gamma}(t) = b_1(\gamma(t))v^1 + \cdots + b_k(\gamma(t))v^k,$$

and solve for the fictitious inputs, $v^i(t)$. This is where gait controllability is a necessary condition, albeit in a slightly different form that in the previous chapter. If the system is not gait controllable, then it not be possible to construct an extended system such that the b_i span the entire tangent space. In that case, it will not be possible to follow an arbitrary $\gamma(t)$. The difference between controllability here and in the previous chapter is that because of Assumption 5.5, we are able to consider Lie brackets between vector fields defined on different strata. This is not possible in the more general case considered in Chapter 4 since the vector fields were not considered in one common space.

Equating the formal equations

$$\dot{S}(t) = S(t)(b_1 v^1 + \cdots + b_k v^k) = e^{h_k b_k} e^{h_{k-1} b_{k-1}} \cdots e^{h_1 k_1}$$

(again, as detailed in Section 3.4), gives the backward Philip Hall coordinates, $h_i(t)$. Equating the coefficients of basis elements in

$$e^{h_k b_k} e^{h_{k-1} b_{k-1}} \cdots e^{h_1 b_1} = e^{\tilde{h}_1 b_1} e^{\tilde{h}_2 b_2} \cdots e^{\tilde{h}_k b_k}$$

gives the forward Philip Hall coordinates, from which, we can approximate individual Lie bracket as in Equation 2.2. The only complicating detail is that control inputs to switch among strata must be interspersed with the regular inputs to ensure that any flow occurs on its appropriate stratum. We will illustrate this in Section 5.3.

With regard to gait efficiency, note that the straight–forward application of the method of Section 3.3 may result in an inordinate amount of strata switches. That is because the sequence of flows in Equation 3.7 are arranged by *order*, and, from a gait efficiency point of view, it is desirable to have them arranged by strata. It is possible to regroup this sequence of flows by strata if the Lie bracket between any vector fields (considered restricted to the bottom stratum) from different strata are zero. If this is true, then, as already clearly illustrated in the Examples 5.1, 5.2, 5.3 and 5.4, it will be possible to reorder the flows to obtain the same net result. In particular, then, flows corresponding to the same stratum could be grouped together. In physical terms, this will reduce the amount that a particular foot would have to be lifted and replaced in and out of contact with the ground. Note that the example in Section 5.3 is not a problem of this type.

5.2 Gait Stability

An important issue for quasi–static legged robotic locomotion is gait stability; namely, ensuring that the quasi–static assumption remains valid. Note that there is not an inherent mechanism in the direct application of the method in Section 3.4 to guarantee the stability of the gait. Recall that the main approach of the method was to pick a trajectory for the extended system, $\gamma(t)$, from which to determine the fictitious inputs. Then, using the fact that any flow can be decomposed to individual flows along the Philip Hall basis vector fields, the real inputs could be determined. The important point to note is that the actual trajectory will, in general, *not* be $\gamma(t)$. Thus, merely picking an initial trajectory $\gamma(t)$ which is always stable is not sufficient. What also must be guaranteed is that deviations from the initial trajectory be within the stability bounds as well.

Our approach to this is as follows. Assume that there is a means for determining the stability of the system. Typically, this may be a scalar–valued function of the configuration, $\Psi(x)$. For convenience, assume that when $\Psi(x)$ has a negative value, the system is unstable, when $\Psi(x)$ has a positive value, the system is stable, and when $\Psi(x) = 0$, the system is on the boundary between stability and instability. In the trajectory generation method, then, we must pick the initial trajectory, $\gamma(t)$ such that it does not does not intersect any unstable regions and also such that it does not intersect the stability boundary, *i.e.*, $\Psi(\gamma(t)) > 0$, $t \in [0, 1]$.

The overall approach is to take steps that are "small enough" to ensure that the system remains stable. Since we are considering small motions and need a norm to provide a measure of the length of a flow, we will consider the system locally in $\mathbb{R}^n$. Given a desired step along the trajectory, $\gamma(t)$, $t \in [0, 1]$, let $\mathcal{R} = \min\{\|x - c\|, \quad c \in \Psi^{-1}(0)\}$, *i.e.* the distance from the starting point to the closest point on the stability boundary.

The goal is to ensure that the trajectory of the system does not intersect the set $\Psi^{-1}(0)$. If x denotes the starting point, and x_f the final point, let $\gamma(t) = x + t(x_f - x)$ be the desired straight line path between the starting and end points. Also, let $\Delta = \|x_f - x\|$. Recall that the fictitious inputs, v^i were determined by solving the equation $\dot{\gamma}(t) = g_1(\gamma(t))v^1 + \cdots + g_s(\gamma(t))v^s$ for the v^i. Then $\|v^i\| < C\|\dot{\gamma}(t)\| = C\Delta$, for some fixed constant C. By the method of construction of the real inputs from the fictitious inputs, then, $\|u^i\| < C\Delta^{1/k}$, where k is the degree of nilpotency of the system, or the degree of the nilpotent approximation.

Now, pick a ball, $\mathcal{B}$, of radius $\mathcal{R}$, and let K be the maximum norm of all the (first order) vector fields, g_i for all points in the ball $\mathcal{B}$. Recall that the real inputs, u^i were given by a sequence of inputs which approximate the flow of the extended system. Denote this sequence by u^i_j, where the superscript indexes which input it is, and the subscript indexes its position in the sequence. The maximum distance that the system can possibly flow from the starting point, x, is given by the sum of the distances of the individual flows. Let $x_{max} = max_{t\in[0,1]}\{\|x(t) - x\|\}$ denote the point in the flow that is maximally distant from the starting point. (Note that this is not necessarily the final point, x_f). To guarantee stability, we want to show that $\|x_{max} - x\| < \mathcal{R}$. However, this distance, $\|x_{max} - x\|$ is necessarily bounded by the sum of the norms of each individual flow associated with one real control input, u^i_j, *i.e.*,

$$\|x_{max} - x\| \leq \sum_{i,j} \| \int_0^1 g_i u^i_j dt\|.$$

However, $\|u^i_j\| \leq C\Delta^{1/k}$ and $\|g_i(x)\| \leq K \ \forall x \in \mathcal{B}$. Thus,

$$\|x_{max} - x\| \leq \sum_{i,j} KC\Delta^{1/k}, \tag{5.8}$$

and since $\Delta = \|x_f - x\|$, by choosing the desired final point close enough to the starting point, the trajectory will not intersect the stability boundary.

Note that because Δ is raised to the power of $1/k$, if k is large, then it may be necessary to make Δ exceedingly small in order to ensure stability. However, the bound expressed in Equation 5.8 is itself very conservative since it sums the length of a bound on each individual flow in the series. Thus an appropriate step length may be best determined experimentally.

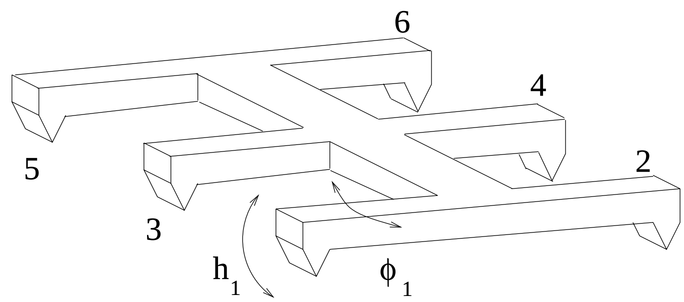


Figure 5.4. Six legged robot.

Finally, note that these very same observations apply to obstacle avoidance. If the robot is traversing an environment with obstacles, assume that the nominal trajectory is designed by a given holonomic or rigid body motion planner in such a manner that it avoids all the obstacles (*e.g.*, Latombe (1990)). Then, ensuring that the actual trajectory avoids the obstacles as well, amounts, in the exact same manner as the stability analysis above, to requiring that the nominal trajectory is broken into sufficiently small steps to ensure that the actual trajectory remains sufficiently close to it. Section 5.3 illustrates both the stability and obstacle avoidance problems.

5.3 Example

We illustrate the application of our approach by generating control inputs which will steer the hexapod robot model from Section 4.3, illustrated again in Figure 5.4. Assume that the robot walks with a tripod gait, alternating movements of legs 1–4–5 with movements of legs 2–3–6.

Recall that the equations of motion are

$$
\begin{aligned}
\dot{x} &= \cos\theta\,(\alpha(h_1)u_1 + \beta(h_2)u_2) \\
\dot{y} &= \sin\theta\,(\alpha(h_1)u_1 + \beta(h_2)u_2) \\
\dot{\theta} &= l\alpha(h_1)u_1 - l\beta(h_2)u_2 \\
\dot{\phi}_1 &= u_1 \\
\dot{\phi}_2 &= u_2 \\
\dot{h}_1 &= u_3 \\
\dot{h}_2 &= u_4
\end{aligned}
$$

where (x, y, θ) represents the planar position of the center of mass, ϕ_i is the front to back angular deflection of the legs and h_i is the height of the legs off the ground and the functions $\alpha(h_1)$ and $\beta(h_2)$ are defined by

$$
\alpha(h_1) = \begin{cases} 1 & \text{if } h_1 = 0 \\ 0 & \text{if } h_1 > 0 \end{cases} \qquad \beta(h_2) = \begin{cases} 1 & \text{if } h_2 = 0 \\ 0 & \text{if } h_2 > 0 \end{cases}.
$$

Since the robot walks in a tripod gait, stability on flat terrain is ensured if the center of mass of the robot remains above the triangle defined by the three feet of the robot which are in contact with the ground. Denote the length of the body by l_b, and consider the motion of legs 1–4–5. The center of mass of the robot must be at least a distance $b = \frac{l_b}{4} + l\sin\phi_1$ from the front of the robot to ensure stability. Figure 5.5 schematically illustrates this geometry. Alternatively, if the center of mass is located a distance b from the front of the robot, then stability is ensured if

$$
\phi_1 < \sin^{-1}\left(\frac{b - l_b/4}{l}\right) \qquad \text{and} \qquad \phi_2 > -\sin^{-1}\left(\frac{3l_b/4 - b}{l}\right)
$$

during its motion to remain stable.

