GEOMETRIC ANALYSIS AND CONTROL OF UNDERACTUATED MECHANICAL SYSTEMS

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Abstract

by

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Geometric analysis and control of underactuated mechanical systems is a multidisciplinary field of study that overlaps diverse research areas in engineering and applied mathematics. These areas include differential geometry, geometric mechanics and nonlinear control theory. Many challenging applications exist such as robotics, autonomous aerospace and marine vehicles, multi-body systems, constrained systems and legged locomotion. These systems are characterized by the fact that one or more degrees of freedom are unactuated. The unactuated nature gives rise to many interesting control problems which require fundamental nonlinear approaches. This thesis contains contributions to modeling, analysis and algorithm design for underactuated mechanical systems.

We provide two novel differential geometric formulations of the nonlinear control models for underactuated mechanical systems. The key feature of each formulation is the partitioning of the equations of motion into those associated with the actuated and unactuated dynamics. Both formulations are constructed using control forces and the kinetic energy metric inherent in the classic problem formulation. Interestingly, each formulation gives rise to an intrinsic vector-valued symmetric bilinear form that can be associated with an underactuated mechanical control system.

The first formulation models an underactuated mechanical system evolving on an affine foliation of the tangent bundle. The affine foliation decomposes the velocity curve of the underactuated system into affine and linear components. We show that the affine component represents the unactuated velocity states and the linear component represents the actuated velocity states. In this framework, the ability to move from leaf to leaf in the affine foliation is characterized by the definiteness of the intrinsic symmetric bilinear form.

The second formulation utilizes two linear connections. Specifically, we introduce the actuated and unactuated connections which provide a coordinate-invariant representation of the actuated and unactuated dynamics. We show that feedback linearization of the actuated dynamics gives rise to a control-affine system whose drift vector field is the geodesic spray of the unactuated connection. We call this control-affine system the geometric normal form for underactuated mechanical systems. The geometric normal form is the starting point for our reachability analysis and motion algorithms for mechanical systems underactuated by one.

Our main analytical contribution is a unique characterization of the set of reachable velocities from an arbitrary initial configuration and velocity (possibly nonzero velocity) for mechanical systems underactuated by one control. The characterization is computable and dependent upon the definiteness of the intrinsic symmetric bilinear form. The proof of the existence of a control law that will drive a mechanical system underactuated by one control from velocity to velocity is constructive. Therefore, our main result gives rise to a velocity to velocity motion planning algorithm. The algorithm is applied to various examples of nonlinear mechanical systems underactuated by one control.

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CHAPTER 1

INTRODUCTION

Mechanics and control theory are two well developed fields of study. However, their intersection still provides a rich and challenging research area commonly referred to as GEOMETRIC CONTROL OF MECHANICAL SYSTEMS. Underactuated mechanical systems or mechanical control systems with fewer actuators than degrees of freedom form a large and important subclass. Whenever fewer input forces are available than degrees of freedom, various control questions arise. The linear approximation around equilibrium points may, in general, not be controllable. These systems require fundamental nonlinear approaches. The areas of application of control theory for underactuated mechanical systems are diverse and challenging. Such areas include autonomous aerospace and marine vehicles, robotics, mobile robots, constrained systems and legged locomotion. The formalism of linear connections and distributions on a Riemannian manifold provides an elegant framework for modeling, analysis and control Lewis [42].

1.1 Motivating Example

As a concrete example, take the planar ice skater illustrated in Figure 1.1. The schematic drawing illustrates the kinematics and actuator locations of the model. Note that each leg is composed of two links which are connected by

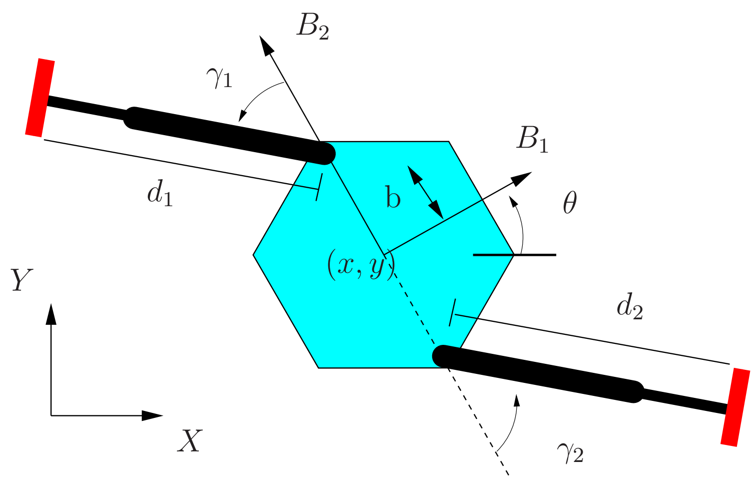

Figure 1.1. A schematic of the planar ice skater.

a translation joint at the knee and a pin joint at the hip. The foot is an ice skate which is constrained to the plane in such a way that prohibits motion of the foot perpendicular to the blade. Technically speaking, the skate blade forms a NONHOLONOMIC constraint with the plane and gives rise to interesting geometries that can be modeled using the affine connection formalism.

A single actuator capable of generating torque in both the clockwise and counterclockwise directions is placed at each pin joint or hip. Another set of linear actuators are placed at each translation joint or knee. The planar ice skater has five degrees of freedom and only four actuators. This is an example of an UNDERACTUATED control system. Unactuated states give rise to many interesting control questions. For instances, it is not immediately clear whether the moving ice skater can be "stopped" using the limited control authority. If it cannot be stopped, then the set of reachable velocities does not include zero velocity. *In this, and other underactuated mechanical systems, existing geometric control theory does not provide a general test for stopping and more generally speaking, the set* *of reachable velocities from a nonzero velocity is not well understood.* The modern development of geometric control of mechanical systems has been limited, for the most part, to the zero velocity setting. Yet the underlying mathematical structure is that of second-order dynamics where the state of the system is defined by a configuration and velocity. Theoretical results that are limited to zero velocity states do not provide an adequate characterization of the behavior of mechanical systems and limit the development of motion planning algorithms for the larger class of hybrid or stratified nonlinear mechanical systems Bullo and Zefran [12], Bullo and Zefran [14], Zefran et al. [70], Goodwine and Burdick [28], Goodwine and Burdick [27].

1.2 Statement of Contribution

The fundamental approach of this thesis is to exploit the inherent geometric structure for the purpose of characterizing the set of reachable velocities for an underactuated mechanical system. This thesis is motivated by the following two open research questions:

1. Starting from an arbitrary configuration and velocity, what new velocities can be reached?

2. If we can characterize the set of reachable velocities, is it possible to design a velocity to velocity algorithm?

In general, the set of reachable states from states with nonzero velocity is not currently well understood, but preliminary results can be found Martinez and Cortes [47], Zefran et al. [70]. Our strategy is to partition the equations of motion associated with an underactuated mechanical system into the actuated and unactuated dynamics. This partitioning gives rise to an intrinsic symmetric bilinear form that represents the coupling between the actuated and unactuated velocity states. We use the definiteness of the intrinsic symmetric bilinear form as sufficient conditions for a general test for velocity reachability. We focus on a constructive solution that naturally gives rise to a velocity to velocity algorithm. This thesis contains contributions to modeling, analysis and algorithm design for underactuated mechanical systems.

We provide two novel differential geometric formulations of the nonlinear control models for underactuated mechanical control systems. It is well-known that the choice of representation for mechanical control systems can be a key step in confronting any problem. For example, mechanical control systems with constraints can be described by a coordinate-invariant affine connection Lewis [41]. The coordinate-invariant model is elegant and provides a natural link to previous results for unconstrained mechanical systems Lewis and Murray [44]. However, the explicit representation of the so-called *constrained affine connection* requires cumbersome differentiation of a tensor field. An alternative representation was developed a few years later Bullo and Zefran [13]. This simplification led to a more efficient method of computing and ultimately interpreting the Christoffel symbols of the connection. The Christoffel symbols play an important role in computing symmetric products which are used to characterize the structure of the accessibility distribution at zero velocity. The accessibility distribution can then be used to characterize the reachable set of velocities and configurations.

The key feature shared by both of our formulations is the partitioning of the equations of motion into the actuated and unactuated dynamics. Both formulations are constructed using control forces and the kinetic energy metric inherent in the classic problem formulation. Interestingly, each formulation gives rise to an intrinsic vector-valued quadratic (symmetric bilinear) form that can be associated with an underactuated mechanical control system. The following subsections detail each contribution of this thesis.

1.2.1 Affine Foliation for Underactuated Mechanical Systems

We develop an alternative representation of the equations of motion for the general class of underactuated mechanical systems by constructing an affine foliation of the tangent bundle. We use the Riemannian metric along with the control forces to construct an orthonormal frame on the tangent bundle using the input distribution $\mathcal{Y}$ and the Riemannian metric $\mathbb{G}$ included in the basic problem formulation. Though Riemannian geometry is a classic technique in modeling underactuated mechanical systems, affine foliations and affine subbundles are not. In general, we think of an underactuated mechanical system as moving from leaf to leaf in the affine foliation. Each leaf in the affine foliation is parameterized by a family of one-forms referred to as the affine parameters. We will show that the affine parameters represent the *unactuated* velocity states. Each leaf in the affine foliation can also be associated with an affine subbundle. The linear part of the affine subbundle is parameterized by a second family of one-forms referred to as the linear parameters. We will show that the linear parameters represent the *actuated* velocity states. We demonstrate that the characterization of the affine parameters along system trajectories correspond to the unactuated dynamics while the characterization of the linear parameters along system trajectories correspond to the actuated dynamics. Our modeling leads to two important observations. First, the actuated dynamics can be linearized using partial feedback linearization. This creates a linear subsystem that we will use to influence the unactuated velocity states. Second, the unactuated dynamics give rise to an intrinsic vector-valued quadratic form. The quadratic form characterizes the influence the actuated velocity states have on the unactuated velocity states. Interestingly, the quadratic structure has also been shown to be a novel way of characterizing dynamic singularities in mechanisms which has implications in the field of mechanism design Goodwine and Nightingale [29].

This intrinsic vector-valued quadratic form can be associated with large class of underactuated mechanical systems. A significant advantage of this characterization is that the definiteness of the symmetric form is independent of the choice of basis for the input distribution. In addition, it has been observed that vector-valued *quadratic* forms arise in a variety of areas in control theory which has motivated a new initiative to understand the geometry of these forms Bullo et al. [15]. Several efforts have been made to obtain conditions in the zero velocity setting from properties of a certain intrinsic vector-valued quadratic form which does not depend upon the choice of basis for the input distribution Bullo and Lewis [9], Hirschorn and Lewis [31]. A significant advantage of this formulation is that it is still valid for underactuated mechanical systems with linear velocity constraints. Often times, the most interesting geometries for underactuated mechanical systems arise when linear velocity constraints exist. Our unique representation provides the foundation for our velocity reachability analysis and constructive velocity to velocity algorithm for mechanical systems underactuated by one.

1.2.2 Partitioning Connections for Underactuated Mechanical Systems

A common starting point for treatments of underactuated mechanical systems is to assume that there exists a set of coordinates $q = (q^1, \ldots, q^n)$ such that the local expression for the governing equations of motion are

$$
\begin{aligned}
M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + F_1(q, \dot{q}) &= B(q)u && (1.1) \\
M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + F_2(q, \dot{q}) &= 0 && (1.2)
\end{aligned}
$$

where $q_1 \in \mathbb{R}^m$ is the first m-components of $q \in \mathbb{R}^n$ and represents the actuated degrees of freedom, $q_2 \in \mathbb{R}^{n-m}$ is the remaining $n-m$-components of $q \in \mathbb{R}^n$ and represents the unactuated degrees of freedom, and $M_{ij}(q)$ represents $n \times n$ inertia matrix Spong [59], Reyhanoglu et al. [57], Olfati-Saber [53]. The basic idea is that only the first m degrees of freedom are actuated. Equation (1.1) represents the actuated dynamics, while Equation (1.2) represents the unactuated dynamics. A known limitation of this formulation for underactuated mechanical systems is that it requires that the input codistribution to be integrable Bullo and Lewis [10]. It is not always physically valid to assume that the input codistribution is integrable for a general underactuated mechanical system. Many of the mechanical systems considered in this body of research have a single actuator which always gives rise to integrable codistributions. For example, the forced planar rigid body and various constrained systems considered in this thesis do not satisfy this assumption.

This thesis contains an alternative formulation for underactuated mechanical systems that utilizes partitioning connections. We introduce two linear connections that provide a coordinate invariant representation that partitions the actuated and unactuated dynamics. Our formulation does not require that the input codistribution be integrable, therefore can be viewed as a generalization of the partitioning used in existing literature on underactuated mechanical systems Spong [59], Reyhanoglu et al. [57], Olfati-Saber [53]. We show that feedback linearization of the actuated dynamics gives rise to a control-affine system whose drift vector field is the geodesic spray of the unactuated connection associated with unactuated dynamics. We call this control-affine system the geometric normal form for underactuated mechanical systems. The geometric normal form is the starting point for our reachability analysis and motion algorithms for mechanical systems underactuated by one. Similar to the affine foliation formalism, the unactuated connection gives rise to an intrinsic vector-valued symmetric bilinear (quadratic) form. Again, a significant advantage of the partitioning connections is that the formulation is still valid for the extended class of underactuated mechanical systems with linear velocity constraints.

1.2.3 Characterization of Reachable Velocities for Mechanical Systems Underactuated by One

One of the fundamental problems in control theory is determining the set of states reachable from an initial state. Problems of this nature are commonly referred to as controllability. A detailed review of controllability and existing results for underactuated mechanical systems can be found in Section 1.3. The matter of determining the general structure of states reachable from a nonzero velocity state is currently unresolved Bullo and Lewis [10], Cortes et al. [21], Bullo and Zefran [14]. We provide a general test for mechanical systems underactuated by one control that depends on the definiteness of an intrinsic symmetric bilinear form that determines the system's ability to reach a specified velocity from a nonzero velocity state. In other words, we provide a sufficient condition dependent on the definiteness of a symmetric bilinear form for velocity to velocity motion planning. A significant advantage of our result is that it applies to mechanical systems underactuated by one control with linear velocity constraints. Underactuated mechanical systems with linear velocity constraints give rise to nontrivial geometries that are challenging to analyze and control. Here is an informal statement of our main result.

Theorem 1.2.1 (Reachability for Mechanical Systems Underactuated by One Control)**.** *Consider a mechanical system underactuated by one control (possibly with linear velocity constraints) whose intrinsic symmetric bilinear form is indefinite at the given configuration and velocity. For any* $\epsilon, \alpha, \Delta > 0$ *and any target velocity there exists a piecewise control law that will drive the system to any* ϵ*-ball of the target velocity in time less than* Δ *while staying within an* α*-ball of the initial configuration.*

Though our main result can be applied to nonzero velocity targets, we also consider the problem of reaching rest which can be viewed as a form of stabilization. This test is applicable to both constrained and unconstrained systems. Here is the statement of our corollary for stopping.

Corollary 1.2.2 (Stopping for Mechanical Systems Underactuated by One Control)**.** *Consider a mechanical system underactuated by one control (possibly with linear velocity constraints) whose intrinsic symmetric bilinear form is indefinite at the given configuration and velocity. For any* $\epsilon, \alpha, \Delta > 0$ *there exists a piecewise control law that will drive the system to any* ϵ*-ball of rest in time less than* Δ *while staying within an* α*-ball of the initial configuration.*

Our theoretical results are useful for two reasons. First, such results are necessary conditions for velocity to velocity motion algorithms. In terms of stopping, if zero velocity is not contained in the set of reachable velocities, then it is impossible to specify a control law that will drive the system to rest. Second, these results are useful design tools which provide constructive strategies for actuator assignment and help to make the control scheme robust to actuator failure Tafazoli [65]. The task of actuator assignment is always a balance between the sophistication of the system design and the associated complexity of the controller. For example, a system which is fully actuated requires a simple control scheme to drive it to rest. In contrast, if the system is underactuated even by just one control, a control scheme must take into account the underlying geometry or nonlinearities of the geometric model. Such a control scheme is theoretically challenging due to nonzero drift which indicates a component of the dynamics that is not directly controlled or unactuated.

There has been preliminary work done on stopping underactuated mechanical systems. It has been shown that the *roller racer* and the *robotrikke* could not be stopped given a single control input from an arbitrary initial configuration and velocity Krishnaprasad and Tsakiris [37], Chitta et al. [18]. It is important to note that the existing investigations into the roller racer and robotrikke have focused on a particular instance of a mechanical system underactuated by one control and cannot be easily extended to different systems in the same class. Further, we show that given certain conditions on the symmetric bilinear form and the relationship between the initial unactuated velocity state and the targeted velocity state that the roller racer can be driven arbitrarily close to rest.

It is true that nonlinear mechanical systems underactuated by one control is the simplest case next to fully actuated systems. However, these systems are not feedback linearizable and thus not amendable to standard techniques in control theory Isidori [33]. The literature on the analysis and control of mechanical systems underactuated by one control is vast. Such systems include underactuated ships Do [23], gymnastic robots Xin and Kaneda [68], the Harrier which is a planar vertical/short take-off and landing (V/STOL) aircraft in the absence of gravity Sastry [58], a hovercraft type vehicle Tanaka et al. [66] and a planar rigid body with two thrusters moving on a flat horizontal plane M'Closkey [48].

1.2.4 Velocity to Velocity Algorithm for Mechanical Systems Underactuated by One

The problem of general motion planning for underactuated mechanical systems is still not well understood Martinez and Cortes [47], Bullo and Lewis [10]. Due to the challenging nature of these problems, many of the existing results have been limited for example to gait generation algorithms applicable only to the specific systems Ostrowski et al. [55], Chitta and Kumar [17], Chitta et al. [18], configuration to configuration algorithms with zero-velocity transitions between feasible motions for specific systems Bullo and Lewis [8], Bullo and Zefran [14] and numerically generated optimal trajectories J.P. Ostrowski and Kumar [56]. In contrast, we demonstrate the utility of our alternative formulations and symmetric bilinear form by constructing a general velocity to velocity algorithm. The algorithm is a natural consequence of the constructive proof of our main result on velocity reachability. The use of the intrinsic symmetric form as a constructive tool for motion algorithms for underactuated mechanical systems in this thesis is a new contribution to existing control literature, although preliminary results can be found in Nightingale et al. [51], Nightingale et al. [50], Nightingale et al. [49]. Illustrative examples of the control algorithm can be found in Chapter 6.

1.3 Literature Review

This thesis has been inspired by a differential geometric approach to control theory. Here we review the role that geometry has played in the development of control theory and the influence it has had on modeling, analysis and control of mechanical systems.

In general, control theory is the study of the manipulation of a dynamical system in order to obtain a desired objective. The dynamical laws governing these systems are not fixed as in classical physics, rather they depend on parameters referred to as controls. Roughly speaking, a "mechanical control system" is a system of second-order differential equations defined on the tangent bundle of the configuration manifold in which the control function appears as parameters. An important geometric observation is that the natural dynamics (geodesic spray) and each control (external force) determines a vector field on the tangent bundle, and thus a mechanical control system can be viewed as a family of vector fields on the tangent bundle *some of which* are parameterized by controls. A trajectory of such a system is a continuous curve made up of finitely many segments of integral curves of the vector fields in the family.

The formalism of affine connections and distributions (geometric) have been shown to provide an adequate geometric framework for modeling, analysis and control given zero initial velocity Bullo and Lewis [10]. If the initial velocity of the control system is zero, then we may associate the family of vector fields with a distribution. The distribution can then be used to derive controllability results.