Denote the stratum when all the feet are in contact $(\alpha = \beta = 1)$ by S_{12}, the stratum when leg one is in contact $(\alpha = 1, \beta = 0)$, by S_1, the stratum when leg two is in contact $(\alpha = 0, \beta = 1)$, by S_2 and the stratum when no legs are in contact

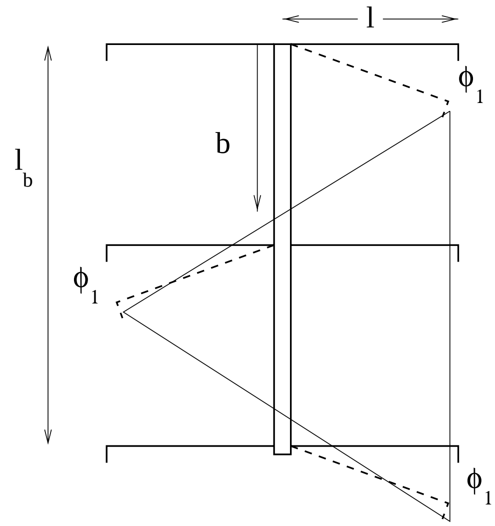


Figure 5.5. Stability margin for hexapod tripod gait.

$(\alpha = \beta = 0)$, by S_0.

Note that this system satisfies the requirements of Assumption 5.5 since, regardless of the values of α and β, the vector fields moving the foot out of contact with the ground are of the form $\left\{\frac{\partial}{\partial h_i}\right\}$ for each foot. Also, the rest of the equations of motion are independent of the foot heights, h_i.

The equations of motion in the bottom strata, S_{12} (where all the feet maintain ground contact), are:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta \\ \sin\theta & \sin\theta \\ l & -l \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \tag{5.9}$$

where (x, y, θ) represents the planar position of the robot's center. ϕ_1 is the angle of legs 1–4–5 and ϕ_2 is the angle of legs 2–3–6. The variables u^3 and u^4 are constrained to be 0 (so that the legs maintain ground contact). Let $g_{12,1}$ and $g_{12,2}$ represent the first and second columns in Equation 5.9.

If legs 1–4–5 are in contact with the ground, but legs 2–3–6 are not in contact, the equations of motion are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 0 & 0 \\ l & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^4 \end{pmatrix} \tag{5.10}$$

where h_i is the height of the corresponding set of legs and u^3 is constrained to be 0. Let columns one, two and three in Equation 5.10 be labeled $g_{1,1}, g_{1,2}$ and $g_{1,3}$, respectively. This higher stratum will be called S_1. If legs 2–3–6 are in ground contact and legs 1–4–5 are not, the equations of motion are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{h}_1 \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta & 0 \\ 0 & \sin\theta & 0 \\ 0 & -l & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \tag{5.11}$$

where u^4 is constrained to be 0. The columns in Equation 5.11 will be denoted $g_{2,1}, g_{2,2}$ and $g_{2,3}$, respectively, and this higher stratum is S_2.

We need enough vector fields to span the tangent space of the bottom stratum, S_{12}. A simple calculation shows that the set of vector fields,

$$\{g_{12,1}, g_{12,2}, g_{1,2}, g_{2,1}, [g_{1,1}, g_{2,2}]\}$$

spans $T_x S_{12}$ for all $x \in S_{12}$. Note that $[g_{1,1}, g_{2,2}] = (-2l\sin\theta, 2l\cos\theta, 0, 0, 0)^T$. This Lie algebra is *not* nilpotent, and thus the extended system will only be a nilpotent approximation.

Now, construct the *stratified extended system* by constructing the extended system using the vector fields from all strata.

$$\dot{x} = g_{12,1}v^1 + g_{12,2}v^2 + g_{1,2}v^3 + g_{2,1}v^4 + [g_{1,1}, g_{2,2}]v^5, \tag{5.12}$$

or, in greater detail,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta & 0 & 0 & -2l\sin\theta \\ \sin\theta & \sin\theta & 0 & 0 & 2l\cos\theta \\ l & -l & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \end{pmatrix}. \tag{5.13}$$

Let the starting and ending configurations be:

$$\begin{aligned} p &= (x, y, \theta, \phi_1, \phi_2, h_1, h_2) = (0, 0, 0, 0, 0, 0, 0) \\ q &= (x, y, \theta, \phi_1, \phi_2, h_1, h_2) = (1, 1, 0, 0, 0, 0, 0) \end{aligned}.$$

A path that connects these points is $\gamma(t) = (t, t, 0, 0, 0, 0, 0)$. Equating $\dot{\gamma}(t)$ with with the stratified extended system and solving for the fictitious controls yields

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \end{pmatrix} = \frac{1}{2l} \begin{pmatrix} l(\cos\theta + \sin\theta) \\ l(\cos\theta + \sin\theta) \\ -l(\cos\theta + \sin\theta) \\ -l(\cos\theta + \sin\theta) \\ (\cos\theta - \sin\theta) \end{pmatrix},$$

or, since $\theta(t) = 0$, and if $l = 1$,

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

For a system which is nilpotent of order 2, from Equation 3.9 for the extended system on S_{12}

$$\begin{aligned} \dot{h}_1 &= v^{S_{12},1}, \quad \dot{h}_2 = v^{S_{12},2}, \\ \dot{h}_3 &= v^{S_{12},3} + h_1, v^{S_{12},2} \end{aligned}$$

which yields.

$$h_1(1) = \frac{1}{2} \quad h_2(1) = \frac{1}{2} \quad h_3(1) = \frac{3}{4}. \tag{5.14}$$

Since the nilpotent approximation is of order two, there is no need to transform to forward Philip Hall coordinates. The control sequence is

$$\sqrt{\frac{3}{4}}(v^{S_{12},1} \circ v^{S_{12},2} \circ -v^{S_{12},1} \circ -v^{S_{12},2})$$

to get $e^{h_3 B_3}$, and $\frac{1}{2} v^{S_{12},2} \circ v^{S_{12},1}$ to get $e^{h_2 B_2} e^{h_1 B_1}$. Hence, the complete sequence is

$$\sqrt{\frac{3}{4}}(v^{S_{12},1} \circ v^{S_{12},2} \circ -v^{S_{12},1} \circ -v^{S_{12},2}) \circ \frac{1}{2} v^{S_{12},2} \circ v^{S_{12},1}.$$

Figure 5.6 shows the path of the robot's center as it follows a straight line trajectory, which is broken into four equal segments. Due to the nilpotent approximation, there is some small final error. Better accuracy can be obtained by use of a higher order nilpotent approximation or a second iteration of the algorithm from the robot's

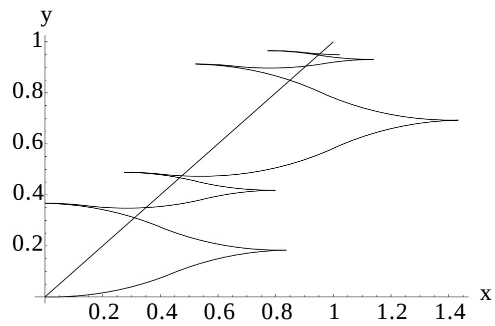


Figure 5.6. Straight trajectory.

ending position. The "zig–zag" pattern results from the piecewise inputs and the strata switching.

There is no inherent limitation in the method which requires the trajectory to be broken down into subsegments, however, there are two reasons to do so. First, since the method fundamentally is based upon decomposing a desired trajectory into flows along the Philip Hall basis vector fields, the final trajectory is only related to the desired trajectory in that the end points are the same (or approximately the same, in the case of a nilpotent approximation). Second, robot stability requirements may also demand smaller steps.

The approach is general enough that arbitrary trajectories are possible. Figure 5.7 shows the hexapod tracking an ellipse while maintaining a constant angular orientation. Figure 5.8 shows the results when a smaller step size is used. In the first simulation, the the elliptical trajectory is broken down into 30 segments. In the second, it is broken down into 60 segments. In this example, part of the trajectory tracking error is due to the nilpotent approximation, but another contribution to the trajectory error is the simplicity of the hexapod mechanical model. Essentially, some directions are more "difficult" for the hexapod to execute than others. In these simulations, because of the simplicity of the model, which eliminates "crab–like" gates, when the robot has to move sideways, its tracking error is greater because this direction corresponds to a Lie bracket direction.

Also plotted along with the trajectory is the stability criterion. In each case,

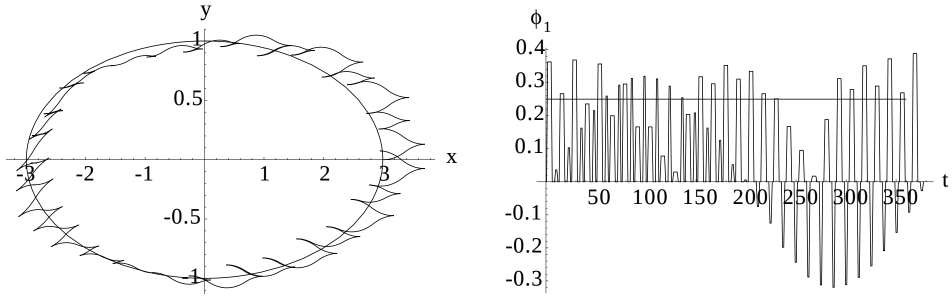


Figure 5.7. Elliptical trajectory and stability criterion for the hexapod robot.