Controllability is a fundamental problem in control theory. Many design methodologies rely on some hypotheses that concerns controllability Bullo and Murray [11], Bullo [7]. The problem of controllability is essentially one of describing the nature of the set of states reachable from an initial state. The development of this theory can be decomposed into two characteristics. The first characteristic is commonly referred to as ACCESSIBILITY, which means that the reachable set has a nonempty interior. Sussmann and Jurdjevic [62] describes the fundamental approach to accessibility for nonlinear control systems. The characteristic of CONTROLLABILITY extends accessibility by further asking that the initial state lies in the interior of the reachable set. The works of Sussmann, beginning with Sussmann [60] and ultimately the general results of Sussmann [64] are key contributions to controllability.

Most of the literature on geometric control of mechanical systems is a hybrid of analytic methods and differential geometric *ideas*. We emphasize ideas because the distinct feature of this approach is the adoption of a *differential geometric point of view* rather than specific structures of differential geometry Sussmann [61]. Though it is the general language and distinctive philosophy of differential geometry that frames the approach of a geometric control theoretician, many of the existing results are arrived at via computations and analytic arguments. In most cases, the analytic results do not have a clear geometric interpretation; however, there does exist a common theme among the exceptions. These analytic results point towards the identification of the smallest invariant subset containing the image of the control system's inputs. The remainder of this section contains a review of key analytic results on controllability and the known limitations of these results. We provide a geometric interpretation when it exists.

In the early 1960's, Kalman [35] challenged the accepted approach to control theory of that period (*i.e.*, Laplace transforms and the frequency-based methods) by showing that the basic control problems could be studied efficiently through the notion of a state of the system that evolves in time according to ordinary differential equations in which controls appear as parameters. Let us consider a LINEAR CONTROL SYSTEM:

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t)$$

where $m \leq n$, $x \in \mathbb{R}^n$ is the state parameter, $u \in \mathbb{R}^m$ is the control parameter, $A : \mathbb{R}^n \to \mathbb{R}^n$ is the system dynamics, and $B : \mathbb{R}^m \to \mathbb{R}^n$ is the control dynamics. It is natural to ask what states can be reached given an initial state $x = 0$. Let us denote the reachable set from $0 \in \mathbb{R}^n$ by $\mathcal{R}(0)$. For linear systems there exists two equivalent answers.

- $\mathcal{R}(0) = \text{span}_{\mathbb{R}}\{[B|AB|\ldots|A^{n-1}B]\}$;
- $\mathcal{R}(0)$ is the smallest A-invariant subspace containing image(B).

The first answer known as matrix controllability was given by Lee and Markus [38]. It is computationally efficient; however, the truthfulness of this result is not obvious. In contrast, the second answer immediately appears "justifiable" and it provides insight into how the components of the control system (A, B) combine to provide the set of reachable points. Let us consider the trivial case when $A = 0$. The reachable set is the image(B). Now consider the nontrivial case when $A \neq 0$. The reachable set is a subspace containing image(B) that is invariant to the system dynamics A. The second answer was derived by Kalman et al. [34].

For linear systems, many of the basic controllability questions have been answered. The matter of providing general conditions for determining the structure of the reachable set for a general nonlinear control system is currently unresolved, however there have been many deep and insightful contributions.

In 1963, Hermann [30] related Chow's theorem [19] to control theory. Let us consider the following DRIFTLESS NONLINEAR SYSTEM:

$$\dot{x}(t) \quad = \quad u^1(t)g_1(x) + \cdots + u^m(t)g_m(x)$$

where $x \in M$ is the state parameter, M is a smooth manifold, $u : \mathbb{R} \to \mathbb{R}^m$ is the control parameter, and $\{g_1, \ldots, g_m\}$ is a family of control vector fields on M. Loosely speaking, the family of vector fields can be associated with a distribution $\mathcal{D}$ on M. A distribution $\mathcal{D}$ on M is a smooth assignment of a subspace $\mathcal{D}_x$, for each $x \in M$, of the tangent space T_xM. Chow's theorem implies that the closure of the distribution $\mathcal{D}$ under the Lie bracket, denoted by $\mathrm{Lie}^{(\infty)}(\mathcal{D})$, is the smallest invariant subspace of the tangent space containing the image($\mathcal{D}$). Provided that the set of inputs u satisfy certain restrictions, the geometric interpretation is that the reachable set is the submanifold $S \subset M$ such that $T_xS = \mathrm{Lie}^{(\infty)}(\mathcal{D})$ for each $x \in M$. The driftless control system is small-time locally controllable if $T_xM = \mathrm{Lie}^{(\infty)}(\mathcal{D})$ for each $x \in M$.

The most general class of nonlinear control systems presented in this thesis is commonly referred to as CONTROL-AFFINE SYSTEMS. The problem of determining controllability for underactuated control-affine systems is difficult. Let us consider the following control-affine system:

$$\dot{x}(t) \quad = \quad f(x) + u^1(t)g_1(x) + \cdots + u^m(t)g_m(x)$$

where $x \in M$ is the state parameter, M is a smooth manifold, $u : \mathbb{R} \to \mathbb{R}^m$ is the control parameter, f is the drift vector field on M, and $\{g_1, \ldots, g_m\}$ is a family of control vector fields on M. The extreme challenge of deriving controllability conditions for this class of nonlinear control systems is a consequence of the **drift vector field**. The drift vector field represents system dynamics that are not parameterized by controls or unactuated dynamics. As mentioned earlier, Sussmann and Jurdjevic [62] have characterized the fundamental approach to accessibility for control-affine systems. It is the case that accessibility for control-affine systems has a geometric interpretation in the context of orbits. In contrast, local controllability for control-affine systems has only been characterized analytically.

Sussmann [64] provides sufficient conditions for small-time local controllability for control-affine systems that follow a Lie series approach which incorporates the ideas of Crouch and Byrnes [22] concerning input symmetries. The formal proof employs the use of free Lie algebras. Note that a detailed statement of these results requires the introduction of a significant amount of notation that the uninitiated reader can expect to devote some time to understanding due to the use of free Lie algebras. There are three well-known limitations to the results by Sussmann [64]:

1. The general sufficient conditions for local controllability of a control-affine system are restricted to an equilibrium point.

2. The general conditions are dependent upon the choice of basis for the input distribution and thus sufficient.

3. The general sufficient conditions for local controllability of a control-affine system gives rise to a geometric growth in the number of elements to test.

Despite these limitations, Sussmann's work [64] on sufficient conditions for small-time local controllability forms the cornerstone of many existing analyzes of mechanical control systems. *In contrast to the vast majority of literature on controllability for underactuated mechanical systems, this thesis is not an application of the results of Sussmann [64].*

Let us consider the following mechanical control system:

$$\Psi'(t) \quad = \quad Z(v) + u^1(t)Y_1^{\mathrm{vlft}}(v) + \cdots + u^m(t)Y_m^{\mathrm{vlft}}(v)$$

where $\Psi \in TM$ is the state parameter, TM is the tangent bundle, $u : \mathbb{R} \to \mathbb{R}^m$ is the control parameter, Z is the geodesic spray of the Levi-Civita connection or drift vector field on TM, and $\{Y_1^{\mathrm{vlft}}, \ldots, Y_m^{\mathrm{vlft}}\}$ is a family of control vector fields on M vertically lifted to TM. A mechanical control system can be identified with a control-affine system on TM, and thus the results of Sussmann [64] on controllability will apply. However, mechanical control systems carry an additional *metric* or *connection* structure which simplifies their analysis. Lewis and Murray [44] study this class of nonlinear control systems because their unique structure had been underexploited in literature. Relying on the results of Sussmann [64], Lewis and Murray [44] provide computable sufficient conditions for SMALL-TIME CONFIGURATION CONTROLLABILITY for a class of mechanical systems. Configuration controllability is strictly concerned with the reachable set of configuration states and not velocity states. Lewis and Murray [44] focus on simple mechanical systems, which forms an important subset of all mechanical systems. Simple mechanical systems are characterized by the Lagrangian equal to the difference between kinetic energy and potential energy. Note that the results obtained by Lewis and Murray [44] inherited the limitations associated with the original work Sussmann [64]. However, Lewis and Murray [44] were able to show that the geometric growth in the number of elements to test can be pruned by using the unique Riemannian or affine connection structure associated with simple mechanical systems. There are two key features associated with the results of Lewis and Murray [44]:

1. The general sufficient conditions for accessibility and small-time local controllability of simple mechanical control systems is limited to initial states with zero velocity.

2. The general conditions are dependent upon the choice of basis for the input distribution and thus sufficient.

These results were extended by Lewis [41] to affine connection control systems with constraints and used to provide a decomposition for affine connection control systems Lewis and Murray [43]. Affine connection control systems form a subclass of simple mechanical systems where the Lagrangian is strictly kinetic energy Bullo and Lewis [10]. Finally, the results of Lewis and Murray [44] have been extended to affine connection control systems with dissipation Cortes et al. [20] and to the larger class of simple mechanical control systems (*i.e.*, nonzero potential) with dissipation Kang et al. [36]. Note that each of these extensions inherit the limitations of the original results Sussmann [64] and are restricted to initial states with zero velocity.

Let us return to the second limitation of Sussmann [64]. It implies that the conditions are not invariant under input transformations. The consequences of the lack of feedback invariance can be seen even in simple examples, where the system can fail the sufficient condition test, but still be controllable. This indicates the need to develop controllability tests independent of the choice of basis for the input distribution. There have been several attempts to sharpen the configuration controllability results using the Riemannian or affine connection structure associated with mechanical systems. Lewis [39] investigated the single-input case from rest, building on previous results for general scalar-input systems Sussmann [63]. However, mechanical control systems with a single-input are special cases.

The results of Lewis and Murray Lewis and Murray [44] provide an analytic description of the reachable set. The geometric interpretation of the reachable set was obtained by Lewis [40] at a later date. Lewis [40] introduces the notion of a GEODESICALLY INVARIANT distribution and provides a product of vector fields (symmetric product) which allows one to test for geodesic invariance in the same way one uses the Lie bracket to test for integrability. A distribution $\mathcal{D}$ is geodesically invariant if and only if $\mathcal{D} \subset TM$ is invariant under the geodesic flow. Geometrically, a geodesically invariant distribution plays the same role in interpreting the reachable set that the "smallest A-invariant subspace containing the image(B)" does for linear control systems. Loosely speaking, the geodesically invariant distribution $\mathcal{D}$ is a distribution on the tangent bundle of the phase manifold and represents all possible velocities that can be reached from rest. The identification of this invariant distribution was the key insight into the geometric interpretation of the reachable set for affine connection control systems.

This thesis is most closely related to the work of Bullo and Lewis [8],Hirschorn and Lewis [31], Tyner and Lewis [67],Hirschorn and Lewis [32],Bullo et al. [15]. These papers mark a shift in literature towards a geometric, rather than analytic, investigation into properties of local controllability. Hirschorn and Lewis [31] study the basic geometric properties of local controllability for control-affine systems. They contend that in a geometric point of view, a nonlinear control system, affine in the controls, can be thought of as an affine subbundle of the tangent bundle of the state space. Further, Hirschorn and Lewis [31] derive geometric conditions dependent upon the properties of the affine subbundle that either ensure or prohibit local controllability. These results are limited to second-order conditions and affine subbundles containing zero velocity. The advantage of this approach, at least for low-order controllability, is that the conditions are independent of the basis representing the input distribution. The controllability results by Bullo and Lewis [8] bear strong resemblance to the more general conditions of Hirschorn and Lewis [31]. However, Bullo and Lewis [8] are able to provide more detail in this case because they restrict their attention to affine connection control systems. They obtain low-order controllability results using a certain intrinsic vector-valued quadratic form that can be associated to an affine connection control system. Additional uses of vector-valued quadratic forms in control theory are outlined by Bullo et al. [15].

1.4 Outline of Thesis

A brief outline of the content of the various chapters is as follows:

Chapter 1. Here we provide a motivating example, statement of the contributions and literature review.

Chapter 2. Here we review necessary tools from differential geometry and Riemannian geometry. We include numerous local coordinate expressions that are required to analyze and numerical simulate specific examples.

Chapter 3. Here we review the formulation of mechanical control systems on Riemannian manifolds.

Chapter 4. Here we present the first modeling contribution of this thesis. We construct an affine foliation of the tangent bundle for underactuated mechanical systems. We use the affine foliation to partition the actuated and unactuated dynamics. We provide a characterization of an underactuated mechanical systems ability to move from leaf to leaf in the affine foliation.

Chapter 5. Here we present the second modeling contribution of this thesis. We construct two partitioning connections for underactuated mechanical systems. We use the two connections to partition the actuated and unactuated dynamics. We also introduce a partial feedback linearization control law that gives rise to our geometric normal form. The geometric normal form serves as a starting point for our reachability analysis and velocity to velocity algorithm.

Chapter 6. Here we present the main analytical contribution of this thesis. We provide a unique characterization of the reachable set of velocities from an arbitrary initial configuration and velocity that depends on the definiteness of an intrinsic symmetric bilinear form. A natural consequence of the constructive proof of our main result is a velocity to velocity algorithm. The algorithm is applied to the forced planar rigid body, roller racer, snakeboard and three link manipulator. Numerical simulations are included to illustrate the velocity to velocity algorithm.

Chapter 7. Here we make concluding remarks and state possible directions of future research.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

This thesis examines mechanical control systems in the context of differentiable manifolds and vector bundles. This chapter contains a review of necessary tools from differential and Riemannian geometry. For an introduction to linear and multilinear algebra see Abraham et al. [2]. For an introduction to Riemannian geometry see Carmo [16], Gallot et al. [24], Boothby [6], Yano and Ishihara [69]. For an introduction to geometric mechanics see Arnold [4], Abraham and Marsden [1] and Oliva [54].

2.1 Differentiable Manifolds

2.1.1 Topological and Differentiable Structure

A n-dimensional **topological manifold** M is a set that is locally homeomorphic to Euclidean space, *i.e.*, there exists a homeomorphism from an open set of M to an open set of $\mathbb{R}^n$. A homeomorphism ϕ_α is a one-to-one map where ϕ_α and its inverse are continuous. A pair (U_α, ϕ_α) is called a **system of coordinates** or **coordinate chart** of M at $q \in M$ where U_α is an open set of M containing q and ϕ_α is a continuous bijection from U_α to $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$. The homeomorphism ϕ_α defined on $U_\alpha \subset M$ is composed of n **local coordinate functions** $(x^1(q), \ldots, x^n(q))$. For the point $q \in M$, the n-tuple $(x^1, \ldots, x^n)$ of $\phi_\alpha(q)$ in $\mathbb{R}^n$ is called the **coordinate of the point** q. The local properties of a manifold can be described by the local coordinate system. We use the coordinate system to write explicit coordinate-dependent expressions even though the coordinate system itself has no geometric significance.

In general, it is not possible to cover the whole manifold M with a single chart. If we need more than one coordinate system $\{(U_\alpha, \phi_\alpha)\}$ to cover M then we require that $\bigcup_\alpha U_\alpha = M$. The collection of opens sets $\{U_\alpha\}$ is called the **open covering** of M. The family of all coordinate charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ is called the **atlas** of M. If we further require that ϕ_α be a smooth bijection that satisfies the usual compatibility condition then the family $\{(U_\alpha, \phi_\alpha)\}$ is called a **differentiable structure**. In other words, if two open sets U_α and U_β in the collection of open sets $\{U_\alpha\}$ overlap, i.e., $U_\alpha \cap U_\beta \neq 0$, then $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ must be a diffeomorphism. The overlap map $\phi_\alpha \circ \phi_\beta^{-1}$ is a diffeomorphism from $\phi_\beta(U_\alpha \cap U_\beta) \mapsto \phi_\alpha(U_\alpha \cap U_\beta)$ if it is a homeomorphism and the map along with its inverse are smooth. A **smooth manifold** M is a topological manifold endowed with a C^∞ differentiable structure. Intuitively, a manifold's differentiable structure measures its *smoothness* and shows how different open sets in an open covering of the manifold are patched together.

2.1.2 Tangent Vector, Tangent Space and Tangent Bundle

Let $C^\infty(M)$ denote the set of all **smooth functions** $f : M \to \mathbb{R}$. Let $\gamma(t)$ be a **smooth curve** through a point $q \in M$ defined by the map

$$\gamma : (-\epsilon, \epsilon) \subset \mathbb{R} \to M$$

where $t = 0$ is mapped to $\gamma(0) = q$. If we restrict f to the smooth curve $\gamma(t)$ then we obtain a differentiable function $f(\gamma(t))$ with respect to the parameter t. The rate of change of the function along the curve at point q is given by

$$\frac{d}{dt} f(\gamma(t))\bigg|_{t=0}.$$

We define the **tangent vector** X_q along the curve $\gamma(t)$ at the point $q \in M$ to be the **linear differential operator** $\frac{d}{dt}|_q$ that acts on functions along a curve on the manifold. Tangent vectors defined this way can be thought of as a generalization of the directional derivative on $\mathbb{R}^n$.

The **tangent space** $T_q M$ to the manifold M is the set of all differential operators $X_q : C^\infty(M) \to \mathbb{R}$ along all curves on the manifold passing through q that satisfy the Leibniz rule and linearity. Note that $T_q M$ is isomorphic to $\mathbb{R}^n$. This implies that there is a well-defined notion of adding two or more tangent vectors that live in the same tangent space or multiplying a tangent vector by a real number. Tangent vectors that live in different tangent spaces cannot be combined or compared in a natural way. This requires us to either define a one parameter Lie transformation group or we must introduce additional geometric structure called a connection.

Let $\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ be the local coordinate functions in the neighborhood $U_\alpha \subset M$ containing q. If we take a curve through point q chosen along the coordinate direction x^i, i.e. $x^i = t$, then the rate of change of a function f along the coordinate curve at point q is

$$\frac{d}{dt} f(x^i(t))\bigg|_{t=0}.$$

We can expand this expression by applying the chain rule to get

$$\frac{d}{dt} f(x^i(t))\bigg|_{t=0} = \frac{\partial f}{\partial x^i}\bigg|_q \frac{\partial x^i}{\partial t}\bigg|_{t=0}.$$

By definition, we have $x^i = t$ which further reduces the expression to

$$\frac{d}{dt} f(x^i(t))\bigg|_{t=0} = \frac{\partial}{\partial x^i}\bigg|_q f.$$

We see that the tangent vector $X_q^i \in T_qU_\alpha$ along the coordinate curve x^i is $\frac{\partial}{\partial x^i}\big|_q$. In fact, the tangent vector $X_q \in T_qU_\alpha$ along an arbitrary curve or direction can be expressed as a linear combination of $\{\frac{\partial}{\partial x^1}\big|_q, \ldots, \frac{\partial}{\partial x^n}\big|_q\}$. The set $\{\frac{\partial}{\partial x^1}\big|_q, \ldots, \frac{\partial}{\partial x^n}\big|_q\}$ is called the **local coordinate frame** or **natural basis** for T_qU_α. Any tangent vector $X_q \in T_qU_\alpha$ can be written $X_q = X^i \frac{\partial}{\partial x^i}\big|_q$ where $X^i \in \mathbb{R}$ are called the **components** of X_q with respect to the local coordinate frame. The local expression for the tangent vector X_q along the curve $\gamma(t)$ is

$$f \mapsto \frac{d\gamma^i}{dt}\bigg|_{t=0} \frac{\partial}{\partial x^i}\bigg|_q f$$

where $\gamma^i(t) = x^i \circ \gamma(t)$. The components $\frac{d\gamma^i}{dt}\big|_{t=0}$ of the tangent vector X_q along the curve $\gamma(t)$ with respect to the local coordinate frame are the **velocity components** of $\gamma(t)$ at $t = 0$.