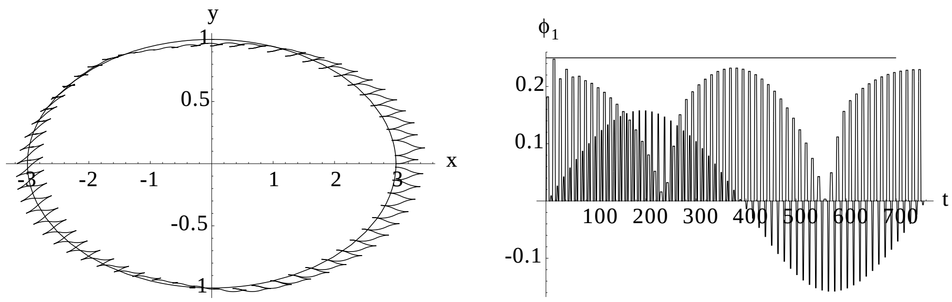
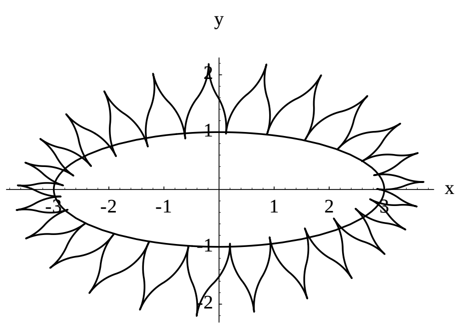


Figure 5.8. Elliptical trajectory and stability criterion for the hexapod robot taking smaller steps.

take the length of the body to be 2 units of length and let the center of mass be located a distance of 0.75 units from the front of the robot. Then, the stability criterion is $\phi_1 < 0.25$ [rad] and $\phi_2 > -.85$ [rad]. In Figures 5.7 and 5.8 the stability limits for ϕ_1 are indicated by the straight horizontal lines. In the first case, where the robot takes bigger steps, the stability condition is violated. However, in the second case it is not.

Figure 5.9 shows the hexapod following the same ellipse while also rotating at a constant rate. Figure 5.10 plots the robot's angular orientation as the simulation progresses.

Finally, consider the obstacle avoidance issue. As a starting point, the nominal



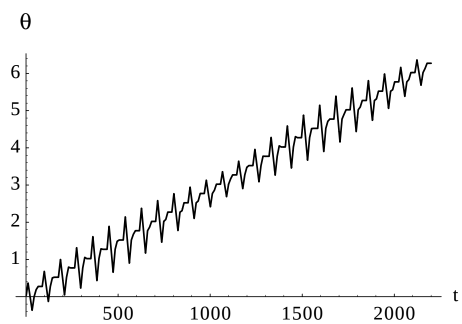


Figure 5.9. Elliptical path with rotation.

Figure 5.10. Hexapod orientation during execution of path seen in Figure 5.9.

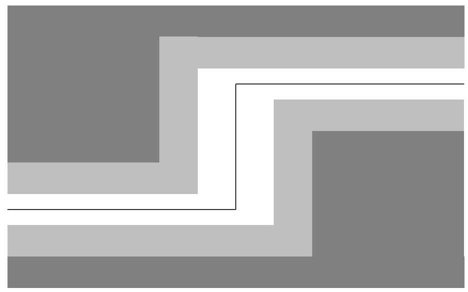


Figure 5.11. Nominal trajectory for obstacle avoidance problem.

trajectory $\gamma(t)$ must *a priori* avoid any obstacles. Unfortunately, this still does not guarantee that the actual motion will avoid the obstacles. As discussed previously, however, if the trajectory is divided into sufficiently small step sizes, then the actual motion will not significantly deviate from the nominal trajectory. Figure 5.11 shows a nominal path between a set of obstacles. The desired trajectory is indicated by the black line and the "walls" of the environment by dark grey regions. Since the plots will show the path of the center of mass of the robot, it is not enough for the center of mass to avoid the walls, the center of mass must not come within a certain distance from the walls so that the other portions of the body do not come into contact with the wall. The lighter grey regions indicate "buffer zone" near the walls that the center of mass can not enter to ensure that other parts of the robot do not hit the real walls (dark grey).

To make the problem more challenging and realistic, assume that the robot rotates at a uniform rate as it follows the nominal trajectory. A real–world scenario in which this might be desirable would be some sort of patrol robot, that must constantly scan in all directions. Figure 5.12 shows the path of the center of mass of the robot when the trajectory is subdivided into 100 subtrajectories. The path of the center of mass intersects the lighter grey regions during its motion both on the lower and upper horizontal trajectory portions. However, if the nominal trajectory is subdivided into 300 subtrajectories, then the robot avoids the walls, as illustrated

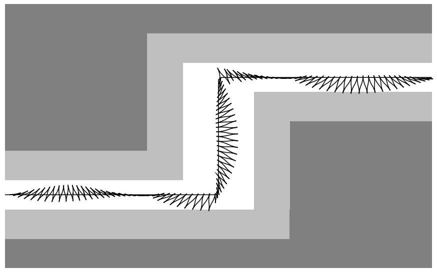


Figure 5.12. Obstacle contact.

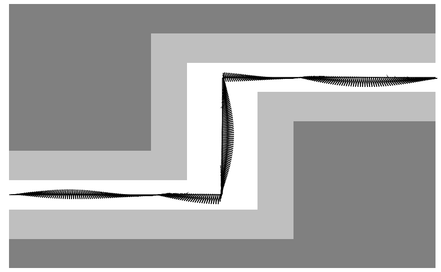


Figure 5.13. Obstacles avoided.

in Figure 5.13.

One possible implementation improvement would be an adaptive step size scheme wherein the robot only shortens step lengths in "critical" regions. In the trajectories illustrated in Figures 5.12 and 5.13, these critical regions correspond to when the robot is attempting to walk in a "crab–like" manner. These motions are when the desired motion is almost directly along a Lie bracket direction, which requires the greatest excursion from the nominal trajectory.

5.4 The "Efficiency" of Stratified Systems

Primarily by way of example, this section illustrates that even for systems that are controllable without making and breaking contact, a more efficient control strategy may be to utilize the possibility intermittent contact. The example is related to grasping, and appears in Murray, Li, and Sastry (1994) and in Murray and Sastry (1990). Essentially the example is that of a sphere rolling on a plane, and is intended to model a finger–tip contacting a grasped object. In Murray, Li, and Sastry (1994) and Murray and Sastry (1990) this model is used as an example of how a finger–tip can reorient itself by exploiting its nonlinear surface geometry. Here, we modify the example by allowing the finger to lift off of the surface. In such a case, "inefficient" high order Lie bracket motions can be replaced by low order motions when the finger is not in contact with the object.

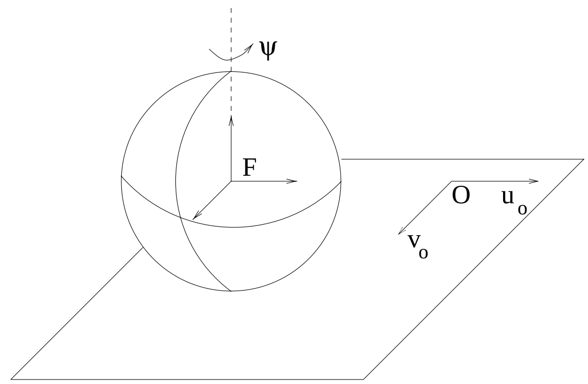


Figure 5.14. Spherical finger rolling on a plane.

Example 5.6: (Spherical finger rolling on a plane.) Consider the sphere rolling on a plane shown in Figure 5.14. Parameterize the plane (the "object") with local coordinates $c_o(u_o, v_o) = (u_o, v_o, 0)$ and locally parameterize the sphere (the "finger") with the chart

$$c_f(u_f, v_f) = \begin{pmatrix} \rho \cos u_f \cos v_f \\ -\rho \cos u_f \sin v_f \\ \rho \sin u \end{pmatrix},$$

where ρ is the radius of the sphere, $u_f \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $v_f \in (-\pi, \pi)$. Denote the distance of the finger from the object by h. Thus, the configuration space for the system is parameterized by the variables $x = (u_f, v_f, u_o, v_o, \phi, h)$.

Montana (1988) provides the equations of motion for such contacting systems in terms of the surface geometry of the finger and object. If the control inputs are the angular velocities of the sphere, it is easy to show (Murray, Li, and Sastry (1994)) that the equations of motion when the finger is in contact with the object are

$$\begin{pmatrix} \dot{u}_f \\ \dot{v}_f \\ \dot{u}_o \\ \dot{v}_o \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ \sec\, u_f \\ -\rho\sin\psi \\ -\rho\cos\psi \\ -\tan u_f \end{pmatrix} \omega_x + \begin{pmatrix} -1 \\ 0 \\ -\rho\cos\psi \\ \rho\sin\psi \\ 0 \end{pmatrix} \omega_y = g_{1,1}(x)u^1 + g_{1,2}(x)u^2.$$

This system is controllable because the vector fields

$$\{g_1, g_2, [g_1, g_2], [g_1, [g_1, g_2]], [g_2, [g_1, g_2]]\}$$

span the tangent space to the configuration space.

Also consider when the finger is lifted off of the surface. When the finger is not in contact with the object, the equations of motion are

$$\begin{aligned} \begin{pmatrix} \dot{u}_f \\ \dot{v}_f \\ \dot{u}_o \\ \dot{v}_o \\ \dot{\psi} \\ \dot{h} \end{pmatrix} &= \begin{pmatrix} 0 \\ \sec\, u_f \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \omega_x + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \omega_y + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u^3 \\ &= g_{0,1}(x)u^1 + g_{0,2}(x)u^2 + g_{0,3}(x)u^3. \end{aligned}$$

$\square$

We will use two standard methods applicable to smooth systems to reorient the finger tip, and then compare the results when the stratified nature of the system is exploited. As will be clear shortly, explicitly exploiting the stratified nature of the system may make it easier to avoid obstacles or maintain stability.

5.4.1 Motion Planning Using Sinusoids

There exists a canonical class of nonlinear systems called *chained systems*, for which motion planning can be accomplished using sinusoidal inputs with integrally related frequencies (see Murray and Sastry (1993)). In particular, a *one–chain system* is a two–input system of the form:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2 u_1 \\
\dot{x}_4 &= x_3 u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1} u_1.
\end{aligned}$$

Murray and Sastry (1993) present a general method for steering such systems. Rather than present the most general results, here simply note that if the inputs are

$$\begin{aligned}
u_1(t) &= a \cos t \\
u_2(t) &= b \cos 2t,
\end{aligned}$$

then $x_1(2\pi) = x_2(2\pi) = x_3(2\pi) = 0$ and

$$x_4(2\pi) = -\frac{a^2 b \pi}{4}, \tag{5.15}$$

so by appropriately choosing a and b the inputs u_1 and u_2 steer the system in the x_4–direction.

Now, say we wish to move the finger to a point on the object above the current point of contact. Let the initial configuration be $x_0 = (0, 0, 0, 0, 0)$ and specify the final desired configuration to be $x_f = (0, 0, 0.1, 0, 0)$. Solving Equation 5.15 yields

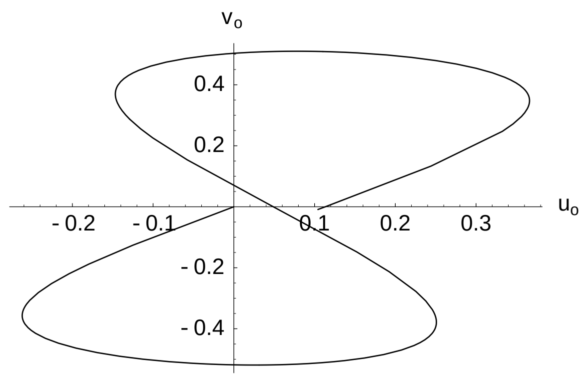


Figure 5.15. Trajectories for object contact coordinates: sinusoidal inputs.

the inputs:

$$\begin{aligned} u_1(t) &= 0.5 \cos t \\ u_2(t) &= 0.51 \cos 2t \end{aligned} \tag{5.16}$$

Figure 5.15 shows the time evolution of the object contact coordinates, and Figure 5.16 shows the evolution of the finger contact coordinates. The trajectory for the finger contact coordinates return to their initial conditions, but there is a net displacement in the u_o variable, as desired. All the other variables return to their initial conditions as well.

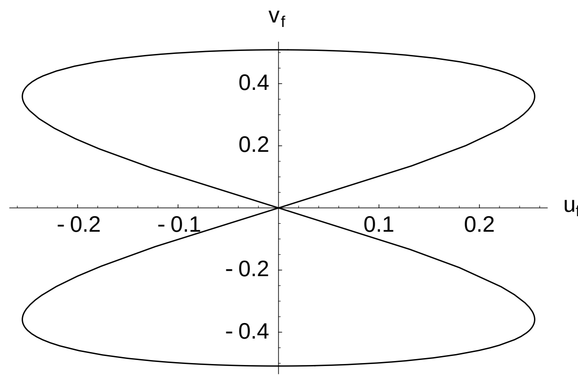


Figure 5.16. Trajectories for finger contact coordinates: sinusoidal inputs.

5.4.2 Motion Planning Using Piecewise Constant Inputs

Now use the method outlined in Section 5.1 to steer the system to the same final position. Since the final motion is along only the vector field

$$-[g_1, [g_1, g_2]] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

the method simply amounts to determining the sequence of piecewise constant inputs which approximate the bracket. One possibility is the following sequence of flows:

$$\phi_\epsilon^{-g_1} \circ \phi_\epsilon^{-g_2} \circ \phi_\epsilon^{g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1} \circ \phi_\epsilon^{g_1}(x_0). \tag{5.17}$$

Figure 5.17 shows the time evolution of the object contact coordinates, and Figure 5.18 shows the evolution of the finger contact coordinates. Again, the trajectory

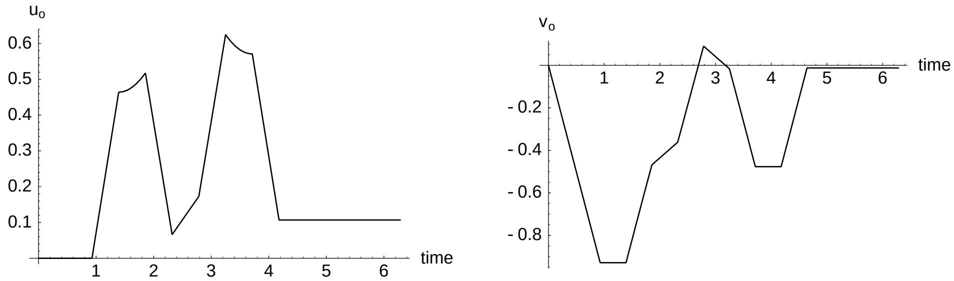


Figure 5.17. Trajectories for object contact coordinates: piecewise constant inputs.

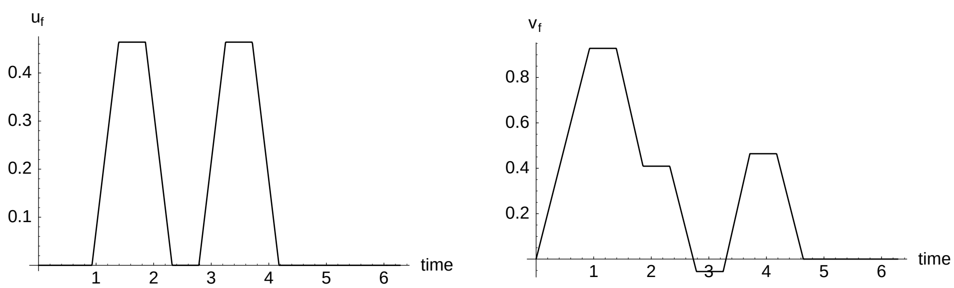


Figure 5.18. Trajectories for object contact coordinates: piecewise constant inputs.

for the finger contact coordinates return to their initial conditions, but there is a net displacement in the u_o variable. All the other variables return to their initial conditions as well.

5.4.3 Example of Constrained Motion

Now consider one additional aspect of the problem shown in Figure 5.14; namely, the fact that some objects have edges, off of which the finger may roll. From Figure 5.15 it is clear that if the dimensions of square side of the object are less than 0.6 units of length, then the motion above (designed to only move the finger 1/3 of the way to the edge of the object) will cause the finger to fall off of the object.

The remedy for this is, just as for the obstacle avoidance analysis in Section 5.3, is to break the trajectory into a sequence of small segments. As an illustration of

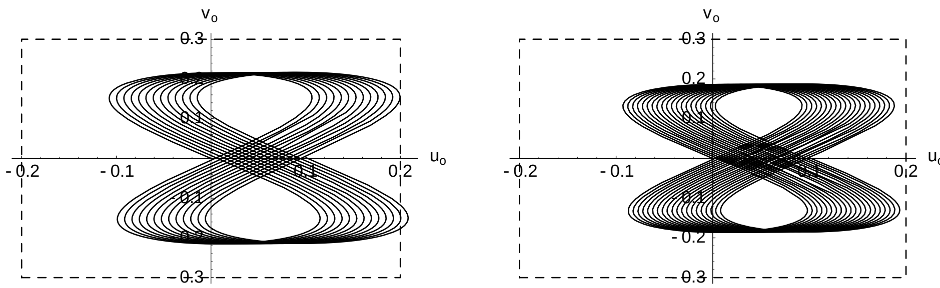


Figure 5.19. 13 and 20 segment motions: sinusoidal inputs.

the difficulties associated with this problem, consider the following examples. For the following examples, assume that the square side of the object is 0.4 units of length square. Note that the order of the maximum net deflection scales with the desired net motion as $u_{\max} \sim u_{\text{net}}^{(1/3)}$. (The 1/3 factor comes from the order of the a^2b term in the numerator of the expression in Equation 5.15).

To break the trajectory into sufficiently small subsegments, we must consider the maximum displacements in the u_o and v_o directions. By inspecting Figure 5.15, the magnitude of the periodic motion is approximately 0.26. Now, the goal is for the $u_o < 0.2$. However, since the motion is designed to take that coordinate to $u_o = 0.1$, the maximum magnitude must be less than 0.1. So, the maximum step length must be on the order of $(0.1/0.26)^{1/3} = 0.057$, which requires the trajectory to be split into about $1/0.057 = 17.5$ subsegments.