The **tangent bundle** TM is the disjoint union

$$TM = \bigcup_{q \in M} T_qM$$

of all tangent spaces. The tangent bundle is a $2n$-dimensional manifold, which is locally a product manifold. The coordinate charts (U_α, ϕ_α) on the manifold M

give rise to **natural charts** on the tangent bundle $(TU_\alpha, T\phi_\alpha)$ where

$$TU_\alpha = \bigcup_{q \in U_\alpha} T_q U_\alpha$$

and $T\phi_\alpha : TU_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$. The local expression for $T\phi_\alpha$ is

$$(q, v) \mapsto \left((x^1(q), \ldots, x^n(q)), \left(\frac{\partial x^1(q)}{\partial x^j} v^j, \ldots, \frac{\partial x^n(q)}{\partial x^j} v^j \right) \right)$$

where $(v^1, \ldots, v^n)$ are the components of v with respect to the natural basis for $T_q U_\alpha$. The coordinates for a point $(q, v) = v_q \in TM$ with respect to the natural chart on TM will be denoted by $((x^1, \ldots, x^n), (v^1, \ldots, v^n)) \in \mathbb{R}^n \times \mathbb{R}^n$. The **tangent bundle projection** is the map $\pi_{TM} : TM \to M$ defined by $\pi_{TM}(v_q) = q$ when $v_q \in T_q M$. The local expression for π_{TM} associated with the natural chart on TM is

$$\mathbb{R}^n \times \mathbb{R}^n \ni ((x^1, \ldots, x^n), (v^1, \ldots, v^n)) \mapsto (x^1, \ldots, x^n) \in \mathbb{R}^n.$$

2.1.3 Covector, Cotangent Space and Cotangent Bundle

We define the **differential** of a smooth function f at a point $q \in M$ to be the linear map $df|_q$ that takes a tangent vector $X_q \in T_q M$ to $\mathbb{R}$. The differential of a smooth function $df|q$ is an example of a geometric objected called a **covector** ψ_q. The set of all covectors $\psi_q : T_q M \to \mathbb{R}$ at the point q on M is called the **cotangent space** $T_q^* M$. Let (U_α, ϕ_α) be a coordinate chart on M with the local coordinate functions $(x^1(q), \ldots, x^n(q))$ on $U_\alpha \subset M$. We can take the differential of the coordinate functions at a point $q \in U_\alpha$ to get the covectors $(dx^1|_q, \ldots, dx^n|_q) \in T_q^* U_\alpha$. We say that the set of covectors $\{dx^1|_q, \ldots, dx^n|_q\}$ are the **dual basis** to

$\{\frac{\partial}{\partial x^1}\big|_q, \ldots, \frac{\partial}{\partial x^n}\big|_q\}$ because $dx^j|_q \cdot \frac{\partial}{\partial x^i}|_q = \delta_i^j$ at each point $q \in U_\alpha$ where δ_i^j is the **Kronecker delta**. The Kronecker delta δ_i^j is 1 when i and j are equal, and 0 otherwise. The cotangent space is also isomorphic to $\mathbb{R}^n$. Any covector $\psi_q \in T_q^*U_\alpha$ can be expressed as a linear combination of $\{dx^1|_q, \ldots, dx^n|_q\}$ written $\psi_q = \psi_i dx^i|_q$ where $\psi_i \in \mathbb{R}$ are components ψ_q with respect to the dual basis for $T_q^*U_\alpha$. The local expression for the differential of a smooth function $df|_q$ at a point $q \in U_\alpha$ is

$$C^\infty(M) \ni f \mapsto \left.\frac{\partial f}{\partial x^i}\right|_q dx^i|_q \in T_q^*U_\alpha$$

where $\frac{\partial f}{\partial x^i}|_q \in \mathbb{R}$ is the component of $df|_q$ with respect to the dual basis for $T_q^*U_\alpha$.

The **cotangent bundle** T^*M is the disjoint union

$$T^*M = \bigcup_{q\in M} T_q^*M$$

of all cotangent spaces. The cotangent bundle is a $2n$-dimensional manifold, which is locally a product manifold. The coordinate charts (U_α, ϕ_α) on the manifold M give rise to natural charts on the tangent bundle $(T^*U_\alpha, T^*\phi_\alpha)$ where

$$T^*U_\alpha = \bigcup_{q\in U_\alpha} T_q^*U_\alpha$$

and $T^*\phi_\alpha : T^*U_\alpha \to \mathbb{R}^n \times \mathbb{R}^n$. The local expression for $T^*\phi_\alpha$ is

$$(q, \psi) \mapsto \left((x^1(q), \ldots, x^n(q)), \left(\frac{\partial x^j(q)}{\partial x^1}\psi^j, \ldots, \frac{\partial x^j(q)}{\partial x^n}\psi^j\right)\right)$$

where $(\psi^1, \ldots, \psi^n)$ are the components of ψ with respect to the dual basis for T_q^*M. The coordinates for a point $(q, \psi) = \psi_q \in T^*M$ with respect to the natural chart associated with T^*M will be denoted by $((x^1, \ldots, x^n), (\psi_1, \ldots, \psi_n)) \in \mathbb{R}^n \times \mathbb{R}^n$.

The **cotangent bundle projection** is the map $\pi_{T^*M} : T^*M \to M$ defined by $\pi_{T^*M}(\psi_q) = q$ when $\psi_q \in T_q^*M$. The local expression for π_{T^*M} associated with the natural chart on T^*M is

$$\mathbb{R}^n \times \mathbb{R}^n \ni ((x^1, \ldots, x^n), (\psi_1, \ldots, \psi_n)) \mapsto (x^1, \ldots, x^n) \in \mathbb{R}^n.$$

2.1.4 Vector Field, Lie Derivative and Integral Curve

A **vector field** X on M is a smooth map that associates to each point $q \in M$ a tangent vector $X_q \in T_qM$. We can also think of X on M as a linear differential operator that maps

$$C^\infty(M) \ni f \mapsto X \cdot f \in C^\infty(M).$$

We can pair the differential of a smooth function df with X to get a useful object called the **Lie derivative of a function**. The Lie derivative of f with respect to X is defined by the map

$$q \mapsto df(q) \cdot X(q).$$

Given the local coordinate function $\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ in the neighborhood $U_\alpha \subset M$ containing q, we can define n unique vector fields denoted by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ on U_α using the Lie derivative of the local coordinate functions with respect to these vector fields. We define $\frac{\partial}{\partial x^i}$ to be

$$\mathcal{L}_{\frac{\partial}{\partial x^i}} x^j = \delta_i^j$$

where $i, j \in \{1, \ldots, n\}$ and δ_j^i is the Kronecker delta. At each point $q \in U_\alpha$ these vector fields are linearly independent and give rise to the natural basis for T_qU_α. We can write $X = X^i(q)\frac{\partial}{\partial x^i}$ for functions $X^i(q) \in C^\infty(M)$ called the **components** of X with respect to the chart (U_α, ϕ_α). Further, the local expression for the Lie derivative of a function f with respect to the vector field X denoted by $\mathcal{L}_X f$ in the chart (U_α, ϕ_α) is

$$C^\infty(M) \ni f \mapsto X^i(q)\frac{\partial f}{\partial x^i} \in C^\infty(M).$$

Let $\Gamma(TU_\alpha)$ be the set of all smooth vector fields on $U_\alpha \subset M$ and $\Gamma(T\mathbb{R}^n) \simeq \Gamma(\mathbb{R}^n \times \mathbb{R}^n)$ be the set of all smooth vector fields on $\mathbb{R}^n \times \mathbb{R}^n$. Given the natural chart $(T\phi_\alpha, TU_\alpha)$ on TM, $T\phi_\alpha$ naturally induces a mapping $T\phi_\alpha : \Gamma(TU_\alpha) \to \Gamma(T\mathbb{R}^n) \simeq \Gamma(\mathbb{R}^n \times \mathbb{R}^n)$ given by the expression

$$\Gamma(TU_\alpha) \ni X^i(q)\frac{\partial}{\partial x^i} \mapsto X^i(x^1, \ldots, x^n)\mathbf{e}^i \in \Gamma(\mathbb{R}^n \times \mathbb{R}^n)$$

where the set of vectors $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ is the standard basis for $\mathbb{R}^n$. It follows that $T\phi_\alpha$ takes the set of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ into the standard basis on $\mathbb{R}^n$.

Let $\Gamma(TM)$ be the set of all smooth vector fields on M. The addition of two or more vector fields is well-defined. In addition, there is a well-defined product between two vector fields called the Lie bracket. For any $X, Y \in \Gamma(TM)$, the vector field $[X, Y]$ defined by

$$\mathcal{L}_{[X,Y]}f = \mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f,$$

is the **Lie bracket** of X and Y, or the **Lie derivative of a vector field** Y with respect to X which is also denoted by $\mathcal{L}_X Y$. Given the local coordinate function $\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ in the neighborhood $U_\alpha \subset M$ containing q, the local components for $[X, Y]$ with respect to the set of vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ on U_α

are

$$[X,Y]^i = \frac{\partial Y^i}{\partial x^j} X^j - \frac{\partial X^i}{\partial x^j} Y^j.$$

The Lie bracket of two vector fields is still a vector field $[X,Y] \in \Gamma(TM)$. In fact, the set $\Gamma(TM)$ is a space of vector fields with a Lie algebraic structure. A **Lie algebra** is an algebra where the product is the Lie bracket. The Lie bracket operation satisfies two fundamental properties: skew symmetry

$$[X,Y] = -[Y,X]$$

and the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

An **integral curve** of a vector field X with initial condition $q_0 \in M$ is a smooth curve $c : I \to M$ where I is an open interval about 0, $c(0) = q_0$ and $\frac{dc}{dt}(t) = X(c(t))$ for $t \in I$. Basically, the tangent vector to the curve c is equal to the tangent vector specified by the vector field at each point along the curve. In local coordinates, the condition that c be the integral curve of X is equivalent to a system of first-order ordinary differential equations. Given the local coordinate function $\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ in the neighborhood $U_\alpha \subset M$ containing q, let $(c^1(t), \ldots, c^n(t))$ and $(X^1(x^1, \ldots, x^n), \ldots, X^n(x^1, \ldots, x^n))$ be the local representations for c and X where $c^i(t) = x^i \circ c(t)$ is a curve on $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ and $X^i(x^1, \ldots, x^n) \in C^\infty(\mathbb{R}^n)$ for $i \in \{1, \ldots, n\}$ are the components of $T\phi_\alpha(X) \in \Gamma(\mathbb{R}^n \times \mathbb{R}^n)$ with respect to the standard basis $\{\mathbf{e}^1, ..., \mathbf{e}^n\}$ for $\mathbb{R}^n$. If we assume that $\frac{dc}{dt}(t) = X(c(t))$ is true, then

$$\begin{aligned}
\dot{c}^1(t) &= X^1(c^1(t), \dots, c^n(t)) \\
\vdots &= \vdots \\
\dot{c}^n(t) &= X^n(c^1(t), \dots, c^n(t))
\end{aligned}$$

where "$\cdot$" means derivative with respect to the parameter t. In general, it is not possible to explicitly solve for $c(t)$.

Finally, we introduce notation for the derivative of the curve $c : I \to M$. We say that the curve $c' : I \to TM$ is the **velocity curve** of c. Given the chart (U_α, ϕ_α) the curve c can be written locally $t \mapsto (c^1(t), \dots, c^n(t))$ where $c^i(t) = x^i \circ c(t)$ for $i \in \{1, \dots, n\}$. The local expression for the velocity curve c' is defined to be

$$t \mapsto ((c^1(t), \dots, c^n(t)), (\dot{c}^1(t), \dots, \dot{c}^n(t))).$$

In coordinates, c' is the usual velocity *along with* the curve c.

2.1.5 Vector Bundle, Vertical Subspace and Vertical Lift

A **fiber bundle** is given by a surjective submersion $\pi : M \to B$ which has the property of being locally trivial. A special class of fiber bundles are **vector bundles** whose fibers have a vector space structure. A **section** of a vector bundle $\pi : E \to M$ is a map $\xi : M \to E$ so that $\pi \circ \xi = id_M$. The set of sections of a vector bundle E will be typically denoted by $\Gamma(E)$. If $\pi : E \to M$ is a vector bundle, then M can be naturally realised as a submanifold of E by identifying $q \in M$ with the zero vector in $\pi^{-1}(q)$. We will denote this submanifold by $Z(E)$ and call it the **zero section** of E. For each $q \in M$, we denote by 0_q the corresponding point in the zero section of E.

The tangent bundle is a specific example of a vector bundle. Intuitively, the tangent bundle consists of a **total space** (TM), a **base space** (M) and a **projection** π_{TM}. The **fiber** of a point in the base space (T_qM) is the preimage of the point under the projection map. Again, the tangent bundle is a vector bundle since the fiber for each point q of the base space is a vector space. A vector field X on M is an element of $\Gamma(TM)$ or section of the tangent bundle TM.

Given the local coordinate function $T\phi_\alpha$ that takes

$$v_q \mapsto \left((x^1(q), \ldots, x^n(q)), \left(\frac{\partial x^1(q)}{\partial x^j} v^j, \ldots, \frac{\partial x^n(q)}{\partial x^j} v^j\right)\right)$$

in the neighborhood $TU_\alpha \subset TM$ containing v_q, the natural coordinates of $v_q \in TM$ are $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$. Using the natural coordinates for TM, we can construct a natural basis for the **tangent space to the tangent bundle** $T_{v_q}TU_\alpha$. If we pick a curve on $TU_\alpha \subset TM$ through the point v_q that is along the coordinate direction x^i, i.e. $x^i = t$, then the tangent vector $W^i_{v_q} \in T_{v_q}TU_\alpha$ along the coordinate curve x^i is $\frac{\partial}{\partial x^i}|_{v_q}$ for $i \in \{1, \ldots, n\}$. Note that "tangent vector" $\frac{\partial}{\partial x^i}|_q \in T_qU_\alpha$ is not the same "tangent vector" $\frac{\partial}{\partial x^i}|_{v_q} \in T_{v_q}TU_\alpha$ because they live in different spaces. Similarly, if we pick a curve on $TU_\alpha \subset TM$ through the point v_q that is along the coordinate direction v^i then the tangent vector $V^i_{v_q} \in T_{v_q}TU_\alpha$ along the coordinate curve v^i is $\frac{\partial}{\partial v^i}|_{v_q}$ for $i \in \{1, \ldots, n\}$. All tangent vectors $W_{v_q}, V_{v_q} \in T_{v_q}TU_\alpha$ along an arbitrary curve or direction can be expressed as a linear combination of $\{(\frac{\partial}{\partial x^1}|_{v_q}, \ldots, \frac{\partial}{\partial x^n}|_{v_q}), (\frac{\partial}{\partial v^1}|_{v_q}, \ldots, \frac{\partial}{\partial v^n}|_{v_q})\}$. This set is a natural basis for $T_{v_q}TU_\alpha$ which is isomorphic to $\mathbb{R}^{2n}$. The natural coordinates for a tangent vector $W_{v_q} \in T_{v_q}TU_\alpha$ are $((x^1, \ldots, x^n), (v^1, \ldots, v^n), (w^1, \ldots, w^n), (u^1, \ldots, u^n))$ where $w^i \in \mathbb{R}$ are the components of W_{v_q} with respect to the basis tangent vectors

$\frac{\partial}{\partial x^i}|_{v_q}$ and $u^i \in \mathbb{R}$ are the components of W_{v_q} with respect to the basis tangent vectors $\frac{\partial}{\partial v^i}|_{v_q}$.

Recall that π_{TM} denotes the projection map $TM \mapsto M$. Given the natural coordinates $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ associated with the chart $(TU_\alpha, T\phi_\alpha)$ containing $v_q \in TM$, the local expression for π_{TM} is $((x^1, \ldots, x^n), (v^1, \ldots, v^n)) \mapsto (x^1, \ldots, x^n)$. The projection map π_{TM} naturally induces the map

$$\pi_{(TM)^*} : T_{v_q}(TM) \to T_q M$$

where $T_{v_q}(TM)$ is the tangent space to the tangent bundle at $v_q \in TM$. The local expression for $\pi_{(TM)*}$ is

$$((x^1, \ldots, x^n), (v^1, \ldots, v^n), (w^1, \ldots, w^n), (u^1, \ldots, u^n)) \mapsto$$

$$((x^1, \ldots, x^n), (w^1, \ldots, w^n)).$$

Any curve γ in M has a **natural lift** to TM given by the curve $t \mapsto \gamma'(t)$ where $\gamma'(t)$ is the tangent vector to γ at $\gamma(t)$. This is the velocity curve introduced in the previous section. A vector field on TM whose integral curves are velocity curves or natural lifts of curves on M is called a **second-order differential equation field**. The namesake follows from the fact that the projections of its integral curves onto M are the solutions of a system of second-order differential equation given in local coordinates. Let us show that if a vector field Z on TM is a second-order differential equation field, then it satisfies the condition $\pi_{(TM)^*} Z_{v_q} = (q, v)$ for all $v_q \in TM$. We begin with the assumption that $Z(\gamma'(t)) = \frac{d}{dt}\gamma'(t)$ holds, which is equivalent to saying that the velocity curve $\gamma'(t)$ is an integral curve of Z. Let us take the natural chart $(TU_\alpha, T\phi_\alpha)$ on TM along with the associated local coordinate frame $\{(\frac{\partial}{\partial x^1}|_{v_q}, \ldots, \frac{\partial}{\partial x^n}|_{v_q}), (\frac{\partial}{\partial v^1}|_{v_q}, \ldots, \frac{\partial}{\partial v^n}|_{v_q})\}$ for $T_{v_q}TU_\alpha$. Recall that the local expression for $\gamma'(t)$ is

$$t \mapsto ((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1, \ldots, \dot{\gamma}^n(t)))$$

where $\gamma^i(t) = x^i \circ \gamma(t)$ in the given chart. The local representation for $Z(\gamma'(t)) = \frac{d}{dt}\gamma'(t)$ is the system of $2n$ ordinary differential equations given by

$$\begin{aligned}
\dot{\gamma}^1(t) &= Z^1((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))) \\
\vdots &= \vdots \\
\dot{\gamma}^n(t) &= Z^n((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))) \\
\ddot{\gamma}^1(t) &= Z^{n+1}((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))) \\
\vdots &= \vdots \\
\ddot{\gamma}^n(t) &= Z^{2n}((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)))
\end{aligned}$$

where " $\ddot{}$ " means the second derivative with respect to the parameter t, and $(Z^1, \ldots, Z^{2n})$ are the local components of Z with respect to the standard basis on $\mathbb{R}^{2n}$. Given the natural coordinate chart, we can write

$$Z_{\gamma(t)} = ((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)), (\ddot{\gamma}^1(t), \ldots, \ddot{\gamma}^n(t))).$$

Now we apply $\pi_{(TM)^*}$ to $Z_{\gamma(t)}$ to get

$$((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)), (\ddot{\gamma}^1(t), \ldots, \ddot{\gamma}^n(t))) \mapsto$$

$$((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)))$$

which is clearly the local expression for the velocity curve $\gamma'(t)$.

An element $w_{v_q} \in T_{v_q}(TM)$ satisfies $\pi_{(TM)^*} w_{v_q} = 0_q$ if and only if it is tangent to the fiber $\pi_{TM}^{-1}(q)$. The set of all $w_{v_q} \in T_{v_q}(TM)$ satisfying this condition is referred to as the **vertical subspace** $V_{v_q}(TM) \subset T_{v_q}(TM)$ and the elements of $V_{v_q}(TM)$ are called **vertical vectors**. A vector field W is said to be vertical if W_{v_q} is vertical for each $v_q \in TM$. Any element X_q of T_qM determines a vertical vector at any point v_q in the fiber over q called the **vertical lift** to v_q. The vertical lift of X_q at the point v_q is denoted by $X_{v_q}^{\text{vlft}}$ and is the tangent vector at $t = 0$ to the curve $t \mapsto v_q + tX_q$ on the fiber $\pi_{TM}^{-1}(q) = T_qM$ of the point $q \in M$. In addition, $\cdot^{\text{vlft}} : T_qM \to V_{v_q}(TM)$ is an isomorphism which is analogous to the canonical isomorphism of a finite-dimensional real vector space with its tangent space at any point. Finally, the vertical lift X^{vlft} of a vector field X on M is the vertical vector field defined by $X_{v_q}^{\text{vlft}} = (X_q)_{v_q}^{\text{vlft}}$ which is constant along the fibers, i.e., $X_{v_q}^{\text{vlft}}$ does not depend on the v of $v_q \in TM$. Though the definition of the vertical subspace $V_{v_q}(TM) \subset T_{v_q}(TM)$ is natural, we will need additional geometric structure called the connection to completely split $T_{v_q}(TM)$ into its vertical and horizontal subspaces.

2.1.6 Distribution, Integrability and Orbit

A **distribution** $\mathcal{D}$ on M is a subset $\mathcal{D} \subset TM$ having the property that for each $q \in M$ there exists a family of vector fields $\mathcal{V} = \{X_1, \ldots, X_m\}$ on M so that for each $q \in M$ we have

$$\mathcal{D}_q \equiv \mathcal{D} \cap T_qM = \operatorname{span}_{\mathbb{R}}\{X_1(q), \ldots, X_m(q)\}.$$

We refer to the family of vector fields $\mathcal{V}$ as **generators** for $\mathcal{D}$. A distribution is called **regular** if the rank K is constant. The **rank** of a distribution is the dimension of the subspace $\mathcal{D}_q$. We assume that distributions are regular unless specified and that it is possible to find a family of smooth vector fields that locally span them.

A distribution $\mathcal{D}$ is **involutive** if for any pair of smooth vector fields X and Y taking values in $\mathcal{D}$ it holds that the vector field $[X, Y]$ also takes values in $\mathcal{D}$ for each $q \in M$. Given a set of generators for $\mathcal{D}$, involutivity of $\mathcal{D}$ can be checked by showing that

$$[X_i, X_j](q) = b_{ij}^k(q) X_k(q)$$

for some functions $b_{ij}^k \in C^\infty(M)$, $i, j, k \in \{1, \ldots, m\}$. The notion of involutivity is closely related to the notion of integrability. A **local integral manifold** through q_0 for $\mathcal{D}$ is an immersed submanifold S of a neighborhood U of q_0 with the property that, for each $q \in S$, $T_qS \subset \mathcal{D}_q$. A local integral manifold is said to be **maximal** if $T_qS = \mathcal{D}_q$ for each $q \in S$. Finally, the distribution $\mathcal{D}$ is **integrable** if there exists a maximal local integral manifold through each $q \in M$. Note that TS is a subbundle of rank K of the tangent bundle TM. The classical result of Frobenius equates integrability and involutivity.

The set of all vector fields on M is a Lie algebra which we denote by $\Gamma(TM)$. Since $\Gamma(TM)$ is a Lie algebra, the **smallest Lie algebra** of $\Gamma(TM)$ which contains a family of vector fields $\mathcal{V}$ is the set of vector fields on M generated by repeated Lie brackets of elements in $\mathcal{V}$. We will denote the smallest Lie algebra of V by $\mathrm{Lie}^{(\infty)}(\mathcal{V})$.

Related to integrable distributions are foliations. Loosely speaking, a **foliation**, $\mathcal{F}$, of a differentiable manifold M is a collection of disjoint immersed submanifolds of M whose disjoint union equals M. Each connected submanifold $\mathcal{F}$ is called a **leaf** of the foliation. Given an integrable distribution D, the collection of maximal integral manifolds for D defines a foliation of M. This foliation is denoted by $\mathcal{F}_D$.

A foliation, $\mathcal{F}$, of M defines an equivalence relation on M such that two points in M are equivalent if they lie in the same leaf of $\mathcal{F}$. The set of equivalence classes is denoted $M/\mathcal{F}$ and will be called the **leaf space** of $\mathcal{F}$. A foliation $\mathcal{F}$ is said to be **simple** if $M/\mathcal{F}$ inherits a manifold structure so that the projection from M to $M/\mathcal{F}$ is a surjective submersion.

Let $\mathcal{D}$ be a distribution on M, and let us denote the vector fields $\{X_1, \ldots, X_n\}$ that generate $\mathcal{D}$ by $\mathcal{V}$. Let us also denote by $\mathrm{Diff}(\mathcal{D})$ the set of diffeomorphisms of M generated by diffeormorphisms of the form

$$\Phi_{t_1}^{X_1} \circ \cdots \circ \Phi_{t_k}^{X_k}, \quad t_1, \ldots, t_k \in \mathbb{R},\ X_1, \ldots, X_k \in \mathcal{V},\ k \in \mathbb{N}.$$

We say that Φ^{X^i} is the **flow** of the vector field X^i on M. Therefore, a diffeomorphism of this form, applied to q, sends q to the point obtained by flowing along X_k for time t_k, then along X_{k-1} for time t_{k-1}, and so on, down to flowing along X_1 for time t_1. The **$\mathcal{D}$-orbit** through q_0 is the set

$$\mathcal{O}(q_0, \mathcal{D}) = \{\Phi(q_0) \mid \Phi \in \mathrm{Diff}(\mathcal{D})\}.$$

Loosely speaking, the $\mathcal{D}$-orbit through q_0 are those points in M that can be reached from q_0 by finite concatenations of curves $\gamma_1, \ldots, \gamma_k$, defined on the intervals $[0, t_1], \ldots, [0, t_k]$, satisfying $\gamma_i'(t) \in \mathcal{D}_{\gamma(t)}, t \in [0, t_i], i \in \{1, \ldots, k\}$, and for which the concatenated curve is continuous.

The smallest involutive distribution containing $\mathcal{D}$ is called the **involutive closure** of $\mathcal{D}$ and is denoted by $\mathrm{Lie}^{(\infty)}(\mathcal{D})$. We may compute $\mathrm{Lie}^{(\infty)}(\mathcal{D})$ using the following algorthim. We denote $\mathrm{Lie}^{(0)}(\mathcal{D}) = \mathcal{D}$, and inductively define distributions $\mathrm{Lie}^{(l)}(\mathcal{D})$ on M by

$$\begin{aligned} \mathrm{Lie}^{(l)}(\mathcal{Y})_q &= \mathrm{Lie}^{(l-1)} + \mathrm{span}_{\mathbb{R}}\{[X,Y](q) \mid \\ X &\in \Gamma(\mathrm{Lie}^{(l_1)}(\mathcal{Y})), Y \in \Gamma(\mathrm{Lie}^{(l_2)}(\mathcal{Y})), l_1 + l_2 = l-1\}. \end{aligned}$$

for $l \in \mathbb{N}$. The following result is from the paper of Sussmann and Jurdjevic [62].

Theorem 2.1.1 (Orbit Theorem for distributions)**.** *If $\mathcal{D}$ is an analytic distribution on M and $q_0 \in M$, then the following statements hold:*

(i) $\mathcal{O}(q_0, \mathcal{D})$ is an analytic immersed submanifold;

(ii) for each $q \in \mathcal{O}(q_0, \mathcal{D})$, $T_q(\mathcal{O}(q_0, \mathcal{D})) = \mathrm{Lie}^{(\infty)}(\mathcal{D})_q$;

(iii) M is the disjoint union of all orbits of $\mathcal{D}$.

The $\mathcal{D}$-orbit through each point q foliates M. We say that two points in M lie on the same leaf if they lie on the same $\mathcal{D}$-orbit. Each $\mathcal{D}$-orbit forms an equivalence class of the leaf space. The orbit theorem for distributions is a generalization of Chow's theorem [19].

Corollary 2.1.2 (Chow's theorem)**.** *If M is connected and if, for a distribution $\mathcal{D}$ on M, $\mathrm{Lie}^{(\infty)}(\mathcal{D}) = TM$, then $\mathcal{O}(q_0, \mathcal{D}) = M$.*

2.1.7 One-form, Codistribution and Annihilator

A **one-form** or **covector field** ψ on M associates to each point $q \in M$ a covector $\psi_q \in T_q^*M$. From a vector bundle perspective, $\psi \in \Gamma(T^*M)$ is a section of the cotangent bundle T^*M that pairs with a vector field $X \in \Gamma(TM)$ to give an element of $C^\infty(M)$, i.e., $\psi : \Gamma(TM) \to C^\infty(M)$. Let (U_α, ϕ_α) be a coordinate chart for M with the local coordinate functions $(x^1(q), \ldots, x^n(q))$. We showed in a previous section that the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is a basis for T_qU_α when evaluated at $q \in U_\alpha$. Recall that the Lie derivative of the local coordinate functions with respect to each of the basis elements $\{\frac{\partial}{\partial x^i}\}$ is $\mathcal{L}_{\frac{\partial}{\partial x^i}} x^j = \delta_i^j$. We also know that by definition $\mathcal{L}_{\frac{\partial}{\partial x^i}} x^j = dx^i \cdot \frac{\partial}{\partial x^j}$. Therefore the set of covector fields $\{dx^1, \ldots, dx^n\}$ is the dual basis to $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ when evaluated at $q \in U_\alpha$. For any $\psi \in \Gamma(T^*M)$ we write $\psi = \psi_i(q)dx^i$ for the functions $\psi_i : U_\alpha \to \mathbb{R}$ called the **components** of ψ with respect to the chart (U_α, ϕ_α).

Similar to the notion of a distribution, a **codistribution** Λ on M is a subbundle of T^*M. A codistribution Λ on M is a subset $\Lambda \subset T^*M$ having the property that for each $q \in M$ there exists a family of one-forms $\Psi = \{\psi^1, \ldots, \psi^m\}$ on M so that for each $q \in M$ we have

$$\Lambda_q \equiv \Lambda \cap T_q^*M = \mathrm{span}_{\mathbb{R}}\{\psi^1(q), \ldots, \psi^m(q)\}.$$

We refer to the family of one-forms Ψ as **cogenerators** for Λ. The rank of Λ at $q \in M$ is the dimension of the subspace Λ_q. Given a distribution $\mathcal{D}$ on a manifold M, its **annihilator** $\mathrm{ann}(\mathcal{D})$ is defined to be the set of one-forms ψ such that $\psi \cdot X = 0$, for all $X \in \Gamma(\mathcal{D})$. Similarly, given a codistribution Λ on M, its **coannihilator** $\mathrm{coann}(\Lambda)$ is defined to be the set of vector fields X such that $\psi \cdot X = 0$, for all $\psi \in \Gamma(\Lambda)$. We say that a regular codistribution Λ is integrable when the distribution $\mathrm{coann}(\Lambda)$ is integrable.

2.2 Riemannian Geometry

2.2.1 Metric Structure and Musical Isomorphisms

A **Riemannian metric** (or **Riemannian structure**), $\mathbb{G}$, is a smooth assignment of an inner-product $\mathbb{G}(\cdot, \cdot)$ on the tangent space $T_q M$ at each point $q \in M$. Recall that an inner-product is a symmetric, bilinear, positive-definite form. A **Riemannian manifold** is a pair, $(M, \mathbb{G})$, where M is a smooth manifold and $\mathbb{G}$ is a Riemannian metric on M. Let $(TU_\alpha, T\phi_\alpha)$ be a coordinate chart for TM with the local coordinates $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ and the local coordinate frame $\{\frac{\partial}{\partial x^1}|_q, \ldots, \frac{\partial}{\partial x^n}|_q\}$ for $T_q U_\alpha$. Given the tangent vectors $X_q = X^i \frac{\partial}{\partial x^i}|_q \in T_q U_\alpha$ and $Y_q = Y^i \frac{\partial}{\partial x^i}|_q \in T_q U_\alpha$, the local expression for $\mathbb{G}$ at the point q is

$$T_q U_\alpha \times T_q U_\alpha \ni (X_q, Y_q) \mapsto \mathbb{G}_{ij}(x^1, \ldots, x^n) X^i Y^j \in \mathbb{R}$$

where $\mathbb{G}_{ij}(x^1, \ldots, x^n) = \mathbb{G}(\mathbf{e}^i, \mathbf{e}^j)_{(x^1, \ldots, x^n)}$ are the n^2 component of $\mathbb{G}$ at the point $(x^1, \ldots, x^n)$ with respect to the standard basis $\{\mathbf{e}^1, \ldots, \mathbf{e}^n\}$ for $\mathbb{R}^n$.

Given a Riemannian metric $\mathbb{G}$, there are two natural isomorphisms: $\mathbb{G}^\sharp : T^*M \to TM$ and $\mathbb{G}^\flat : TM \to T^*M$ defined by

$$\psi_q \cdot \mathbb{G}^\sharp(\omega_q) = \mathbb{G}^{-1}(\psi_q, \omega_q)$$

$$\mathbb{G}^\flat(X_q) \cdot Y_q = \mathbb{G}(X_q, Y_q)$$

where $X_q, Y_q \in T_q M$ and $\psi_q, \omega_q \in T_q^* M$. These isomorphisms are commonly referred to as **musical isomorphisms**. The namesake follows from the raising ($\mathbb{G}^\sharp$) or lowering ($\mathbb{G}^\flat$) of the component indices associate with image of a covector and tangent vector under the appropriate musical isomorphisms. Using the natural coordinates

$$((x^1, \ldots, x^n), (v^1, \ldots, v^n))$$

on TM and the natural coordinates

$$((x^1, \ldots, x^n), (\psi_1, \ldots, \psi_n))$$

on T^*M, the local representation for $\mathbb{G}^\flat$ is

$$((x^1, ..., x^n), (v^1, \ldots, v^n)) \mapsto$$

$$((x^1, ..., x^n), (v^i \mathbb{G}_{i1}(x^1, \ldots, x^n), \ldots, v^i \mathbb{G}_{in}(x^1, \ldots, x^n)))$$

and the local representation for $\mathbb{G}^\sharp$ is

$$((x^1, ..., x^n), (\psi_1, \ldots, \psi_n)) \mapsto$$

$$((x^1, ..., x^n), (\psi_j \mathbb{G}^{1j}(x^1, \ldots, x^n), \ldots, \psi_j \mathbb{G}^{nj}(x^1, \ldots, x^n)))$$

where $\mathbb{G}^{ij}(x^1, \ldots, x^n)$ is the inverse of $\mathbb{G}_{ij}(x^1, \ldots, x^n)$.

Let f be a smooth function on M. We define the **gradient** of f at $q \in M$ to be $\mathrm{grad}(f)|_q = \mathbb{G}^\sharp(df|_q)$. Using the natural coordinates $(x^1, ..., x^n)$ on M and the dual basis $\{dx^1|_q, \ldots, dx^n|_q\}$, the local representation for $\mathrm{grad}(f)$ at $q \in M$ is

$$\left((x^1, ..., x^n), \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)\right) \mapsto$$

$$\left((x^1, \ldots, x^n), \left(\frac{\partial f}{\partial x^j} \mathbb{G}^{1j}(x^1, \ldots, x^n), \ldots, \frac{\partial f}{\partial x^j} \mathbb{G}^{nj}(x^1, \ldots, x^n)\right)\right)$$

where $\mathrm{grad}(f)|_q \in T_qM$.

If we do not specify a point q on the base manifold M, then we can think of $\mathbb{G}^\sharp$ and $\mathbb{G}^\flat$ as the isomorphisms: $\mathbb{G}^\sharp : \Gamma(T^*M) \to \Gamma(TM)$ and $\mathbb{G}^\flat : \Gamma(TM) \to \Gamma(T^*M)$ defined by

$$\psi \cdot \mathbb{G}^\sharp(\omega) = \mathbb{G}^{-1}(\psi, \omega)$$

$$\mathbb{G}^\flat(X) \cdot Y = \mathbb{G}(X, Y)$$

where $X, Y \in \Gamma(TM)$ and $\psi, \omega \in \Gamma(T^*M)$. Given the local coordinate function $\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ in the neighborhood $U_\alpha \subset M$ containing q and the resulting set of vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ on U_α, the local components for $\mathbb{G}^\sharp(\psi)$ are

$$\mathbb{G}^\sharp(\psi)^i(q) = \mathbb{G}^{ij}(q)\psi_j(q)$$

for $i, j \in \{1, \ldots, n\}$. Given the local coordinate function $\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ in the neighborhood $U_\alpha \subset M$ containing q and the resulting set of dual one-forms $dx^1, \ldots, dx^n$ on $\Gamma(TU_\alpha)$, the local components for $\mathbb{G}^\flat(X)$ are

$$\mathbb{G}^\flat(X)_j(q) = \mathbb{G}_{ij}(q)X^i(q)$$

for $i, j \in \{1, \ldots, n\}$. Now we define the grad$(f)$ to be the vector field $\mathbb{G}^\sharp(df)$ with components

$$\mathbb{G}^\sharp(df)^i(q) = \mathbb{G}^{ij}(q)\frac{\partial f}{\partial x^j}(q)$$

relative to the local coordinate frame $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ on U_α.

2.2.2 Affine Connection and Christoffel Symbols

An **affine connection** ∇ on a smooth manifold M is a mapping $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ denoted by $(X, Y) \mapsto \nabla_X Y$ that satisfies the following properties:

1. $\mathbb{R}$-linear in both X and Y, and
2. $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_X fY = f\nabla_X Y + (\mathcal{L}_X f)Y$ for each $f \in C^\infty(M)$.

Let $(TU_\alpha, T\phi_\alpha)$ be a coordinate chart for TM with the local coordinates $((x^1,\ldots,x^n),(v^1,\ldots,v^n))$ and the family of vector fields $\{\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\}$ on U_α that, when evaluated at the point $q \in U_\alpha$, generate the local coordinate frame for T_qU_α. Given the two vector fields $X = X^i(q)\frac{\partial}{\partial x^i}$ and $Y = Y^i(q)\frac{\partial}{\partial x^i}$ on $U_\alpha \subset M$, the local expression for the affine connection is

$$\Gamma(TU_\alpha) \times \Gamma(TU_\alpha) \ni (X,Y) \mapsto$$

$$\left(\frac{\partial Y^k(q)}{\partial x^i}X^i(q) + \Gamma^k_{ij}(q)X^i(q)Y^j(q)\right)\frac{\partial}{\partial x^k} \in \Gamma(TU_\alpha)$$

where $\Gamma^k_{ij}(q) \in C^\infty(M)$ is the n^3 component functions of the affine connection. The component functions of the affine connection $\Gamma^k_{ij}(q)$ are called the **Christoffel symbols**. The Christoffel symbols are defined to be the local components of the vector field

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}(q)\frac{\partial}{\partial x^k}$$

defined on U_α. Note that the definition of the Christoffel symbols is not coordinate invariant and is therefore not a coordinate invariant geometric object.

Suppose there exists family of vector fields $\mathcal{V} = \{X_1,\ldots,X_n\}$ on $U_\alpha \subset M$ such that $\mathcal{V}$ evaluated at each $q \in U_\alpha$ forms a basis for T_qU_α. We define the **generalized Christoffel symbols**, $\widehat{\Gamma}^k_{ij}$, $i,j,k \in \{1,\ldots,n\}$, for ∇ on U_α to be

$$\nabla_{X_i}X_j = \widehat{\Gamma}^k_{ij}X_k.$$

Let (U_α, ϕ_α) be a coordinate chart for M with the family of vector fields

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$$

on U_α that, when evaluated at the point $q \in U_\alpha$, generate the local coordinate frame for $T_q U_\alpha$. If we set $X_i = \frac{\partial}{\partial x^i}$ for $i \in \{1, \ldots, n\}$, then the generalized Christoffel symbols are the usual Christoffel symbols.