Figure 5.19 show the trajectories in the object contact coordinates for a simulation with 13 segments (where the finger still rolls off the object) and 20 segments (where the finger stays on the object). As before, all the other variables return to their initial conditions the finger contact coordinates return to the initial conditions.

Figure 5.20 shows the similar results for the piecewise constant input case. These simulations have the same nominal trajectory as before, but uses the piecewise constant inputs rather than the sinusoidal inputs, again with 13 segments and 20 segments.

Now, for the stratified system, it is a simple matter to lift the finger off of the object. The desired motion is along the difference of the vector fields $g_{1,2} - g_{0,2}$

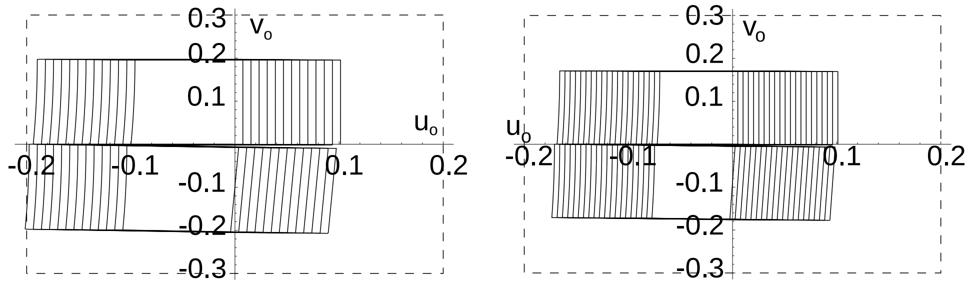


Figure 5.20. 13 and 20 segment motions: piecewise constant inputs.

(when $\psi = 0$), and so the simple sequence of flows

$$\phi^{0.1}_{g^1_2} \circ \phi^{0.1}_{g^0_3} \circ \phi^{0.1}_{-g^0_2} \circ \phi^{0.1}_{-g^0_3}$$

gives the desired motion. The result is shown in Figure 5.21. The dotted parallel lines represent two edges of the object (the other "edge" coordinate is not plotted).

Of course, for the particular example given, there is some flexibility in choosing the "aspect ratio" of the motion by appropriately choosing the values of a and b in Equation 5.15 or by not choosing equal length piecewise constant inputs in Equation 5.17. However, the main point is that a higher order brackets require motions that are large compared with the desired final net motion. For a stratified system, this can often be avoided by switching strata, thus resulting in more "efficient" motion planning.

5.5 Summary

This chapter extended a general motion planning algorithm for smooth systems to the stratified case. Basically, if the vector fields on different strata satisfied certain conditions, then it is possible to consider some, or nearly all, of the vector fields defined on higher strata as defined on the bottom stratum. This greatly increases the number of vector fields available in the motion planning algorithm. Additionally,

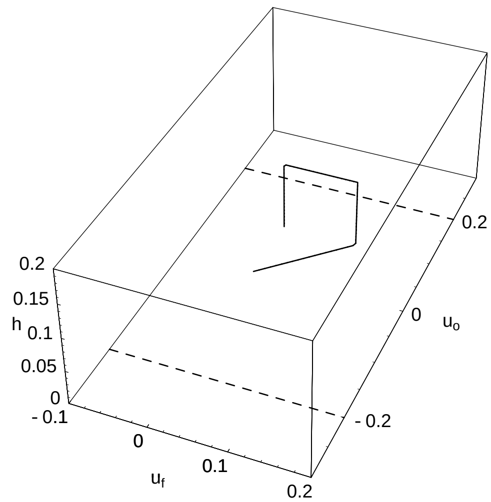


Figure 5.21. "Stratified" input.

we illustrated that for systems which are completely controllable on the bottom stratum, higher order motions which correspond to higher order brackets may be replaced by lower order motions. This is desirable because the motions associated with higher order brackets cause large excursions from the nominal trajectory, which may cause stability or obstacle avoidance problems.

Chapter 6

Unilateral Controllability

This chapter considers a distinct, but closely related topic to the controllability results in Chapter 4. A basic assumption in the controllability results in Sections 3.2.1, 3.2.2 and 3.2.3 was that the control inputs could be both positive and negative. In other words, it was possible to flow in both the positive and negative directions of a vector field. This chapter considers control systems where some of the control inputs are restricted to be strictly positive. Another important fact is that the results in this chapter are not restricted to driftless systems.

The relationship between the results in this chapter and the previous chapters in this dissertation is that one example of a stratified system is the problem of so–called *nonprehensile manipulation*. This is the problem of trying to manipulate an object by pushing on it. Clearly, the intermittent physical contact present in this problem makes it a stratified system. However, since the object is not fixed, in contrast to the terrain in a legged locomotion problem, this gives rise to vector fields along which the system can only flow in one direction. Physically, this corresponds to the fact that physical contact can only "push" on an object, and not "pull" it.

Unfortunately, a simple general theory does not exists for systems with unilateral inputs. Therefore, a prerequisite to considering stratified unilateral systems is to develop a test for smooth unilateral system. This is the subject of Sections 6.1, and 6.2. Section 6.3 considers the stratified generalization.

6.1 Systems with Unilateral Control Inputs

This section considers control systems of the form

$$\dot{x} = f(x) + h_i(x)v^i + g_j(x)u^j \tag{6.1}$$

where $i = 1, \ldots, m$, $j = 1, \ldots, n$, $v^i \in [0, 1)$ $\forall i$, $u^j \in (-1, 1)$ $\forall j$, $x \in M$, M a manifold and f, h_i and g_j vector fields on M; that is, the control inputs v^i are restricted to be non–negative.

Let $\mathcal{C}$ be the smallest subalgebra of $V^{\infty}(M)$ (the Lie algebra of smooth vector fields on M) that contains $f, h_1, \ldots, h_m, g_1, \ldots, g_n$, and let C be the *accessibility distribution* generated by $\mathcal{C}$:

$$C(x) = \text{span}\{X(x) : X \in \mathcal{C}\}, \quad x \in M.$$

If $\text{dim}C(x_0) = \text{dim}M$, then the system satisfies the *Lie Algebra Rank Condition* ("LARC") at x_0. This is the same definition of LARC as before, except this definition includes the drift term and unilateral inputs as well.

Let $\mathbf{X} = \{X_0, X_1, \ldots, X_m, X_{m+1}, \ldots X_{m+n}\}$ and $\mathbf{f} = \{f, h_1, \ldots, h_m, g_1, \ldots, g_n\}$ so that $\text{Ev}(\mathbf{f})$ takes X_0 to f, X_i to h_i for $i = 1, \ldots, m$ and X_j to g_{j-m} for $j = m+1, \ldots, m+n$. The f, h_i and g_j above correspond to the vector fields in the Equation 6.1. Let $\text{Br}(\mathbf{X})$ be the set of "brackets" of elements from $\mathbf{X}$ and $\delta^i(B)$ be the number of occurrences of X_i in $B \in \text{Br}(\mathbf{X})$.

Consider the automorphism generated by σ_i, $i = m+1, \ldots, m+n$ where σ_i sends X_j to X_j if $j \neq i$ and X_i to $-X_i$. Clearly, a Λ_0–fixed element of $L(\mathbf{X})$ cannot have an odd number of each $X \in \{X_{m+1}, \ldots, X_{m+n}\}$. Thus, we can consider only elements with an even number of each $X \in \{X_{m+1}, \ldots, X_{m+n}\}$, so we will call an element $B \in \text{Br}(\mathbf{X})$ *bad* if δ^b is even for each $b = m+1, \ldots, m+n$ and $\delta^0 + \sum_{a=1}^{m} \delta^a$ is odd. A bracket is *good* if it is not bad. Let S_m denote the permutation group on m symbols. For $\pi_m \in S_m$ and $\pi_n \in S_n$ define $\overline{\pi}(B)$ to be the bracket obtained by fixing X_0, sending X_a to $X_{\pi_m(a)}$ for $i = 1, \ldots m$ and sending X_b to $X_{(\pi_n(b-m))+m}$

for $b = m+1, \dots, m+n$. Now define the symmetrization operator

$$\beta(B) = \sum_{\pi_n \in S_n} \sum_{\pi_m \in S_m} \overline{\pi}(B). \tag{6.2}$$

Let $\theta \geq 1$ be a real number, and define $\Delta(\rho)$ by

$$\begin{aligned} \Delta(\rho) : (X_0, \dots, X_{m+n}) \mapsto& \qquad (6.3)\\ (\rho X_0, \rho^\theta X_1, \dots, \rho^\theta X_m, &\rho X_{m+1}, \dots, \rho X_{m+n}). \end{aligned}$$

This dilation is compatible with $\hat{S}(X, K)$ by construction. Note also that this dilation makes each $\overline{\pi}$ a graded linear map. The Δ–degree of a bracket B is given by

$$\delta_\theta(B) = \delta^0(B) + \theta \sum_{i=1}^{m} \delta^{h_i}(B) + \sum_{i=1}^{n} \delta^{i+m}(B).$$

The following is the main result of this chapter.

Proposition 6.1 *Consider the bijection $\phi : \mathbf{X} \to \mathbf{f}$ which sends X_0 to f, X_a to h_a for $a = 1, \dots, m$, and X_b to g_{b-m} for $b = m+1, \dots m+n$. Suppose that the system described by Equation 6.1 is such that every bad bracket $B \in Br(\mathbf{X})$ has the property that*

$$Ev_x(\phi)(\beta(B)) = \sum_{a=1}^{k} \xi^a Ev_x(\phi)(C_a) \tag{6.4}$$

where $\xi^i \in \mathbb{R}$ and $\delta_\theta(C_a) < \delta_\theta(B)$ for $a = 1, \dots k$. Also suppose that 6.1 satisfies the LARC at x. Then the system described by Equation 6.1 is STLC at x.