An affine connection is the second mapping that we have introduced that takes two vector fields $(X, Y) \in \Gamma(TM) \times \Gamma(TM)$ and returns a vector field $\nabla_X Y \in \Gamma(TM)$. Recall that the first of such mappings was the Lie derivative of a vector field Y with respect to the vector field X denoted by $\mathcal{L}_X Y$. Not only are $\nabla_X Y$ and $\mathcal{L}_X Y$ different, we can see from their local components that $\nabla_X Y$ is **punctual** in X and **local** in Y whereas $\mathcal{L}_X Y$ is local in X and Y. In otherwords, the components of $\nabla_X Y$ only depend on the value of the vector field X at the point $q \in M$ (i.e. not on the extension of X on U_α) whereas the components of $\nabla_X Y$ depend on the local extension of Y on U_α. In contrast, $\mathcal{L}_X Y$ depends on the local extension of X and Y on U_α.

2.2.3 Covariant Derivative, Parallel and Geodesic Spray

An affine connection can be used to define a method for comparing two tangent vectors that live in different tangent spaces. This method depends on a generalization of the usual notion of parallelism encountered in flat Euclidean space. The generalization takes into account the curvature of the manifold or the tendency of parallel lines to converge towards or diverge away from each other as lines are extended. Curvature is the central topic of differential geometry. It is important to understand the interpretation of an affine connection in the context of parallelism and curvature. Let us begin by defining a closely related object called the covariant derivative.

Given (M, ∇) there exists a unique correspondence which associates to a vector field V along a smooth curve $\gamma : I \to M$ another vector field $\frac{D}{dt}V$ along γ called the **covariant derivative** of V along γ. The covariant derivative is linear

$$\frac{D}{dt}(V + W) = \frac{D}{dt}V + \frac{D}{dt}W$$

and satisfies the property that

$$\frac{D}{dt}fV = \left(\frac{d}{dt} \cdot f\right) V + f\frac{D}{dt}V$$

where f is a smooth function restricted to the curve γ and $\frac{d}{dt}$ is the **tangent vector field** along γ. The tangent vector field $\frac{d}{dt}$ along γ is also a linear differential operator that acts on functions along the curve on the manifold. Let $\Gamma(\gamma'(t))$ be the set of all vector fields along the curve γ and $(TU_\alpha, T\phi_\alpha)$ be a coordinate chart for TM that induces the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ on U_α. Given the vector field $V = V^i(t)\frac{\partial}{\partial x^i}$ along the curve $\gamma \subset U_\alpha$, the local expression for the covariant derivative of V along the curve γ is

$$\Gamma(\gamma'(t)) \ni V \mapsto \left(\dot{V}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{\gamma}^i(t)V^j(\gamma(t))\right) \frac{\partial}{\partial x^k} \in \Gamma(\gamma'(t))$$

where $\dot{\gamma}^i(t)$ are components of the tangent vector field $\frac{d}{dt}$ to the curve γ with respect to the given chart.

A vector field V along a curve $\gamma : I \to M$ is called **parallel** when $\frac{D}{dt}V = 0$ for all $t \in I$. Given (M, ∇) there exists a unique parallel vector field V along γ, such that $V(t_0) = V_0$. We refer to such a $V(t)$ as the **parallel transport** of $V(t_0)$

along γ. A parameterized curve $\gamma : I \to M$ is a **geodesic** if

$$\frac{D}{dt}\gamma'(t) = 0, \ \forall \ t \in I$$

where $\gamma'(t)$ is alternative notation for the tangent vector field along the curve γ. Let $(TU_\alpha, T\phi_\alpha)$ be a coordinate chart for TM that induces the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ on U_α. The local expression for the tangent vector field is

$$\frac{d}{dt} = \dot{x}^i(t)\frac{\partial}{\partial x^i}.$$

The covariant derivative of the tangent vector field $\gamma'(t)$ along the curve $\gamma(t)$ is equivalent to $\nabla_{\gamma'(t)}\gamma'(t)$. Let us derive the local expression that a geodesic satisfies. In coordinates, we can expand $\nabla_{\gamma'(t)}\gamma'(t) = 0$ by substituting $\dot{x}^i(t)\frac{\partial}{\partial x^i}$ into the second appearance of $\gamma'(t)$ and applying the second property of an affine connection to get

$$\nabla_{\frac{d}{dt}}\dot{x}^i(t)\frac{\partial}{\partial x^i} = \left(\frac{d}{dt}\dot{x}^k(t) + \dot{x}^i(t)\nabla_{\gamma'(t)}\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = 0.$$

We now take the derivative of the component $\dot{x}^i(t)$ with respect to the parameter t to get

$$\left(\ddot{x}^k(t) + \dot{x}^i(t)\nabla_{\gamma'(t)}\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = 0.$$

Finally, we substitute $\dot{x}^i(t)\frac{\partial}{\partial x^i}$ for the remaining $\gamma'(t)$ and apply the first property of an affine connection to get

$$\left(\ddot{x}^k(t) + \Gamma^k_{ij}(t)\dot{x}^i(t)\dot{x}^j(t)\right)\frac{\partial}{\partial x^k} = 0.$$

In coordinates, this is equivalent to the following system of n second-order differential equations

$$\begin{aligned}
\ddot{x}^1(t) + \Gamma^1_{ij}(t)\dot{x}^i(t)\dot{x}^j(t) &= 0 \\
\vdots &= \vdots \\
\ddot{x}^n(t) + \Gamma^n_{ij}(t)\dot{x}^i(t)\dot{x}^j(t) &= 0.
\end{aligned}$$

This system of n second-order differential equations corresponds to the local representation of a second-order differential equation field Z on TM whose integral curves is the velocity curve $\gamma'(t)$ that satisifes $\nabla_{\gamma'(t)}\gamma'(t)$. Recall that the local representation for $Z(\gamma'(t)) = \frac{d}{dt}\gamma'(t)$ is the system of $2n$ ordinary differential equations given by

$$\begin{aligned}
\dot{\gamma}^1(t) &= Z^1((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))) \\
\vdots &= \vdots \\
\dot{\gamma}^n(t) &= Z^n((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))) \\
\ddot{\gamma}^1(t) &= Z^{n+1}((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))) \\
\vdots &= \vdots \\
\ddot{\gamma}^n(t) &= Z^{2n}((\gamma^1(t), \ldots, \gamma^n(t)), (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t))).
\end{aligned}$$

Let $(TU_\alpha, T\phi_\alpha)$ be a coordinate chart for TM with the local coordinate functions $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ on TU_α. The local representation of the velocity curve $\gamma'(t)$ is given by

$$t \mapsto ((x^1(t), \ldots, x^n(t)), (v^1(t), \ldots, v^n(t)))$$

where $v^i(t) = \dot{x}^i(t)$. In these coordinates, the local representation for $Z(\gamma'(t)) = \frac{d}{dt}\gamma'(t)$ is

$$\begin{aligned}
\dot{x}^1(t) &= v^1(t) \\
\vdots &= \vdots \\
\dot{x}^n(t) &= v^n(t) \\
\ddot{x}^1(t) &= -\Gamma^1_{ij}(t)v^i(t)v^j(t) \\
\vdots &= \vdots \\
\ddot{x}^n(t) &= -\Gamma^n_{ij}(t)v^i(t)v^j(t)
\end{aligned}$$

where the right hand side of this system of $2n$ first-order differential equations is the local representation of the components of the vector field Z along the velocity curve $\gamma'(t)$ that satisfies $\nabla_{\gamma'(t)}\gamma'(t)$. The vector field $Z_{\gamma'(t)}$ along the velocity curve $\gamma'(t)$ on TM that satisfies $\nabla_{\gamma'(t)}\gamma'(t) = 0$ is called the **geodesic spray**. Let us take the natural chart $(TU_\alpha, T\phi_\alpha)$ on TM along with the associated family of vector fields $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}\}$ that when evaluated at point $v_q \in TU_\alpha$ generate a local coordinate frame for $T_{v_q}TU_\alpha$. The local expression for the geodesic spray $Z_{\gamma'(t)}$ along the velocity curve $\gamma'(t)$ is

$$Z_{\gamma'(t)} = v^i(t)\frac{\partial}{\partial x^i} - \Gamma^i_{jk}(t)v^j(t)v^k(t)\frac{\partial}{\partial v^i}.$$

2.2.4 Compatibility, Symmetry and Levi-Civita Connection

Given $(M, \mathbb{G}, \nabla)$, a connection is called **compatible** with the metric $\mathbb{G}$, when for any smooth curve γ and any pair of parallel vector fields P and P' along γ, $\mathbb{G}(P, P') \in \mathbb{R}$ is constant along γ. A connection ∇ on M is compatible with the metric $\mathbb{G}$ if and only if for any vector fields V and W along the smooth curve γ it holds that

$$\frac{d}{dt}\mathbb{G}(V,W) = \mathbb{G}(\frac{D}{dt}V,W) + \mathbb{G}(V,\frac{D}{dt}W)$$

for all $t \in I$. This is also equivalent to

$$X\mathbb{G}(Y,Z) = \mathbb{G}(\nabla_X Y, Z) + \mathbb{G}(Y, \nabla_X Z)$$

where $X, Y, Z \in \Gamma(TM)$. Finally, an affine connection ∇ on M is said to be **symmetric** when

$$\nabla_X Y - \nabla_Y X = [X,Y]$$

for all $X, Y \in \Gamma(TM)$.

Given $(M, \mathbb{G})$ there exists a unique affine connection ∇ on M such that ∇ is symmetric and compatible. This connection is known as the **Levi-Civita** connection. The Christoffel symbols associated with the Levi-Civita connection are

$$\Gamma^k_{ij}(q) = \frac{1}{2}\mathbb{G}^{kl}(q)\left(\frac{\partial}{\partial x^j}\mathbb{G}_{il}(q) + \frac{\partial}{\partial x^i}\mathbb{G}_{jl}(q) - \frac{\partial}{\partial x^l}\mathbb{G}_{ij}(q)\right)$$

where $\mathbb{G}_{ij}(q) = \mathbb{G}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $\mathbb{G}^{ij}(q)\mathbb{G}_{ij}(q) = \delta^j_i$ for $i, j, k, l \in \{1, \ldots, n\}$.

2.2.5 Poincaré Representation and Restricted Connection

The **Poincaré Representation** for the geodesic equations is the local coordinate representation of the system of n second-order differential equations for $\nabla_{\gamma'(t)}\gamma'(t) = 0$ using generalized Christoffel symbols. Let $\gamma : I \to U_\alpha \subset M$ be a smooth curve and $\mathcal{V} = \{X_1, \ldots, X_n\}$ be a family of vector fields on U_α such that $\mathcal{V}$ evaluated at each $q \in U_\alpha$ forms a basis for T_qU_α. Now let $v^i : I \to \mathbb{R}$ for

$i \in \{1, \ldots, n\}$ be the components of the tangent vector field $\gamma'(t)$ with respect to the family of vector fields $\mathcal{V}$, *i.e.*, $\gamma'(t) = v^i(t)X_i(\gamma(t))$. The local expression for $\nabla_{\gamma'(t)}\gamma'(t)$ with respect to $\mathcal{V}$ is

$$\nabla_{\gamma'(t)}\gamma'(t) = (\dot{v}^k(t) + \widehat{\Gamma}^k_{ij}(\gamma(t))v^i(t)v^j(t))X_k(\gamma(t)).$$

We say that the functions $v^i : I \to \mathbb{R}$ for $i \in \{1, \ldots, n\}$ are the **pseudo-velocities** of the curve $\gamma(t)$ because in general the components $v^i(t)$ associated with the family of vector fields $\{X_1, \ldots, X_n\}$ are not equal to the usual time derivative components $\dot{x}^i(t)$ associated with the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ associated with coordinate chart (U_α, ϕ_α) on M.

Let $\mathcal{D}$ be a distribution on U_α such that $\mathcal{D}$ is generated by the family of vector fields $\{X_1, \ldots, X_K\}$ where K is the rank of $\mathcal{D}$. We say that the affine connection ∇ restricts to $\mathcal{D}$ if $\nabla_X Y \in \Gamma(\mathcal{D})$ for every $Y \in \Gamma(\mathcal{D})$. If $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for each $t \in I$ then the local expression for $\nabla_{\gamma'(t)}\gamma'(t)$ with respect to $\{X_1, \ldots, X_K\}$ is

$$\nabla_{\gamma'(t)}\gamma'(t) = (\dot{v}^r(t) + \widehat{\Gamma}^r_{ap}(\gamma(t))v^a(t)v^p(t))X_r(\gamma(t))$$

where $a, p, r \in \{1, \ldots, K\}$. Further, the system of K second-order differential equations

$$\begin{aligned} \dot{v}^1(t) + \widehat{\Gamma}^1_{ap}(t)v^a(t)v^p(t) &= 0 \\ \vdots &= \vdots \\ \dot{v}^K(t) + \widehat{\Gamma}^K_{ap}(t)v^a(t)v^p(t) &= 0 \end{aligned}$$

is the coordinate representation of the geodesic equations $\nabla_{\gamma'(t)}\gamma'(t) = 0$.

2.2.6 Symmetric Product and Geodesic Invariance

Given a pair of vector fields $X, Y \in \Gamma(TM)$, their **symmetric product** is the vector field defined by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

A distribution $\mathcal{D}$ on M is **geodesically invariant** with respect to an affine connection ∇ if every geodesic $\gamma : I \to M$, with the condition $\gamma'(t_0) \in \mathcal{D}_{\gamma(t_0)}$ for some $t_0 \in I$, satisfies $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in I$. The symmetric product can be used to determine whether or not a distribution is geodesically invariant.

Theorem 2.2.1 (Characterization of geodesic invariance)**.** *A distribution $\mathcal{D}$ on M is geodesically invariant if and only if $\langle X : Y \rangle \in \Gamma(\mathcal{D})$ for all vector fields X, Y taking values in $\mathcal{D}$.*

Proof. Use the definition of generalized Christoffel symbols along with the components of $\nabla_{\gamma'(t)}\gamma'(t)$ to prove the theorem above. $\square$

Let $\mathcal{D}$ be the distribution generated by the family of vector fields $\mathcal{V}$. The closure of the distribution $\mathcal{D}$ under the symmetric product will be denoted $\mathrm{Sym}^{(\infty)}(\mathcal{D})$. A **symmetric algebra** is an algebra where multiplication is the symmetric product. The **smallest symmetric algebra** of $\mathcal{V}$ is the set of vector fields on M generated by repeated symmetric products of elements of $\mathcal{V}$. We will denote the smallest symmetric algebra of $\mathcal{V}$ by $\mathrm{Sym}^{(\infty)}(\mathcal{D})$. The integrable distribution generated by $\mathrm{Sym}^{(\infty)}(\mathcal{D})$ will be denoted $\mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(\mathcal{D}))$. Since this distribution is integrable, through each $q_0 \in M$ there is an immersed maximal integral manifold S_{q_0} with the property that $T_q S_{q_0} = \mathrm{Lie}^{(\infty)}(\mathrm{Sym}^{(\infty)}(\mathcal{D}))$ for each $q \in S_{q_0}$.

2.2.7 Horizontal Subspace and Horizontal Lift

The tangent space of TM at any point $v_q \in TM$ splits into the horizontal and vertical subspace with respect to an affine connection ∇. The split can be written as a direct sum $T_{v_q}TM = H_{v_q}TM \oplus V_{v_q}TM$ where $H_{v_q}TM$ denotes the horizontal subspace and $V_{v_q}TM$ the vertical subspace. Recall that the definition of the vertical subspace is

$$V_{v_q}TM = \{w \in T_{v_q}(TM) \mid \pi_{(TM)^*}w = 0\}$$

where $\pi_{(TM)^*} : T_{v_q}TM \to T_qM$. If $v_q \in TM$ is specified then for any vector $X_q \in T_qM$ there exists a unique vector $X_q^{\text{hlft}} \in H_{v_q}TM$ such that $\pi_{(TM)^*}X_q^{\text{hlft}} = X_q$. We call X_q^{hlft} the **horizontal lift** of X_q to the point $v_q \in TM$. Let us take the natural chart $(TU_\alpha, T\phi_\alpha)$ on TM along with the associated family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\}$ that when evaluated at point $v_q \in TU_\alpha$ generate a local coordinate frame for $T_{v_q}TU_\alpha$. Given a tangent vector $X_q = X^i \frac{\partial}{\partial x^i} \in T_qM$ and an affine connection ∇ on M, the local components for the horizontal lift of X_q at the point v_q are

$$X_q^{\text{hlft}} = X^i \frac{\partial}{\partial x^i} - \Gamma_{ij}^k(q)X^i v^j \frac{\partial}{\partial v^k}$$

for $i, j, k \in \{1, \ldots, n\}$. The natural coordinates for a tangent vector $W_{v_q} \in T_{v_q}TU_\alpha$ are $((x^1, \ldots, x^n), (v^1, \ldots, v^n), (w^1, \ldots, w^n), (u^1, \ldots, u^n))$ where $w^i \in \mathbb{R}$ are the components of W_{v_q} with respect to the basis tangent vectors $\frac{\partial}{\partial x^i}|_{v_q}$ and $u^i \in \mathbb{R}$ are the components of W_{v_q} with respect to the basis tangent vectors $\frac{\partial}{\partial v^i}|_{v_q}$. We define the **horizontal subspace** $H_{v_q}(TM) \subset T_{v_q}(TM)$ to be the set

$$\{W_{v_q} \in T_{v_q}(TM) \mid u^k + \Gamma^k_{ij}(q) w^i v^j = 0\}$$

where Γ^k_{ij} is the usual Christoffel symbols associated with affine connection ∇ on M. Recall that the local expression for the geodesic spray is

$$Z_{\gamma'(t)} = v^i(t)\frac{\partial}{\partial x^i} - \Gamma^i_{jk}(t) v^j(t) v^k(t) \frac{\partial}{\partial v^i}$$

where the coordinates with respect to the local coordinate frame

$$\left\{ \frac{\partial}{\partial x^1}|_{v_q}, \ldots, \frac{\partial}{\partial x^n}|_{v_q}, \frac{\partial}{\partial v^1}|_{v_q}, \ldots, \frac{\partial}{\partial v^n}|_{v_q} \right\}$$

on $T_{v_q}TU_\alpha$ are

$$((x^1(t), \ldots, x^n(t)), (v^i(t), \ldots, v^n(t)),$$

$$(v^1(t), \ldots, v^n(t)), (-\Gamma^1_{ij}(t)v^i(t)v^j(t), \ldots, -\Gamma^n_{ij}(t)v^i(t)v^j(t))).$$

By examining the coordinates for $Z_{\gamma'(t)}$, we see that the geodesic spray when evaluated at a point along the velocity curve $\gamma'(t)$ gives a tangent vector $Z_{\gamma'(t)} \in H_{\gamma'(t)}(TM)$. In addition, the definition of the horizontal lift of a tangent vector X_q at the point $v_q \in TM$ is easily shown to be consistent with the definition of the horizontal subspace $H_{v_q}(TM)$. This can be seen by simply examining the coordinates

$$((x^1, \ldots, x^n), (X^1, \ldots, X^n), (X^1, \ldots, X^n), (-\Gamma^1_{ij} X^i v^j, \ldots, -\Gamma^n_{ij} X^i v^j))$$

of X_q^{hlft} with respect to the local coordinate frame

$$\left\{ \frac{\partial}{\partial x^1}|_{v_q}, \ldots, \frac{\partial}{\partial x^n}|_{v_q}, \frac{\partial}{\partial v^1}|_{v_q}, \ldots, \frac{\partial}{\partial v^n}|_{v_q} \right\}$$

for $T_{v_q}TU_\alpha$. In fact, the horizontal lift of the tangent vector field $\gamma'(t)$ is the geodesic spray which is the vector field $Z_{\gamma'(t)}$ whose integral curves $\gamma'(t)$ satisfy the geodesic equation $\nabla_{\gamma'(t)}\gamma'(t) = 0$.