Proof: Provided that the dilation defined by Equation 6.3 is compatible with $\hat{S}(\mathbf{X}, K)$ and that the collection of all $\overline{\pi}$'s comprise a group which is an input symmetry, then this follows from Theorem 3.9. As previously mentioned, the dilation defined by Equation 6.3 is compatible with $\hat{S}(\mathbf{X}, K)$ by construction.

We need to show that the group comprised of all the $\overline{\pi}$'s is an input symmetry.

Define $\overline{\pi}^{\#}$ by $\overline{\pi}^{\#}(\exp Z) = \exp(\hat{\overline{\pi}}(Z))\ Z \in \hat{L}(\mathbf{X})$, where $\hat{\overline{\pi}} : \hat{L}(\mathbf{X}) \to \hat{L}(\mathbf{X})$ is given by $\hat{\overline{\pi}}(Z) = \sum_{i=1}^{\infty} \overline{\pi}(P_i)$, if $Z = \sum_{i=1}^{\infty} P_i$, where each P_i is homogeneous of degree i. Clearly, $\overline{\pi}^{\#}$ simply fixes X_0, sends X_a to $X_{\pi_m(a)}$ for $i = 1, \dots m$ and sends X_b to $X_{\pi_n(b-m)+m}$ for $b = m+1, \dots, m+n$ for each term in the infinite series.

Now, from Equation 2.11, we can write

$$\overline{\pi}^{\#}(S(t)) = \sum_I \left(\int_o^t u_I \right) X_{\pi(I)}$$

where, for $I = \{i_1, \dots, i_k\}$, $\pi(I) = \{\pi_{mn}(i_1), \dots, \pi_{mn}(i_k)\}$, where mn is either m or n, depending upon whether $i \in \{1, \dots, m\}$ or $i \in \{m+1, \dots, m+n\}$ respectively. However,

$$\sum_I \left(\int_o^t u_I \right) X_{\pi(I)} = \sum_I \left(\int_o^t u_{\pi^{-1}(I)} \right) X_I$$

since the summation is over all possible multi–indices I. Since π^{-1} maps K to K, it follows that $\overline{\pi}^{\#}$ maps $\hat{S}(X, K)$ to $\hat{S}(X, K)$. Therefore the collection of $\overline{\pi}$ is an input symmetry and controllability follows from Theorem 3.9. ■

The difficulty with unilateral systems is that the first order vector fields associated with the unilateral inputs must be neutralized. Therefore, the unilateral vector fields must either satisfy some convexity condition (so that under the action of the symmetrization operator, they are neutralized), or they cannot be used to satisfy the LARC and must be assigned a sufficiently high degree so that they are neutralized by lower degree brackets.

Using the former approach, one corollary is simple to obtain. First, we must specify a particular dilation. For a given Lie bracket X, consider the *degree of a bracket with respect to a vector field* f, h_i or g_j, denoted by $\delta^f(X)$, $\delta^{h_i}(X)$ and $\delta^{g_j}(X)$, respectively, to be the number of times that the superscripted vector field appears in the bracket X. Now consider the *total degree*, $\delta(X)$, to be

$$\delta(X) = \delta^f(X) + (1+\epsilon) \sum_{i=1}^{m} \delta^{h_i}(X) + \sum_{j=1}^{n} \delta^{g_j}(X),$$

where $0 < \epsilon \ll 1$.

Now, call a bracket, X "bad" if δ^{g_j} is even (including 0) for each j, $\delta^f(X) + \sum_{i=1}^m \delta^{h_i}(X)$ is odd and $\sum_{i=1}^m \delta^{h_i}(X) \neq 1$. Otherwise, call the bracket "good."

Corollary 6.2 *Consider the control system described by Equation 6.1. Assume that the system satisfies the LARC and that there exist coefficients λ_i and α_j such that*

$$\sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{j=1}^{n} \alpha_j g_j(x) = 0 \qquad \forall x \in \mathrm{nbd}(x_0), \tag{6.5}$$

where $\lambda_i \in (0,1)$ and $\alpha_j \in \mathbb{R}$. Assume further that any bad bracket can be written as a linear combination of brackets of lower total degree. Then the system is STLC at x_0.

The intuition behind the restriction expressed by Equation 6.5 is simple. Due to the control input restrictions, none of the control inputs v^i can be negative. However, Equation 6.5 can be solved for one $-h_i$ in terms of the other h_j's with positive coefficients and the g_k's with arbitrary coefficients. Thus, allowable control inputs ($v^j > 0$, $j \neq i$) effect the same result as a disallowed control input (one $v^i < 0$).

Proof: First, scale the coefficients $\lambda_i$and α_j in Equation 6.5 so that

$$\sum_i |\lambda_i| + \sum_j |\alpha_j| = 1. \tag{6.6}$$

Now, observe that if the system

$$\dot{x} = f(x) + \tilde{h}_i(x)v^i + \tilde{g}_j(x)u^j \tag{6.7}$$

where $\tilde{h}_i = \lambda_i h_i$ and $\tilde{g}_j = \alpha_j g_j$, where λ_i and α_j are from Equation 6.6, is controllable, then so is the system described by Equation 6.1 (because we have effectively further restricted the set of allowable control inputs).

If B is a bad bracket such that $\sum_{i=1}^m \delta^{h_i}(B) \neq 1$, then Proposition 6.2 is simply a restatement of Proposition 6.1 where, instead of requiring the symmetrization of a bad bracket, defined in Equation 6.2, to be Δ–neutralized, each term in the sum which defines the symmetrization must be individually Δ–neutralized.

Now if B is such that $\sum_{i=1}^{m} \delta^{h_i}(B) = 1$, then $\beta(B)$ is of the form

$$\beta(B) = \sum_{\pi_n \in S_n} \overline{\pi}(\tilde{B})$$

where the single $X \in \{X_1, \ldots, X_m\}$ in B is replaced by $X_1 + \cdots + X_m$ in $\tilde{B}$ by the operation of $\sum_{\pi_m \in S_m} \overline{\pi}(B)$. Recall that, by assumption, $\sum_{i=1}^{m} \tilde{h}_i = -\sum_{j=1}^{n} \tilde{g}_j$. Thus, if we let $\hat{B}$ be the bracket $\tilde{B}$ with the term $X_1 + \cdots + X_m$ replaced by $-(X_{m+1} + \cdots + X_{m+n})$, then, we have $\mathrm{Ev}_x(\phi)(\beta(B)) = \mathrm{Ev}_x(\phi)(\sum_{\pi_n \in S_n} \hat{B})$. Since $\delta_\theta(\sum_{\pi_n \in S_n} \hat{B}) < \delta_\theta(\beta(B))$, the hypotheses of Proposition 6.1 is satisfied. ■

6.2 Example

This section illustrates the application of Corollary 6.2 by way of an example. Consider the controllability of rigid body with "thrusters." Initially consider the body to be centered about orthogonal coordinate axes, and let two thrusters be placed at the points where the x–axis intersects the surface of the body and two more thrusters placed at the points where the y–axis intersects the surface. Let the force of the thrusters on the x–axis be in the negative z–direction, and the force of the thrusters on the y–axis be in the positive z–direction. As a fifth control input, let the thrusters rotate by a small angle, ψ (which can be both positive and negative) about their respective axes so that the thrusters aligned on the x–axis rotate in opposite directions so that if they are both "thrusting" they both contribute to a positive torque about the z–axis, and let the thrusters aligned on the y–axis rotate in opposite directions so that both contribute a negative torque about the z–axis. We will consider a spherical body with unit radius and mass $\frac{5}{2}$ (so that the inertia tensor is the identity) as illustrated in Figure 6.1; although, for a non–spherical body, the following controllability analysis still holds. In Figure 6.1, for clarity, the coordinate system is shown displaced from the center of mass of the sphere, but for

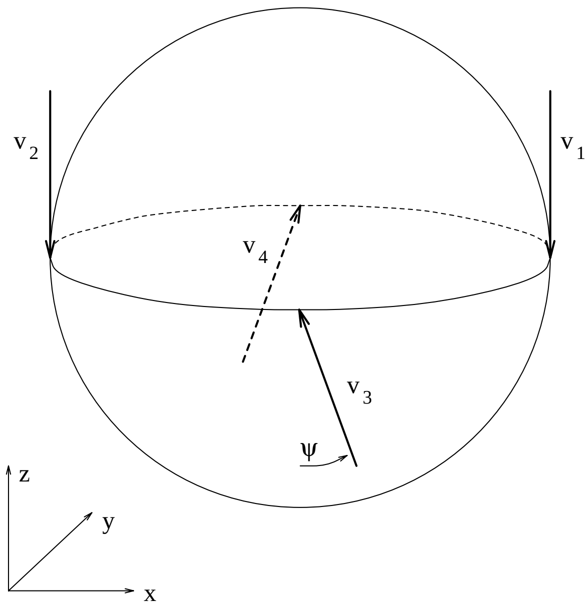


Figure 6.1. Rigid body with thrusters.

the calculations, the origin of the coordinate system is assumed to initially coincide with its center of mass.

Parameterize the configuration space for the system $SE(3) \times S^1$ by the coordinates $X = (x, y, z)$, which are the displacements of the center of mass from the fixed inertial frame, $\Phi = (\phi_1, \phi_2, \phi_3)$, which are the "roll, pitch, yaw" rotations about the x– y– and z–axes, respectively and ψ, which is the rotational angle of the thrusters. Thus, a point in the phase space is given by $q = (X, \Phi, \dot{X}, \dot{\Phi}, \psi)$.