Note that the map $X_q \mapsto X_q^{\text{hlft}}$ is an isomorphism between vector spaces T_qM and $H_{v_q}TM$. Analogously, the map $X_q \mapsto X_q^{\text{vlft}}$ is also an isomorphism between vector spaces T_qM and $V_{v_q}TM$. All tangent vectors $W_{v_q} \in T_{v_q}TM$ can be written in the form $W_{v_q} = X_q^{\text{hlft}} + Y_q^{\text{vlft}}$ where $X_q, Y_q \in T_qM$ are uniquely determined. *The definition of the vertical subspace follows from the assignment of a differential structure to a topological manifold whereas the definition of the horizontal subspace requires the assignment of an affine connection.* However, if we restrict our attention to $Z(TM)$ we may define the horizontal subspace in the following manner. Recall that $Z(TM)$ denotes the zero section of TM. Since M is naturally diffeomorphic to $Z(TM)$, there exists a natural inclusion of T_qM into $T_{0_q}TM$ for each $q \in M$. We may define the image of this inclusion to be the horizontal subspace. Now we have the following decomposition

$$T_{0_q}TM = T_qM \oplus V_{0_q}TM$$

for each $q \in M$. Again, this definition of horizontal is only valid on $Z(TM)$.

2.3 Affine Subbundle and Affine Foliation

An **affine subbundle** on M is a subset $A \subset TM$ having the property that for each $q \in M$ there exists a family of vector fields $\mathcal{V} = \{X_0, \ldots, X_k\}$ so that for each $q \in U$ we have

$$A_q \equiv A \cap T_qM = \{X_0(q)\} + \mathrm{span}_{\mathbb{R}}\{X_1(q), \ldots, X_k(q)\}.$$

The subfamily of vector fields $\{X_1, \ldots, X_k\} \subset \mathcal{V}$ are referred to as **linear generators** of A. Corresponding to an affine subbundle is a distribution $L(A)$ where $L(A)_q$ is the linear part of the affine subspace A_q. If the dimension of A_q is a constant K and $K = n$ for each $q \in M$, then we call A an **affine bundle**.

An **affine section** is a map from the base space M to the total space A with the following property: if ξ_A is an affine section and q is a point in the base space, the ξ_A belongs to the affine fiber of q. An **affine foliation**, $\mathcal{A}$, on TM is a collection of disjoint immersed affine subbundles of TM whose disjoint union equals TM. Each connected affine subbundle A is called an **affine leaf** of the affine foliation.

An affine foliation, $\mathcal{A}$, of TM defines an equivalence relation on TM such that two points in TM are equivalent if they lie in the same leaf of $\mathcal{A}$. The set of equivalence classes is denoted by $TM/\mathcal{A}$ and will be called the **affine leaf space** of $\mathcal{A}$. An affine foliation $\mathcal{A}$ is said to be **simple** if $TM/\mathcal{A}$ inherits a manifold structure so that the projection from TM to $TM/\mathcal{A}$ is a surjective submersion.

CHAPTER 3

MECHANICAL CONTROL SYSTEMS ON RIEMANNIAN MANIFOLDS

In contrast to the classical approach of Goldstein [25], the basic mathematical models of unforced mechanical systems presented in this proposal are formulated using geometric techniques developed by Abraham and Marsden [1], Marsden [45], Marsden and Ratiu [46]. The modern approach to mechanics is commonly referred to as **geometric mechanics**. Geometric mechanics develops the classical notions in the context of differentiable manifolds and vector bundles. These geometric objects provide a more natural mathematical setting for the study of mechanical systems than real vector spaces encountered in classical "vector mechanics". Both the Hamiltonian and Lagrangian viewpoints have benefited tremendously from the renewed attention to the fundamental geometric framework.

Although Bullo and Lewis [10] can be viewed as an adaptation of the methods of nonlinear control theory to mechanical systems, it is also true that their work is an extension of the methods of geometric mechanics to systems with external forces. Prior to their work, the modern development of geometric mechanics had left this important *control* feature out. Another missing piece in geometric mechanics that was developed by Bullo and Lewis [10] was the inclusion of constraints in the formulations. The basic mathematical models of the forced mechanical systems with constraints presented in this thesis follow the development by Bullo and Lewis [10].

3.1 Geometric Mechanics

Geometric mechanics is the study of classical mechanics in the context of manifolds. Physically speaking, **mechanical systems** represent a collection of particles and rigid bodies. We take a **particle** to be a physical object having mass and position but no volume where a **rigid body** is a collection of particles whose position relative to one another is fixed. Mathematically speaking, mechanical systems naturally evolve on a configuration space that is a smooth manifold. This is why differential geometry is such a powerful mathematical tool used to model mechanical systems.

3.1.1 Configuration Manifold

We say that the set of configurations of a mechanical system is in $1-1$ correspondence with a smooth manifold called a **configuration manifold** M. Specifically, the configuration manifold M is an n-dimensional smooth manifold where the dimension n corresponds to the n **degrees of freedom** of the mechanical system.

Recall that a smooth manifold is basically a set that can be locally parameterized by an open set $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ using the local coordinate functions $\phi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$. The behavior of the mechanical system on the open set $\phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is not an approximation. The mapping ϕ_α is a diffeomorphism from $U_\alpha \mapsto \phi_\alpha(U_\alpha)$ which implies that the behavior of the mechanical system on U_α is in $1-1$ correspondence with the behavior of the mechanical system on $\phi_\alpha(U_\alpha)$. This *local correspondence* requires additional analysis to extend results, if applicable, to the global level. Depending on the context, both the local parametrization of the configuration manifold assigned to a mechanical system and the linearization of the nonlinear differential equations of motion governing the time evolution of the mechanical system are referred to as *linearization*. We make the distinction that the latter approach is an approximation to the models we present. These approximations are often useful and easier to understand, however by their very nature are subject to well-known limitations.

The differential geometric approach to modeling mathematical systems places a heavy emphasis on coordinate-invariant formulations of the mathematical model. This allows us to clearly model physical concepts using differential geometric objects such as manifolds, tangent vectors, vector fields, distributions, Riemannian metrics, etc... We believe that the resulting mathematical model represents the *real* structure of the problem. From this perspective, the choice of local parametrization or coordinates is viewed as *ad hoc* or not intrinsic to the problem. Our goal is to use mathematical models where our analysis and results are not limited to a choice of coordinates. This approach is fundamental to geometric mechanics and distinguishes it from classical and analytical mechanics where it is implicit that mechanical systems evolve on Euclidean space $\mathbb{R}^n$. With that said, a choice of parametrization when simulating the behavior of or constructing an explicit control algorithm for a specific mechanical system is unavoidable.

3.1.2 Tangent Bundle to the Configuration Manifold

There are two basic approaches to geometric mechanics. The Hamiltonian approach views mechanical systems evolving on the momentum-phase space (*i.e.*, cotangent bundle) while the Lagrangian approach views mechanical systems evolving on the velocity-phase space (*i.e.*, tangent bundle). Our research focuses on the Lagrangian formulation of mechanical systems. Furthermore, we restrict our attention to SIMPLE MECHANICAL SYSTEMS. Simple mechanical systems are characterized by the fact that the Lagrangian is equal to the difference between kinetic energy and potential energy. Simple does not imply easy: a more appropriate descriptor would be *natural* mechanical systems.

The **tangent bundle** TM to the configuration manifold M is the set of all possible configurations and velocities of a mechanical system. Let $(T\phi_\alpha, TU_\alpha)$ be the natural charts on TM. The natural coordinates for $v_q \in TM$ are

$$((x^1, \dots, x^n), (v^1, \dots, v^n))$$

where v^i are the components associated with the natural basis $\{\frac{\partial}{\partial x1}|_q, \dots, \frac{\partial}{\partial x^n}|_q\}$ for T_qM. This representation of local coordinates implies a local product structure on the tangent bundle. However, the tangent bundle TM is *not* the Cartesian product of the configuration manifold and the set of velocities. We use the subscript notation v_q instead of the product structure notation (q, v) to reinforce that, in general, the tangent bundle is not a product space.

3.1.3 Kinetic Energy Metric

One of the key features to the differential geometric approach to modeling mechanical systems is that the kinetic energy defines a Riemannian metric $\mathbb{G}$ on the configuration manifold M. The Riemannian metric is considered an additional geometric structure and does not naturally follow from the assignment of a differential structure to a topological manifold. In classical mechanics, this structure is commonly referred to as the "inertia matrix" or the "mass matrix". The kinetic energy metric is a mapping that when evaluated at a configuration q of the mechanical system takes two tangent vectors $v_q, u_q \in T_qM$ and returns an element

$\mathbb{G}(v_q, u_q)_q \in \mathbb{R}$. The **kinetic energy** is the function on TM given by

$$KE(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q)_q.$$

Let $(T\phi_\alpha, TU_\alpha)$ be the natural charts on TM. The natural coordinates for $v_q \in TM$ are $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ where v^i are the components associated with the natural basis $\{\frac{\partial}{\partial x1}|_q, \ldots, \frac{\partial}{\partial x^n}|_q\}$ for T_qM. The local expression for the kinetic energy is

$$T\phi_\alpha(U_\alpha) \ni ((x^1, \ldots, x^n), (v^1, \ldots, v^n)) \mapsto v^j \mathbb{G}_{ij}(x^1, \ldots, x^n) v^i \in \mathbb{R}.$$

3.1.4 Potential Energy Function

We use a **potential function** $V \in C^\infty(M)$ to construct a **potential force**

$$F(q) = -dV(q)$$

where $-dV(q)$ is the differential of the potential function V on M. Thus, the potential force is a one-form $-dV(q) \in \Gamma(T^*M)$. Let (U_α, ϕ_α) be a coordinate chart for M with the local coordinate functions $(x^1(q), \ldots, x^n(q))$. The set of one-forms $\{dx^1, \ldots, dx^n\}$ evaluated at $q \in U_\alpha$ is a dual basis for T_q^*M. The local representation for the potential force is

$$-dV(q) = -\frac{\partial V(q)}{\partial x^i} dx^i$$

where $\frac{\partial V(q)}{\partial x^i}$ are the component functions on M relative to the set of one-forms $\{dx^1, \ldots, dx^n\}$. We can use the musical isomorphism $\mathbb{G}^\sharp : \Gamma(T^*M) \to \Gamma(TM)$ to associate a vector field $\mathbb{G}^\sharp(-dV(q))$ with the potential force $-dV(q)$. Recall that the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ is a basis for $T_q U_\alpha$ when evaluated at $q \in U_\alpha$. The local representation for the vector field associated with the potential force is

$$\mathbb{G}^\sharp(-dV(q)) = -\frac{\partial V(q)}{\partial x^j}\mathbb{G}^{ij}(q)\frac{\partial}{\partial x^i}$$

where $-\frac{\partial V(q)}{\partial x^j}\mathbb{G}^{ij}(q)$ are the component functions on M relative to the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$. We denote the vector field associated with the potential force by $-\operatorname{grad} V(q)$. The most common potential force encountered in this research is a gravitational force. A simple mechanical system that evolves on a configuration manifold M naturally carries with it two forms of energy: **potential energy** defined by $V : TM \to \mathbb{R}$ and **kinetic energy** defined by $KE : TM \to \mathbb{R}$. We say that the **natural Lagrangian** is the difference between the kinetic and potential energies.

3.1.5 Euler-Lagrange Equations and Affine Connection

The general **Lagrangian** is a smooth function L on the tangent bundle TM to the configuration manifold M. This function depends on the configuration and velocity of the mechanical system. Let $(T\phi_\alpha, TU_\alpha)$ be the natural charts on TM with the natural coordinates $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ for $v_q \in TM$. In coordinates, the local expression for L is

$$TM \ni v_q \mapsto L(x^1, \ldots, x^n, v^1, \ldots, v^n) \in \mathbb{R}.$$

Let $\gamma : [0, a] \to M$ be a smooth curve on M. A **variation** of γ is a smooth map $\varphi : (-\epsilon, \epsilon) \times [0, a] \to M$ such that:

(i) $\varphi(0, t) = \gamma(t) \ \forall \ t \in [0, a]$,

(ii) $\varphi(s, 0) = \gamma(0) \; \forall \; s \in (-\epsilon, \epsilon)$, and

(iii) $\varphi(s, a) = \gamma(a) \; \forall \; s \in (-\epsilon, \epsilon)$.

For each $s \in (-\epsilon, \epsilon)$, the parameterized curve $\varphi_s : [0, a] \to M$ given by $\varphi_s(t) = \varphi(s, t)$ is called a **curve in the variation**. A variation determines a family $\varphi_s(t)$ of nearby curves of $\varphi_0(t) = \gamma(t)$. The **variational vector field** of φ is the vector field along γ defined by

$$\delta\varphi(t) = \left.\frac{d}{ds}\right|_{s=0} \varphi(s, t) \in \Gamma(T_{\gamma(t)}M).$$

Let M be the configuration manifold with $a \in \mathbb{R}$, $0 < a$ and $q_0, q_a \in M$. We say that

$$C^\infty([0, a], q_0, q_a) = \{\gamma : [0, a] \to M \mid \gamma(0) = q_0, \; \gamma(a) = q_a, \; \gamma \in C^\infty(M)\}$$

is the set of smooth curves on the interval $[0, a]$ that start at q_0 and end at q_a. The **action** for a Lagrangian L on M is the function

$$A_L : C^\infty([0, a], q_0, q_a) \to \mathbb{R}$$

defined by

$$A_L(\gamma) = \int_0^a L(\gamma'(t))dt$$

where $\gamma'(t)$ is the velocity curve of $\gamma(t)$. The fundamental problem in the s- tudy of calculus of variations is finding the curve $\gamma^* \in C^\infty([0, a], q_0, q_a)$ such that $A_L(\gamma^*) \leq A_L(\gamma)$ for all $\gamma \in C^\infty([0, a], q_0, q_a)$. We say that this curve **minimizes** A_L. **Hamilton's principle** states that the motion of a mechanical system from time t_1 to time t_2 is such that the action $A_L(\gamma)$ has a stationary value for the actual path of the motion. We say that the curve γ describing the motion of a mechanical system is an **extremal** for the action A_L. In fact, the curve γ that is an extremal is exactly the curve that satisfies

$$\left.\frac{d}{ds}\right|_{s=0} \int_0^a L\left(\frac{d}{dt}\varphi(s,t)\right) dt = 0$$

for all variations φ of γ. We can now state that Hamilton's principle is a sufficient condition for deriving the Euler-Lagrange equations.

Theorem 3.1.1 (Hamilton's principle and the Euler-Lagrange equations)**.** *Let (U_α, ϕ_α) be any chart where $\gamma(t) \in U_\alpha$ for all $t \in [0,a]$. The local expression for γ in this chart is $t \mapsto (x^1(t), \ldots, x^n(t))$. If such a curve $\gamma \in C^\infty([0,a], q_0, q_a)$ minimizes A_L, then*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0, \ \forall i \in \{1, \ldots, n\}.$$

Proof. If γ is a minimizer for A_L then $s = 0$ should be the minimum of $A_L(\varphi_s)$. Let x_s be the coordinate representation of φ_s. In coordinates, Hamilton's principle states

$$0 = \left.\frac{d}{ds}\right|_{s=0} \int_0^a L(x_s(t), \dot{x}_s(t))dt.$$

If we allow the variational vector field $\left.\frac{d}{ds}\right|_{s=0}$ to act inside the integral on the Lagrangian and apply the chain rule then we get

$$0 = \int_0^a \left(\frac{\partial L}{\partial x^i} \left.\frac{dx_s^i(t)}{ds}\right|_{s=0} + \frac{\partial L}{\partial v^i} \left.\frac{d\dot{x}_s^i(t)}{ds}\right|_{s=0} \right) dt.$$

Note that

$$\left.\frac{d\dot{x}_s^i(t)}{ds}\right|_{s=0} = \left.\frac{d}{ds}\right|_{s=0} \frac{dx_s^i}{dt} = \left.\frac{d}{dt}\frac{dx_s^i}{ds}\right|_{s=0}.$$

Substituting this relationship into the previous expression gives us

$$\begin{aligned} 0 &= \int_0^a \left(\frac{\partial L}{\partial x^i} \left.\frac{dx_s^i(t)}{ds}\right|_{s=0} + \frac{\partial L}{\partial v^i} \frac{d}{dt} \left.\frac{dx_s^i}{ds}\right|_{s=0} \right) dt \\ &= \int_0^a \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i} \right) \right) \left.\frac{dx_s^i}{ds}\right|_{s=0} dt + \left.\left.\frac{\partial L}{\partial v^i} \frac{dx_s^i}{dt}\right|_{s=0}\right|_{t=0}^{t=a} \\ &= \int_0^a \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i} \right) \right) \left.\frac{dx_s^i}{ds}\right|_{s=0} dt \end{aligned}$$

Therefore,

$$0 = \int_0^a \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i} \right) \right) \left.\frac{dx_s^i}{ds}\right|_{s=0} dt$$

for all variations and since

$$\left.\frac{dx_s^i}{ds}\right|_{s=0}$$

is arbitrary then

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i} \right) = 0$$

when evaluated at $(x(t), \dot{x}(t))$ for all $t \in [0, a]$. $\square$

The system of n equations above are called the **Euler-Lagrange equations** with respect to the Lagrangian L. These equations are implict second-order differential equations. The following result describes the solutions to the Euler-Lagrange equations for the Lagrangian $L_{\mathbb{G}}$. This is the first connection made between mechanics and the affine connection.

Theorem 3.1.2 (Euler-Langrange equations on a Riemannian manifold)**.** *Let M be the configuration manifold for a mechanical system, $\mathbb{G}$ denote the kinetic energy of the system, and $L_{\mathbb{G}}$ on TM be the Lagrangian defined by $L_{\mathbb{G}}(v_q) = KE(v_q)$. The solutions of the Euler-Lagrange equations corresponding to $L_{\mathbb{G}}$ are exactly the geodesics of the Levi-Civita connection ∇.*

Proof. Let (U_α, ϕ_α) be the natural chart on M with local coordinates $(x^1, \ldots, x^n)$. The Euler-Lagrange equations are

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$$

where $i = 1, \ldots, n$. Let us derive a local expression for the Euler-Lagrange equations in terms of the Riemannian metric $\mathbb{G}$. First, we substitute the local expression

$$L_{\mathbb{G}}(x^i, \dot{x}^i) = \frac{1}{2}\mathbb{G}_{ij}\dot{x}^i\dot{x}^j$$

into the Euler-Lagrange equations. If we take the partial derivative of $L_{\mathbb{G}}$ with respect $\dot{x}$ we get

$$\frac{\partial L}{\partial \dot{x}^i} = \mathbb{G}_{ij}\dot{x}^j.$$

Now we use the product and chain rule to expand

$$\frac{d}{dt}\mathbb{G}_{ij}\dot{x}^j = \frac{d\mathbb{G}_{ij}}{dt}\dot{x}^j + \mathbb{G}_{ij}\ddot{x}^j = \frac{\partial \mathbb{G}_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j + \mathbb{G}_{ij}\ddot{x}^j.$$

Again, it follows from the chain rule that

$$\frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial \mathbb{G}_{kj}}{\partial x^i}\dot{x}^k\dot{x}^j.$$

We combine these two results to get the following expression

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} &= \frac{\partial \mathbb{G}_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j + \mathbb{G}_{ij}\ddot{x}^j - \frac{1}{2}\frac{\partial \mathbb{G}_{kj}}{\partial x^i}\dot{x}^k\dot{x}^j \\
&= \mathbb{G}_{ij}\ddot{x}^j + \left(\frac{\partial \mathbb{G}_{im}}{\partial x^k}\dot{x}^k\dot{x}^m - \frac{1}{2}\frac{\partial \mathbb{G}_{km}}{\partial x^i}\dot{x}^k\dot{x}^m\right) \\
&= \mathbb{G}_{ij}\ddot{x}^j + \left(\frac{\partial \mathbb{G}_{im}}{\partial x^k} - \frac{1}{2}\frac{\partial \mathbb{G}_{km}}{\partial x^i}\right)\dot{x}^k\dot{x}^m.
\end{aligned}$$

We can then factor $\mathbb{G}_{ij}$ out of the expression above to get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \quad = \quad \mathbb{G}_{ij}\left(\ddot{x}^j + \mathbb{G}^{ij}\left(\frac{\partial \mathbb{G}_{im}}{\partial x^k} - \frac{1}{2}\frac{\partial \mathbb{G}_{km}}{\partial x^i}\right)\dot{x}^k\dot{x}^m\right).$$

Recall that the Riemannian metric $\mathbb{G}$ is symmetric, *i.e.*, $\mathbb{G}_{ij} = \mathbb{G}_{ji}$. This allows us to write

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \quad = \quad \mathbb{G}_{ij}\left(\ddot{x}^j + \mathbb{G}^{ij}\left(\frac{\partial \mathbb{G}_{jm}}{\partial x^k} - \frac{1}{2}\frac{\partial \mathbb{G}_{km}}{\partial x^j}\right)\dot{x}^k\dot{x}^m\right).$$

Now let us examine the term

$$\frac{\partial \mathbb{G}_{jm}}{\partial x^k} - \frac{1}{2}\frac{\partial \mathbb{G}_{km}}{\partial x^j}$$

and recall the expression for the Christoffel symbols. It follows from the symmetry of the Levi-Civita connection that $\Gamma^i_{km} = \Gamma^i_{mk}$. This implies that

$$\Gamma^i_{km} = \frac{1}{2}\mathbb{G}^{ij}\left(\frac{\partial \mathbb{G}_{mj}}{\partial x^k} + \frac{\partial \mathbb{G}_{jk}}{\partial x^m} - \frac{\partial \mathbb{G}_{km}}{\partial x^j}\right) = \mathbb{G}^{ij}\left(\frac{\partial \mathbb{G}_{mj}}{\partial x^k} - \frac{1}{2}\frac{\partial \mathbb{G}_{km}}{\partial x^j}\right).$$

The Euler-Lagrange equations can now be written

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \quad = \quad \mathbb{G}_{ij}(\ddot{x}^j + \Gamma^j_{km}\dot{x}^k\dot{x}^m).$$

Recall that solutions $((x^1(t), \ldots, x^n(t)), (\dot{x}^1(t), \ldots, \dot{x}^n(t)))$ to the Euler-Lagrange equations must satisfy

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0$$

where $i = 1, \ldots, n$. By definition, we know that $\mathbb{G}$ is nondegenerate which implies that solutions $((x^1(t), \ldots, x^n(t)), (\dot{x}^1(t), \ldots, \dot{x}^n(t)))$ to the Euler-Lagrange equations also satisfy the geodesic equations

$$\ddot{x}^j + \Gamma^j_{km} \dot{x}^k \dot{x}^m = 0$$

for $j = 1, \ldots, n$. This proves that the geodesics of the Levi-Civita connection ∇ are in one-to-one correspondence with the solutions of the Euler-Lagrange equations for the Lagrangian $L_{\mathbb{G}}(v_q) = \frac{1}{2}\mathbb{G}(v_q, v_q)$. $\square$

3.1.6 External Force

We consider two classes of force that influence the motion of a mechanical control system. The first class is called an external force such as a potential force, friction or dissipative force. The second class is called a control force. We restrict our attention to control forces that only depend on the position of the mechanical system. In a differential geometric setting for mechanical systems, forces take values in the cotangent bundle T^*M of the configuration manifold M. The reason why we model forces as elements of the cotangent bundle is that a force does work on system as it moves. Let the curve $\gamma : I \to M$ describe the motion of a mechanical system. Suppose that a force $F(\gamma(t))$ is applied to the system. We know that the work done by the force on the system is

$$\text{Work} = \int_I F(\gamma(t)) \cdot \gamma'(t) dt.$$

Note that the work is the integral of the product of force and velocity

$$F(\gamma(t)) \cdot \gamma'(t).$$

Since work is a scalar quantity, we expect that a force is a differential geometric object that when paired with a velocity returns a real number $\mathbb{R}$. This is exactly why a force is modeled as a element of the cotangent bundle.

The **total control force** F differs slightly from an external force in that the total control force is a linear combination of the one-forms $F^1, \ldots, F^m$ with $F = u^a F^a$ where summation is assumed over the repeated indice a. The term u^a is called the **controls** which can be a function of position, velocity, and/or time.

3.1.7 Lagrange-d'Alembert Principle

Let L be a Lagrangian on TM and F be a force that takes values in the cotangent bundle T^*M. A smooth curve $\gamma : [0, a] \to M$ satisfies the **Lagrange-d'Alembert Principle** for the force F and Lagrangian L along the curve γ if for all variations $(-\epsilon, \epsilon) \times [0, a] \to M$ of γ it holds that

$$\left.\frac{d}{ds}\right|_{s=0} \int_0^a L\left(\frac{d}{dt}\varphi(s,t)\right) dt + \int_0^a F(\gamma'(t)) \cdot \delta\varphi(t) dt = 0.$$

The following well-know result describes the motion of a Lagrangian system in the presence of forces.

Theorem 3.1.3 (Forced Euler-Lagrange equations)**.** *Let L be a Lagrangian on M with force F on M. A smooth curve $\gamma : [0, a] \to M$ satisfies the Lagrange-d'Alembert Principle for the force F and Lagrangian L if and only if, for any coordinate chart (U_α, ϕ_α) where $\gamma(t) \in U_\alpha$ for all $t \in [0, a]$, the local expression for the curve γ given by $(x^1(t), \ldots, x^n(t))$ satisfies*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = F_i, \; i \in \{1, \ldots, n\}$$

where $F_1, \ldots, F_n$ are the components of F.

Proof. Let (U_α, ϕ_α) be a coordinate chart such that $\gamma(t) \in U_\alpha$ for all $t \in [0, a]$. Let x_s be the coordinate representation for the curve in variation φ_s of γ. In coordinates, Lagrange-d'Alembert principle states

$$0 = \left.\frac{d}{ds}\right|_{s=0} \int_0^a L(x_s(t), \dot{x}_s(t))dt + \int_0^a F(x(t), \dot{x}(t)) \cdot \left.\frac{d}{ds}\right|_{s=0} x_s(t)dt.$$

We know from our proof of Hamilton's principle being a sufficient condition for the Euler-Lagrange equations that the first term on the right-hand-side of the expression above

$$\left.\frac{d}{ds}\right|_{s=0} \int_0^a L(x_s(t), \dot{x}_s(t))dt$$

is equivalent to

$$\int_0^a \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\right) \left.\frac{dx_s^i}{ds}\right|_{s=0} dt.$$

The second term in the right-hand-side of the coordinate representation of Lagrange-d'Alembert principle

$$\int_0^a F(x(t), \dot{x}(t)) \cdot \left.\frac{d}{ds}\right|_{s=0} x_s(t)dt$$

in component form is equivalent to

$$\int_0^a F^i \left.\frac{dx_s^i}{ds}\right|_{s=0} dt.$$

Combining the new coordinate representations of the first and second term of the right-hand-side of the Lagrange-d'Alembert principle gives

$$0 = \int_0^a \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) + F^i \right) \frac{dx_s^i}{ds}\bigg|_{s=0} dt.$$

Since this must hold for arbitrary variations, we conclude that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = F^i.$$

□

The coordinate-invariant expression for the Lagrange-d'Alembert Principle for the force F and the Lagrangian $L_{\mathbb{G}}$ on Riemannian manifolds is

$$\frac{d}{ds}\bigg|_{s=0} \frac{1}{2}\int_0^a \mathbb{G}(\varphi_s'(t), \varphi_s'(t))dt + \int_0^a F(\varphi_s'(t)) \cdot \delta\varphi(t)dt = 0.$$

Let us define

$$S_\varphi(s,t) = \frac{d}{ds}\varphi(s,t)$$

to be the vector field along the curve φ_s and

$$T_\varphi(s,t) = \frac{d}{dt}\varphi(s,t)$$

to be the tangent vector field along the curve φ_s. We compute the following

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \frac{1}{2}\int_0^a \mathbb{G}(\varphi_s'(t), \varphi_s'(t))dt &= \frac{d}{ds}\bigg|_{s=0} \frac{1}{2}\int_0^a \mathbb{G}(\frac{d}{dt}\varphi(s,t), \frac{d}{dt}\varphi(s,t))dt \\ &= \int_0^a \mathbb{G}(\nabla_{\frac{d}{ds}\varphi(s,t)}\frac{d}{dt}\varphi(s,t), \frac{d}{dt}\varphi(s,t))dt\bigg|_{s=0}. \end{aligned}$$

It follows from the definition of a Lie bracket and a coordinate calculation that

$[S_\varphi(s,t), T_\varphi(s,t)] = 0$ along φ. Therefore, we compute

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_0^a \mathbb{G}(\varphi_s'(t), \varphi_s'(t))dt &= \left.\int_0^a \mathbb{G}(\nabla_{\frac{d}{ds}\varphi(s,t)} \frac{d}{dt}\varphi(s,t), \frac{d}{dt}\varphi(s,t))dt\right|_{s=0}. \\ &= \left.\int_0^a \mathbb{G}(\nabla_{\frac{d}{dt}\varphi(s,t)} \frac{d}{ds}\varphi(s,t), \frac{d}{dt}\varphi(s,t))dt\right|_{s=0}. \end{aligned}$$

We can remove the explicit dependence on the parameter s by evaluating the argument inside the integral at $s = 0$ to get

$$\int_0^a \mathbb{G}(\nabla_{\gamma'(t)}\delta\varphi(t), \gamma'(t))dt.$$

Now we expand the left-hand-side of the expression above using the compatibility of the Levi-Civita connection to get

$$\int_0^a \left(\frac{d}{dt}\mathbb{G}(\delta\varphi(t), \gamma'(t)) - \mathbb{G}(\nabla_{\gamma'(t)}\gamma'(t), \delta\varphi(t)) \right) dt.$$

By definition, $\delta\varphi$ vanishes at the endpoints, so

$$\left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_0^a \mathbb{G}(\varphi_s'(t), \varphi_s'(t))dt = \int_0^a -\mathbb{G}(\nabla_{\gamma'(t)}\gamma'(t), \delta\varphi(t))dt.$$

Now substitute the right-hand-side of the expression above into the coordinate-invariant expression for the Lagrange-d'Alembert Principle to get

$$0 = \int_0^a -\mathbb{G}(\nabla_{\gamma'(t)}\gamma'(t), \delta\varphi(t))dt + \int_0^a F(\gamma'(t)) \cdot \delta\varphi(t)dt \qquad (3.1)$$

for all variations φ. This can be rewritten using the musical isomorphism $\mathbb{G}^\sharp$ to be

$$0 \;=\; \int_0^a \mathbb{G}(\nabla_{\gamma'(t)}\gamma'(t), \delta\varphi(t)) - \mathbb{G}(\mathbb{G}^\sharp(F(\gamma'(t))), \delta\varphi(t))dt. \tag{3.2}$$

Again, this expression must hold for all variations φ which implies that

$$\nabla_{\gamma'(t)}\gamma'(t) = \mathbb{G}^\sharp(F(\gamma'(t)))$$

where ∇ is the Levi-Civita connection. We call this the **coordinate-invariant representation of the equations of motion** for a Lagrangian system $L_\mathbb{G}$ in the presence of an external force F. Let us take the natural chart $(TU_\alpha, T\phi_\alpha)$ on TM with natural coordinates $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ for $v_q \in TM$. Recall that the family of vector fields $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$ on U_α when evaluated at each point $q \in U_\alpha$ defines a natural basis for T_qU_α. In coordinates, the velocity curve is

$$\gamma'(t) = ((x^1(t), \ldots, x^n(t)), (v^1(t), \ldots, v^n(t))).$$

The **local representation of the equations of motion** for a Lagrangian system $L_\mathbb{G}$ in the presence of an external force F is given by

$$\dot{v}^i(t) + \Gamma^i_{jk}(x^1(t), \ldots, x^n(t))v^j(t)v^k(t)$$

$$\shortparallel$$

$$F_j(x^1(t), \ldots, x^n(t))\mathbb{G}^{ij}(x^1(t), \ldots, x^n(t))$$

where F_j are the components of the external force with respect to the family of dual one-forms $dx^1, \ldots, dx^n$ on $\Gamma(TU_\alpha)$ that when evaluated at $q \in M$ form the dual basis for $T_q^*U_\alpha$.

Furthermore, if the force F is the potential force $-dV(q)$ then the coordinate-invariant equations of motion are

$$\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t))$$

where $-\operatorname{grad} V(\gamma(t)) = \mathbb{G}^\sharp(-dV(\gamma(t)))$. We call a mechanical system with kinetic energy Lagrangian $L_\mathbb{G}$ in the presence of a potential force $-dV$ a **simple mechanical system** denoted by the 3-tuple $\{M, \mathbb{G}, V\}$. The local representation of the equations of motion for a simple mechanical system $\{M, \mathbb{G}, V\}$ is given by

$$\dot{v}^i(t) + \Gamma^i_{jk}(x^1(t), \ldots, x^n(t))v^j(t)v^k(t)$$

$$=$$

$$-\frac{\partial V}{\partial x^j}(x^1(t), \ldots, x^n(t))\mathbb{G}^{ij}(x^1(t), \ldots, x^n(t))$$

where $-\frac{\partial V}{\partial x^j}$ are the components of the potential force with respect to the family of dual one-forms $dx^1, \ldots, dx^n$ on $\Gamma(TU_\alpha)$ that when evaluated at $q \in M$ form the dual basis for $T_q^*U_\alpha$.

3.1.8 Linear Velocity Constraint

A **linear velocity constraint** is a distribution $\mathcal{H}$ on the configuration manifold M such that the annihilator associated with $\mathcal{H}$ is a codistribution denoted by $\operatorname{ann}(\mathcal{H})$. We say that a smooth curve $\gamma : I \to M$ is **consistent** with the linear velocity constraint or **constraint distribution** $\mathcal{H}$ on M if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all $t \in I$. In other words, we specify a subspace $\mathcal{H}_q \subset T_qM$ that describes the set of velocities admissible at each configuration $q \in M$. A constraint distribution $\mathcal{H}$ is **holonomic** if $\mathcal{H}$ is integrable. If the constraint distribution $\mathcal{H}$ is not integrable, we say it is **nonholonomic**. This is equivalent to saying that all curves γ that pass through $q \in M$ consistent with the constraint distribution $\mathcal{H}$ have to stay on the maximal integral manifold for $\mathcal{H}$ through q. Finally, we say that a constraint distribution $\mathcal{H}$ is **totally nonholonomic** if the $\mathcal{H}$-orbits through q denoted by $\mathcal{O}(q, \mathcal{H})$ is equal to M for all $q \in M$.

Given a constraint distribution $\mathcal{H}$, the **constraint force** is a force taking values in the annihilator which is the codistribution $\mathrm{ann}(\mathcal{H})$. Let $\gamma : I \to M$ be a smooth curve, we say that a **constraint force along** γ is a one-form $\psi : I \to T^*M$ along γ such that $\psi(t) \in \mathrm{ann}(\mathcal{H})_{\gamma(t)}$ for all $t \in I$. A **constrained simple mechanical system** is the 4-tuple $\{M, \mathbb{G}, V, \mathcal{H}\}$, where M is the configuration manifold, $\mathbb{G}$ is the kinetic energy metric, V is the potential function and $\mathcal{H}$ is the constraint. A curve $\gamma : I \to M$ is a **trajectory for the constrained simple mechanical system** $\{M, \mathbb{G}, V, \mathcal{H}\}$ if the curve γ is consistent with the constraint distribution $\mathcal{H}$ and the curve γ is consistent with the Lagrange-d'Alembert Principle for the force $-dV + \psi$ and Lagrangian $Ł_{\mathbb{G}}$.

3.1.9 Contrained Affine Connection

Let $\{M, \mathbb{G}, V, \mathcal{H}\}$ be a constrained simple mechanical system. Given a constraint distribution $\mathcal{H}$, we may restrict the Levi-Civita affine connection ∇ to $\mathcal{H}$. The constrained solutions are those curves γ that satisfy

$$\begin{aligned} \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) &= \lambda(t) \\ P_{\mathcal{H}}^{\perp}(\dot{\gamma}(t)) &= 0 \end{aligned}$$

where $\lambda \in \Gamma(\mathcal{H}^{\perp})$, $\mathcal{H}^{\perp}$ is the $\mathbb{G}$-orthogonal complement to $\mathcal{H}$ along γ, and $P_{\mathcal{H}}^{\perp} : TM \to TM$ is the orthogonal projection onto $\mathcal{H}^{\perp}$. We may combine the two equations above to eliminate λ and arrive at a single expression

$$\overset{\mathcal{H}}{\nabla}_X Y = \nabla_X Y + (\nabla_X P_{\mathcal{H}}^{\perp})(Y)$$

where γ is a geodesic of the new affine connection $\overset{\mathcal{H}}{\nabla}$ called the **constrained affine connection**.

Let $\{H_1, \ldots, H_K\}$ be the family of vector fields on $q \in U_\alpha$ such that

$$\{H_1(q), \ldots, H_K(q)\}$$

is a $\mathbb{G}$-orthonormal basis for $\mathcal{H}_q \subset T_q U_\alpha$ and the rank of $\mathcal{H}$ be K. The generalized Christoffel symbols for the constrained affine connection $\overset{\mathcal{H}}{\nabla}$ are

$$\widehat{\Gamma}^{\iota}_{\kappa\nu}(q) = \mathbb{G}(\nabla_{H_\kappa} H_\nu(q), H_\iota(q))$$

for $\iota, \kappa, \nu \in \{1, \ldots, K\}$. Given $U_\alpha \subset M$ and the local representation of the velocity curve $\gamma'(t) = v^{\iota}(t) H_\iota(\gamma(t))$, the local components of $\overset{\mathcal{H}}{\nabla}_{\gamma'(t)} \gamma'(t)$ with respect to the family of vector fields $\{H_1, \ldots, H_K\}$ are

$$\dot{v}^{\iota}(t) + \widehat{\Gamma}^{\iota}_{\kappa\nu}(\gamma(t)) v^{\kappa}(t) v^{\nu}(t)$$

for $\iota, \kappa, \nu \in \{1, \ldots, K\}$. Recall that the functions v^{ι} for $\iota \in \{1, \ldots, K\}$ are called pseudo-velocities and are not necessarily locally equivalent to the time derivative of the configuration.