Now, we can write the equations of motion as

$$\begin{pmatrix} \dot{X} \\ \dot{\Phi} \\ \ddot{X} \\ \ddot{\Phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \dot{X} \\ \dot{\Phi} \\ 0 \\ -Q^{-1}\dot{Q}\dot{\Phi} \\ 0 \end{pmatrix} + F \cdot v, \tag{6.8}$$

where Q is the local mapping which takes the derivatives of the roll, pitch and yaw coordinates we chose for our parameterization and gives the body angular velocities, which given by

$$Q = \begin{pmatrix} \cos\phi_2 \cos\phi_3 & \sin\phi_3 & 0 \\ -\cos\phi_2 \sin\phi_3 & \cos\phi_3 & 0 \\ \sin\phi_2 & 0 & 1 \end{pmatrix}.$$

If we let $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ represent the standard unit vectors in the x–, y– and z–directions fixed in the body, R_x and R_y represent the usual rotation matrices representing a rotation by an angle ψ about the x–axis and the y–axis, respectively and R represent the 3×3 rotation matrix that takes body coordinates to spatial coordinates, then

$$F = \left(\begin{matrix} 0 & 0 \\ 0 & 0 \\ -\frac{5}{2}RR_x\hat{e}_3 & -\frac{5}{2}RR_x\hat{e}_3 \\ Q^{-1}((R_x\hat{e}_3) \times \hat{e}_2) & Q^{-1}((-R_x\hat{e}_3) \times \hat{e}_2) \\ 0 & 0 \end{matrix} \right.$$

$$\left. \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{5}{2}RR_y\hat{e}_3 & \frac{5}{2}RR_y\hat{e}_3 & 0 \\ Q^{-1}((R_y\hat{e}_3) \times e_1) & (Q^{-1}(-R_y\hat{e}_3) \times e_1) & 0 \\ 0 & 0 & 1 \end{matrix} \right)$$

and $v = (v^1, \dots, v^4, u)^T$. We will refer to the first four columns in the matrix on the right–hand side of the equation for F as $h_1, \dots, h_4$, the fifth column as g and the first column on the right–hand side of Equation 6.8 as f, to notationally correspond to Equation 6.1, *i.e.*, the thruster forces, v_j are restricted to be non–negative, and the thruster rotation angle, ψ can be either positive or negative.

First we must check that the system satisfies the LARC. Tedious calculations show that the following collection of vector fields spans T_xM everywhere except for the parameterization singularity at $\phi_2 = \frac{\pi}{2}$:

$$\{h_1, h_2, h_3, g, [g, h_1], [g, h_2], [h_1, f], [h_2, f], [h_3, f],$$
$$[g, h_3], [[g, h_1], f], [[g, h_2], f], [[g, h_3], f]\}\,.$$

Now, we note that the hypothesis of Proposition 6.2 expressed by Equation 6.5 is satisfied because

$$\sum_j h_j(x) = 0, \quad \forall x \in M.$$

The LARC is satisfied by brackets with total degree $\leq 3 + \epsilon$, so we need to show that all bad brackets with total degree $\leq 3 + \epsilon$ are spanned by brackets with lower total degree. Note, that the only bad bracket with one element is $f(x)$. However, $f(x) = 0$ if $\dot{X} = \dot{\Phi} = 0$. For brackets with three elements, we note that any bad bracket must have zero or two occurrences of the vector field g. If there are zero g's, there must be one or more h's (since $[f, [f, f]] = 0$). If there is only one h, then it is not a bad bracket. If there are two or more h's, then the total degree of the bracket is greater than $3+\epsilon$. If there are two occurrence of g, then there must either be two g's and one h_i or two g's and one f. In the first case, since there is one h_i, that bracket is not a bad bracket. In the second case, we note that in this example $[g, f](x) = 0$, so that bad bracket can be written as a linear combination of lower order elements. Therefore the system is STLC from any position with zero velocity.

We also verify by way of simulation that the controllability properties of this system. The following graphs are intended to illustrate that, after a sequence of control inputs, and possibly after a complicated series of gyrations, the system, to leading order, has undergone a net motion in a particular direction. We only present results for motion in two directions, but note that it is possible to do so for all directions in the 13–dimensional phase space.

For example, consider motion in the x–direction. We note that $\dot{x} = \frac{1}{2}[[g, h_1], f] - \frac{1}{2}[[g, h_2], f]$. Figure 6.2 illustrates a sequence of motions that the system may un-

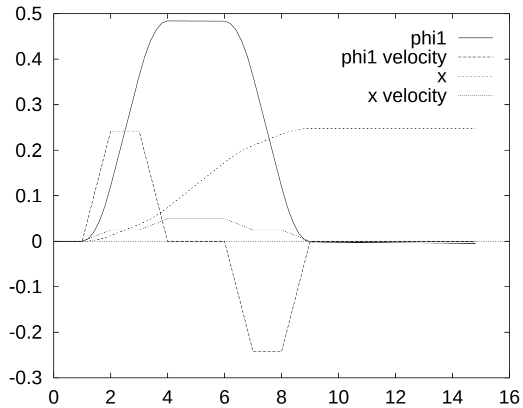


Figure 6.2. Control inputs for x direction.

dergo to move in the x–direction. Note that the system is momentarily displaced in various states other than purely the x–direction; however, the end result is pure displacement in the x–direction.

Also, consider motion in the ϕ_3 direction. Note that $\dot{\phi}_3 = \frac{1}{5}([[g, h_1], f] + [[g, h_2], f])$. Figure 6.3 illustrates a sequence of motions that the system may undergo to move in the ϕ_3–direction. Again, note that the system is momentarily displaced in various states other than purely the ϕ_3–direction; however, the end result is pure displacement in the ϕ_3–direction.

Rather than specifically illustrate how we obtained the sequence of control inputs used for the above simulations, we will discuss the heuristic synthesis technique we used in more general terms. This technique is merely presented to illustrate the means by which we obtained the control inputs for the above examples, and is *not* presented as part of a rigorous theory. (A possible rigorous approach to this synthesis problem would be to extend the averaging results of Leonard (1994) to the context of the problem we are considering.)

Recall that if we denote the flow of a vector field f at time t starting at a point x_0 by $\phi_t^f(x_0)$, we can write

$$\phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(x_0) = x_0 + \epsilon^2[g_1, g_2] + \mathcal{O}(\epsilon^3).$$

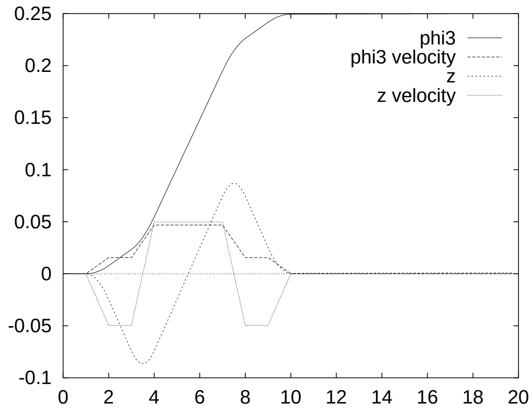


Figure 6.3. Control inputs for ϕ_3 direction.

Thus to "flow" in the direction of a Lie bracket (to leading order), we simply modulate the control inputs associated with two vector fields to execute the sequence of flows illustrated.

Now consider, for example, the third–order bracket $[f, [f, g]]$. Writing this in terms of its approximation by flows we have

$$[f, [f, g]] \sim \phi_\epsilon^{-f} \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^{f} \circ \phi_\epsilon^{g} \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^{g} \circ \phi_\epsilon^{f} \circ \phi_\epsilon^{f}.$$

All the $-f$ terms appearing throughout the above equation are clearly problematic. However, it is actually possible to rewrite the bracket in a manner that will allow it to be executed. First, consider

$$[f, [f, g]] = [-f, [-f, g]] \sim \phi_\epsilon^{f} \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^{g} \circ \phi_\epsilon^{f} \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^{f} \circ \phi_\epsilon^{g} \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^{-f}. \tag{6.9}$$

Now, the first two $-f$ terms do not affect the flow since we assume that $f(x_0) = 0$. There still is one remaining $-f$ flow. However, we note that it corresponds to a set of four flows which are approximating the bracket $[g, -f]$. However, $[g, -f] = [-f, -g]$ and $[-f, -g] \sim \phi_\epsilon^{g} \circ \phi_\epsilon^{f} \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^{-f}$, and when this is substituted into Equation 6.9, we have

$$[f,[f,g]] = [-f,[-f,g]] \sim \phi_\epsilon^g \circ \phi_\epsilon^f \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^f \circ \phi_\epsilon^{-g} \circ \phi_\epsilon^f \circ \phi_\epsilon^g \circ \phi_\epsilon^{-f} \circ \phi_\epsilon^{-f},$$

and so the composed flow $\phi_\epsilon^{-f} \circ \phi_\epsilon^f = 0$, and thus has no effect on the net flow. Finally, to flow along the negative direction of an integral curve of one of the h_i, we simply flow along the positive direction of the vector field $\sum_{j \neq i} h_j$.

These observations allowed us to determine sequences of control inputs which produced displacements in all 13 states of the system, similar to the results illustrated in Figures 6.2 and 6.3

6.3 Unilateral Stratified Systems

Extending the unilateral result to the stratified case is straight–forward. One way to interpret Proposition 6.1 is as follows: the Lie algebra rank condition specifies the dimension of the reachable set of the system and the good/bad brackets requirement tells whether the system is controllable, as opposed to simply accessible. (A system is accessible if the LARC is satisfied, meaning that the reachable set is open. However, this does not mean that the starting point is in the interior of the reachable set.)

A straight–forward extension of the controllability test for a nested sequence of strata would retire that the good/bad bracket test be satisfied on *each* stratum.

Proposition 6.3 *If there exists a nested sequence of submanifolds*

$$x_0 \in S_p \subset S_{(p-1)} \subset \cdots \subset S_1 \subset S_0,$$

such that the associated involutive distributions satisfy

$$\sum_{j=0}^{p} \overline{\Delta}_{S_j}|_{x_0} = T_{x_0}M$$

and on each stratum in the nested sequence, the requirements of either Proposition 6.1 or Corollary 6.2 are met in a neighborhood of x_0 in each stratum, then the *system is STLC (in the topology of S_0) from x_0.*

Proof: By standard accessibility theory (*e.g.*, Nijmeijer and der Schaft (1990)), the dimension of the reachable set is the dimension determined by the LARC. The requirements of Proposition 6.1 specify whether this reachable set contains a neighborhood of the starting point, thus making the system controllable.

If the conditions of Proposition 6.1 are met on the bottom stratum, S_p, the the dimension of the reachable set on S_p, denoted N_p is equal to $\dim\left(\overline{\Delta}_p\right)$. By assumption, the system can move off of S_p into S_{p-1}. Thus, from every point in N_p, the system can reach an $\dim\left(\overline{\Delta}_{p-1}\right)$ submanifold of S_{p-1}. The union of all these submanifolds, denoted, N_{p-1} is a $\dim\left(\overline{\Delta}_p + \overline{\Delta}_{p-1}\right)$ submanifold of S_{p-1} because, both $\overline{\Delta}_p$ and $\overline{\Delta}_{p-1} \in TN_{p-1}$ by construction.

Now proceed by induction. Assume we have constructed a $\dim\left(\sum_{i=k}^{p} \overline{\Delta}_i\right)$ manifold, $N_k \subset S_k$. At each point in N_k, the reachable set in S_{k-1} is an $\dim\left(\overline{\Delta}_{k-1}\right)$ submanifold of S_{k-1}. The union of these submanifolds, denoted N_{k-1} is a $\dim\left(\sum_{i=k-1}^{p} \overline{\Delta}_i\right)$ submanifold of S_{k-1}, because, by construction, $\overline{\Delta}_i \in TN_{k-1}$ for $k = k-1, \ldots, p$. Continuing in this manner gives the desired result. ■

Note that one limitation of Proposition 6.3 is that the additional good/bad bracket condition must be satisified on *each* stratum.

6.4 Summary

This chapter presented results for systems with unilateral inputs. This type of problem arises naturally in stratified systems because one aspect of physical contact is that the normal force between two objects in contact must be positive. One contribution of this chapter was a simpler reformulation of the results of Sussmann (1987) to apply specifically to the stratified case. We also extended this result to the stratified case, and illustrated the application of this controllability test with a very simple example.

Chapter 7

Conclusions

In this dissertation we have extended several standard controllability tests to the case where the configuration space for the control system is stratified. Such a stratified structure provides a means to model many physical systems with governing equations which are discontinuous across subsets of the state space. The general philosophy underlying these extensions was to exploit the particular structure of stratified configuration spaces, which, loosely speaking, allowed us to simultaneously consider the equations of motion for the system on each strata. The examples contained herein illustrated both the steps involved in applying the tests as well as their ease of use.

We also provided a general means to solve the trajectory generation problem for certain types of legged robotic systems and the simulations indicate that the approach is rather simple to apply. The method is independent of the number of legs and is not based on foot placement principles. For a given legged robot mechanism, the deployment of a specifically tuned leg-placement-based algorithm may lead to motions which use fewer steps or results in less tracking error. However, for the purposes of initial design and evaluation of a legged mechanism, our approach affords the robotic design engineer an automated way to implement a realistic trajectory generation scheme for a quasi-static robot of nearly arbitrary morphology. More importantly, we believe that our approach provides an evolutionary path for future research and generalizations. Clearly, this general framework also encompasses other types of systems whose configuration space is similarly stratified. An obvious example would be a robotic grasping problem, where we wish to reorient an object grasped by a robot hand by used of repeated finger repositioning.

The final topic in this dissertation was unilateral controllability. We have presented and illustrated the application of a controllability test for control systems which may have inputs constrained to be non–negative. Although technically difficult to prove, this result is relatively straightforward to use in applications. Roughly, we have treated the vector fields corresponding to the constrained inputs in a manner similar to that for the drift term.

Several avenues of potentially fruitful further work could be based upon the results in this thesis. First, the results in Chapters 4 and 5 are restricted to driftless control systems. Although a much harder problem, controllability tests for smooth systems with drift exist, Sussmann (1987), and could potentially be extended to stratified configuration spaces. One difficulty with simply extending that test is that the test only provides a sufficient condition for controllability. In the case where there is a large number of strata, one is faced with the prospect of the need to satisfy a sufficient condition a large number of times. This is problematic to the extent that sufficient conditions are, generally, too restrictive, in which case, if the test needs to be satisfied multiple times, the restrictive nature of the sufficient conditions are similarly multiplied. Clearly, necessary and sufficient conditions would be preferable, and would provide a more practical basis from which controllability tests for stratified systems could be derived.

Unfortunately, an even more fundamental limitation also exists for systems with drift. A basic hypothesis of the main result in Sussmann (1987) is that controllability is only defined at *equilibrium points*. Therefore, to extend the general results for systems with drift to the stratified case would require that the point of interest be an equilibrium point in all strata (or at least enough strata to satisfy a Lie algebra rank condition type test). Two practical problems illustrate that, at least in the legged robotic context, a point that is an equilibrium point on one stratum will not necessarily be an equilibrium point on another stratum.

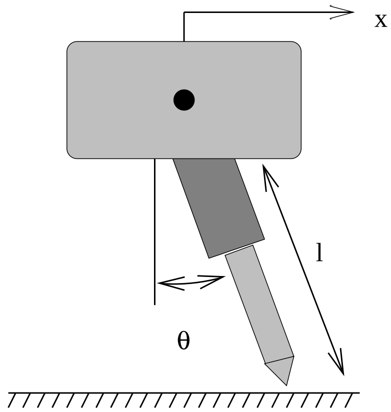


Figure 7.1. Hopping robot.

Figure 7.1 shows the robotic leg used to illustrate the stratified controllability results in Chapter 4. However, now we consider it as a dynamic system, *e.g.*, as a hopping robot, so that when the foot is not in contact with the ground, gravity causes the robot to accelerate downward. If the foot is in contact with the ground and the robot is vertical, then it is at an equilibrium point. However, if the foot is not in contact with the ground, there are no equilibrium points, and so the general controllability results due to Sussmann (1987) are not applicable, and thus, extending those results to the stratified case are problematic.

Another example further illustrates this problem. Consider the biped robot in Figure 7.2, built in the Caltech robotics laboratory by Dr. Shuuji Kajita. Also illustrated is a depiction of the robot's stratified configuration space. As with the hopping robot, if both feet are in contact with the ground and the robot is vertical, then the system is at an equilibrium point. However, if the robot lifts either foot out of contact with the ground, then in either stratum corresponding to one of the two feet out of contact with the ground, the system is no longer in a neighborhood of an equilibrium point. This is because the center of mass of the robot is located vertically above a point between the two feet.

Some progress has been made with stratified dynamic systems with regard to a particular model called the *shimmying wheel* (Goodwine and Stépán (1997)). Un-

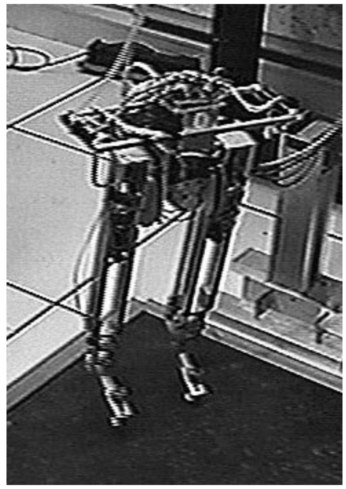


Figure 7.2. Caltech biped.

fortunately, so far the results are specific to the model. Specifically, the shimmying wheel is a model intended as a simplified model of systems such as an aircraft landing gear structure. A common undesirable feature of such systems is that the rolling behavior can be unstable. Since the nonholonomic "rolling without slipping" constraint is imposed by friction, for some trajectories the system will switch from rolling to skidding, which increases the dimension of the phase space by two. The rolling without slipping constraint defines a codimension two submanifold of the phase space, and switches between pure rolling behavior and skidding are switches on and off of the submanifold. Hence, the phase space of this system is a stratified space. One effective approach to control this system is to design a stabilizing controller for the rolling state, and allow the natural dissipative nature of the skidding system drive the system back to the rolling substrata. Due to the complexity of the model, however, the results are primarily numerical, and it is not clear how to generalize them. Some analytic stability results specific to this model were obtained by Žefran and Burdick (1997).

Another avenue of future work would be to make more concrete connections with recent results for hybrid systems. The stratified systems considered here clearly are a subset of all hybrid systems. However, we were able to exploit the particular geometry of the state space to formulate the main controllability results which will not be present in a generic hybrid system. One possible way to do this would be to attempt to generalize the stratified structure present in the systems we consider to encompass a broader class of "switching" systems common in hybrid control.

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