The coordinate-invariant representation for the equations of motion for a constrained Lagrangian $L_{\mathbb{G}}$ system in the presence of the constraint distribution $\mathcal{H}$ and an external force F is

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^{\sharp}(F))$$

where $P_{\mathcal{H}}$ is the $\mathbb{G}$-orthogonal projection onto $\mathcal{H}_q$. Let $U_\alpha \subset M$ with the local coordinates $(x^1, \ldots, x^n)$ and the local representation of the velocity curve be

$$\gamma'(t) = ((x^1(t), \ldots, x^n(t)), (v^1(t), \ldots, v^K(t)))$$

where $v^\iota(t)$ are the pseudo-velocity components of $\gamma'(t)$ with respect to the family of vector fields $\{H_1, \ldots, H_K\}$ generates $\mathcal{H}$. The local representation for the equations of motion for a Lagrangian $L_{\mathbb{G}}$ system in the presence of the constraint distribution $\mathcal{H}$ and an external force F is

$$\dot{v}^\iota(t) + \Gamma^\iota_{\kappa\nu}(x^1(t), \ldots, x^n(t))v^\kappa(t)v^\nu(t)$$

$$\shortparallel$$

$$\mathbb{G}_{ap}(x^1(t), \ldots, x^n(t))F_j(x^1(t), \ldots, x^n(t))\mathbb{G}^{aj}(x^1(t), \ldots, x^n(t))H^p_\iota$$

where $\iota, \kappa, \nu \in \{1, \ldots, K\}$ and $a, p, j \in \{1, \ldots, n\}$. Similarly, the local representation for the equations of motion for a simple mechanical system in the presence of the constraint distribution $\mathcal{H}$ is

$$\dot{v}^\iota(t) + \Gamma^\iota_{\kappa\nu}(x^1(t), \ldots, x^n(t))v^\kappa(t)v^\nu(t)$$

$$\shortparallel$$

$$-\mathbb{G}_{ap}(x^1(t),\ldots,x^n(t))\frac{\partial V}{\partial x^j}(x^1(t),\ldots,x^n(t))\mathbb{G}^{aj}(x^1(t),\ldots,x^n(t))H^p_\iota.$$

3.2 Nonlinear Control Systems

Nonlinear control theory is the study of the manipulation of nonlinear dynamical systems to achieve desired objectives. The dynamical laws governing these systems are not fixed as in classical physics, rather they depend on parameters referred to as controls. The most general class of nonlinear control system that we consider is a control-affine system. We restrict our attention to the class of control-affine systems commonly referred to as simple mechanical control systems. We assume that the set of all possible configurations and velocities of a simple mechanical control system is the tangent bundle TM of a smooth n-dimensional Riemannian manifold $(M, \mathbb{G})$. Furthermore, the dynamics of the system are described by vector fields on TM that may depend on control parameters.

3.2.1 Control-Affine System

A **control-affine system** is the triple $(M, \mathcal{V} = \{f_0, f_1, \ldots, f_m\}, U)$ where M is a smooth manifold, $\mathcal{V}$ is a family of smooth vector fields on M and $U \subset \mathbb{R}^m$. The coordinate-invariant expression for the equations of motion for a control-affine system are

$$\gamma'(t) = f_0(\gamma(t)) + u^a(t) f_a(\gamma(t))$$

where u^a are the components of the map $u : I \to U \subset \mathbb{R}^M$. This mapping u is called the **control** or **input** that takes values in the **control set** U. The smooth manifold M is called the **state manifold**. The curve $\gamma : I \to M$ is the **trajectory** of the system. The vector field f_0 is called the **drift vector field** which represents the natural dynamics of the system (*i.e.* no control). Finally, the family of vector fields $\{f_1, \ldots, f_m\}$ are the **control vector fields** or **input vector fields**. A control-affine system $(M, \mathcal{V} = \{f_0, f_1, \ldots, f_m\}, U)$ is **fully actuated** at the state $q \in M$ if the distribution $\mathcal{F}$ generated by the family of vector fields $\{f_1, \ldots, f_m\}$ is such that $\mathcal{F}_q = T_qM$. Otherwise, we say the control-affine system is **underactuated** at $q \in M$.

A **linear control system** is the triple (V, A, B) where V is a vector space, $A : V \to V$ and $B : \mathbb{R}^m \to V$ are linear maps. We can assign a control-affine system to each linear control system (V, A, B) by setting the state manifold $M = V$, the drift vector field $f_0(x) = A(x)$, the control vector field $f_a(x) = B(\mathbf{e}_a)$ for $a \in \{1, \ldots, m\}$ and $U = \mathbb{R}^m$. The governing equations for a linear control system is

$$\dot{x}(t) = A(x(t)) + B(u(t)).$$

Given a nonlinear control system, we can often assign to it an approximate linear control system using the process of linearization.

3.2.2 Simple Mechanical Control System

A **simple mechanical control systems** is the set of elements containing an n-dimensional **configuration manifold** M; a Riemannian metric $\mathbb{G}$ which is the **kinetic energy metric**; a $\mathbb{R}$-valued function V on M which is the **potential function**; m linearly independent one forms $F^1, \ldots, F^m$ on M which are the **control forces**; and $U \subset \mathbb{R}^m$ is the **control set**. We say that the simple mechanical control system is **fully actuated** if $m = n$, otherwise it is **underactuated**. Note that though we represent the control forces as one forms, we will use the associated dual vector fields $Y_i = \mathbb{G}^\sharp(F^i),\ i = 1, \ldots, m$ in our representation of the governing equations of motion. Given the local coordinate function

$\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$ in the neighborhood $U_\alpha \subset M$ containing q, we take the family of n vector fields denoted by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ on U_α to be the natural basis for T_qM when evaluated at each $q \in U_\alpha$. The local expression for the vector fields dual to the control forces is given by

$$Y_i(q) = F^i_k(q)\mathbb{G}^{jk}(q)\frac{\partial}{\partial x^j}$$

where F^i_k is the kth component of the ith one-form with respect to the dual one-forms $dx^1, \ldots, dx^n$ that form the dual basis for T^*_qM at each $q \in U_\alpha$.

Formally, we denote the control system by the tuple $\Sigma = \{M, \mathbb{G}, V, \mathcal{F}, U\}$ where $\mathcal{F}$ is the **input codistribution** generated by the family of one-forms $\{F^i \mid i = 1, \ldots, m\}$. Analogously, we will refer to $\mathcal{Y}$ as the **input distribution** generated by the family of vector fields $\{Y_i \mid Y_i = \mathbb{G}^\sharp(F^i) \ \forall\ i = 1, \ldots, m\}$ such that

$$\mathcal{Y}_q \equiv \mathcal{Y} \cap T_qM = \operatorname{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}.$$

Note we restrict ourselves to control systems where the input forces are dependent upon configuration and independent of velocity and time. The control forces are linear combinations of the one-forms $F^1, \ldots, F^m$, with the coefficients $u^a : I \to \mathbb{R}$ being the U-valued functions of time.

The equations of motion for a simple mechanical control system follows from the Lagrange-d'Alembert Principle. The global representation of the equations of motion is

$$\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))$$

where summation is assumed over index $a \in \{1, \ldots, m\}$. We can think of a simple mechanical control system as a control-affine system evolving on the state manifold

TM. This requires us to construct a representation of the equations of motion on TM. First, we take the vertical lift of vector field $Y_i = \mathbb{G}^\sharp(F^i)$ along the velocity curve $\gamma'(t)$ to get

$$\mathrm{Y}_i^{\mathrm{vlft}}(\gamma(t)) = \left.\frac{d}{ds}\right|_{s=0} \left(\gamma'(t) + s(\mathrm{Y}_i)_{\gamma(t)}\right)$$

and the vector field associated with the potential force $-\operatorname{grad} V$ to get

$$-\operatorname{grad} V^{\mathrm{vlft}}(\gamma(t)) = \left.\frac{d}{ds}\right|_{s=0} \left(\gamma'(t) + s(-\operatorname{grad} V)_{\gamma(t)}\right).$$

Second, we take the horizontal lift of the tangent vector field $\frac{d}{dt} = \gamma'(t)$ at each point along the velocity curve $\gamma'(t)$ (unavoidable poor notation). Let us take the natural chart $(TU_\alpha, T\phi_\alpha)$ on TM along with the associated family of vector fields $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}\}$ that when evaluated at point $v_q \in TU_\alpha$ generate a local coordinate frame for $T_{v_q}TU_\alpha$. The local components of the tangent vector field are written $\gamma'(t) = \dot\gamma^i(t)\frac{\partial}{\partial x^i}$. In coordinates, we have

$$\gamma'(t)^{\mathrm{hlft}} = \dot\gamma^i(t)\frac{\partial}{\partial x^i} - \Gamma^i_{jk}(\gamma(t))\dot\gamma^j(t)\dot\gamma^k(t)\frac{\partial}{\partial v^i}.$$

By inspection, $\gamma'(t)^{\mathrm{hlft}}$ is the geodesic spray $Z_{\gamma(t)}$ associated with the Levi-Civita connection ∇. Now we combine $Z_{\gamma(t)}$, $-\operatorname{grad} V^{\mathrm{vlft}}(\gamma(t))$ and $Y_i^{\mathrm{vlft}}(\gamma(t))$ to get a system of first-order differential equations on TM. The coordinate-invariant representation is given by

$$\Theta(t) = Z(\gamma'(t)) - \operatorname{grad} V^{\mathrm{vlft}}(\gamma'(t)) + Y_i^{\mathrm{vlft}}(\gamma'(t)).$$

Now we make the following association between the simple mechanical control system $\Sigma = \{M, \mathbb{G}, V, \mathcal{F}, U\}$ and the control-affine system $(M, \mathcal{V} = \{f_0, f_1, \ldots, f_m\})$. Let the drift vector field $f_0 = Z - \operatorname{grad} V^{\mathrm{vlft}}$, the control vector fields $f_a = Y_a^{\mathrm{vlft}}$ for $a \in \{1, \ldots, m\}$ and $U = U$.

Let (U_α, ϕ_α) be the coordinate chart on M with the local coordinates $(x^1, \ldots, x^n)$ and let the local representation of the velocity curve $\gamma'(t)$ be

$$t \mapsto ((x^1(t), \ldots, x^n(t)), (v^1(t), \ldots, v^n(t))).$$

The local representation for the equations of motion of Σ is the system given by

$$\dot{x}^i = v^i \tag{3.3}$$

$$\dot{v}^i = -\Gamma^i_{jk} v^j v^k - \frac{\partial V}{\partial x^j} \mathbb{G}^{ij} + u^a Y^i_a \tag{3.4}$$

where $i, j, k \in \{1, \ldots, n\}$ and $a \in \{1, \ldots, m\}$.

A **controlled trajectory** for Σ is a pair (γ, u) where $u : I \to U$ is locally integrable and $\gamma : I \to M$ satisfying $\dot{\gamma}(t_0) \in H_{\gamma(t_0)}$ for some $t_0 \in I$ such that the local representation of the system defined by Equations (3.3) and (3.4) hold. We denote by $\mathrm{Ctraj}(\Sigma)$ the controlled trajectories for Σ, and by $\mathrm{Ctraj}(\Sigma, T)$ the controlled trajectories defined on $[0, T]$.

3.2.3 Constrained Simple Mechanical Control System

A **constrained simple mechanical control system** is the set

$$\{M, \mathbb{G}, V, \mathcal{F}, \mathcal{H}, U\}$$

where the new element $\mathcal{H}$ is the distribution that represents the **linear velocity constraints** or **constraint distribution**. If we set $\mathcal{H} = TM$, then we have a simple mechanical control system, *i.e.* no linear velocity constraints. The constrained affine connection can be used to express the global representation of the equations of motion. These equations are written

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t)))$$

where $\overset{\mathcal{H}}{\nabla}$ is the constrained affine connection associated with $\mathcal{H}$ and $P_{\mathcal{H}}$ is the $\mathbb{G}$-orthogonal projection mapping $TM \mapsto \mathcal{H}$.

Let us assume that the family of vector fields

$$\{H_1, \ldots, H_K\}$$

are the $\mathbb{G}$-orthonormal set of vector fields that generate the constraint distribution $\mathcal{H}_q$ at each $q \in U_\alpha \subset M$. The **orthonormal Poincaré representation** of the constrained equations of motion is the set of first-order differential equations given by

$$\begin{aligned}
\dot{x}^i &= X^i_\nu v^\nu \\
\dot{v}^\iota &= -\widehat{\Gamma}^\iota_{\kappa\nu} v^\kappa v^\nu + u^a \mathbb{G}_{ip} Y^i_a H^p_\iota - \mathbb{G}_{kp}\frac{\partial V}{\partial x^j}\mathbb{G}^{kj} H^p_\iota
\end{aligned}$$

where $i, p, j, k \in \{1, \ldots, n\}$, $a = 1, \ldots, m$ and $\iota, \kappa, \nu \in \{1, \ldots, K\}$. Although the coordinate representations are easily programmable using symbolic solver software such as Mathematica ®, our choice of an orthonormal set of vector fields used to generate the constraint distribution $\mathcal{H}$ significantly simplifies the symbolic computations required to explicitly write the local equations of motion for a constrained system.

3.3 Motivating Examples

In this section we present some motivating examples. These examples will be revisited throughout this thesis and are intended to illustrate our contributions to modeling, analysis and algorithm design for underactuated mechanical systems.

3.3.1 Planar Rigid Body

In this section we introduce the geometric model of the forced planar rigid body (Figure 3.1). The linearization of this underactuated mechanical system is not controllable.

The configuration manifold for the system is the Lie group $SE(2)$ and the potential function is assumed to be identically zero. Let us use coordinates (x, y, θ) for the planar robot where (x, y) describes the position of the center of mass and θ describes the orientation of the body frame $\{b_1, b_2\}$ with respect to the inertial frame $\{e_1, e_2\}$. In these coordinates, the Riemannian metric is given by

$$\mathbb{G} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{pmatrix},$$

where m is the mass of the body and J is the moment of inertia about the center of mass. The inputs for this system consist of forces applied to a point that is a distance $h > 0$ from the center of mass along the b_1 body-axis and a torque about the center of mass. Physically, the input force can be thought of as a variable-

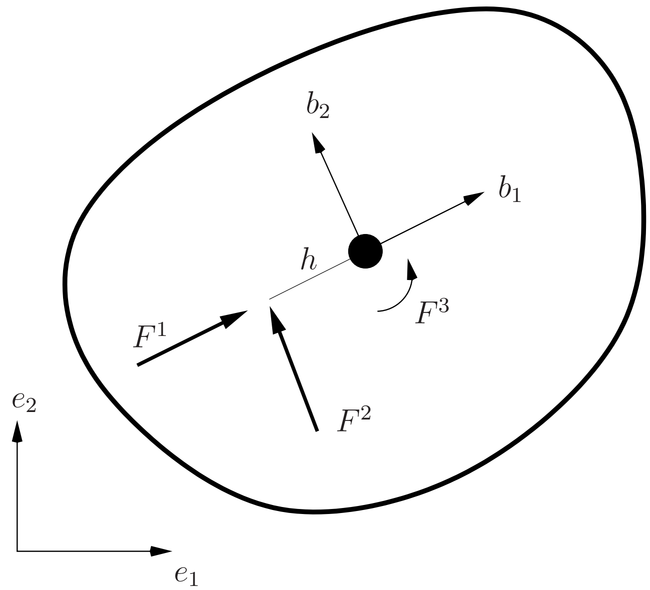

Figure 3.1. A schematic of the forced planar rigid body.

direction thruster on the body which can be resolve into components along the b_1 and b_2 directions. The control inputs are given by

$$
\begin{aligned}
F^1 &= \cos\theta dx + \sin\theta dy, \\
F^2 &= -\sin\theta dx + \cos\theta dy - h d\theta, \\
F^3 &= d\theta.
\end{aligned}
$$

We compute the corresponding control vector fields to be

$$
\begin{aligned}
Y_1 &= \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y}, \\
Y_2 &= -\frac{\sin\theta}{m}\frac{\partial}{\partial x} + \frac{\cos\theta}{m}\frac{\partial}{\partial y} + -\frac{h}{J}\frac{\partial}{\partial\theta}, \\
Y_3 &= \frac{1}{J}\frac{\partial}{\partial\theta}.
\end{aligned}
$$

The governing equations of motion given the control set $\{Y_1, Y_2\}$ is

$$\begin{aligned}
\dot{q}^i(t) &= v^i(t) \\
\dot{v}^1(t) &= \frac{\cos\theta(t)}{m}u^1(t) - \frac{\sin\theta(t)}{m}u^2(t) \\
\dot{v}^2(t) &= \frac{\sin\theta(t)}{m}u^1(t) + \frac{\cos\theta(t)}{m}u^2(t) \\
\dot{v}^3(t) &= -\frac{h}{J}u^2(t)
\end{aligned}$$

for $i = 1, \ldots, 3$. The governing equations of motion given the control set $\{Y_1, Y_3\}$ is

$$\begin{aligned}
\dot{q}^i(t) &= v^i(t) \\
\dot{v}^1(t) &= \frac{\cos\theta(t)}{m}u^1(t) \\
\dot{v}^2(t) &= \frac{\sin\theta(t)}{m}u^1(t) \\
\dot{v}^3(t) &= \frac{h}{J}u^3(t)
\end{aligned}$$

for $i = 1, \ldots, 3$.

3.3.2 Roller Racer

In this section we introduce the geometric model of the roller racer (Figure 3.2).

The configuration manifold for the roller racer is $SE(2) \times \mathbb{S}^1$ and we begin with the local coordinates (x, y, θ, ψ). We assume that the center of mass of the body of the roller racer is located over the wheel axle. The Riemannian metric is

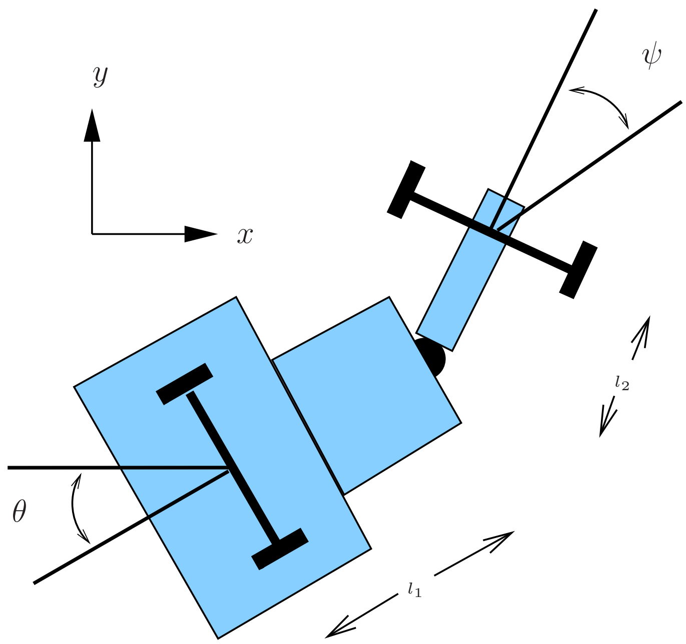

Figure 3.2. A schematic of the roller racer.

given by

$$\mathbb{G} \;=\; \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I_1 + I_2 & I_2 \\ 0 & 0 & I_2 & I_2 \end{pmatrix}$$

where $m > 0$ is the mass of the body of the roller racer, $I_1 > 0$ is the moment of inertia of the body about its center of mass and $I_2 > 0$ is the moment of inertia of the wheel assembly about the pivot point. The constraint one-forms are given by

$$
\begin{aligned}
\omega_1 &= -\sin\theta dx + \cos\theta dy, \\
\omega_2 &= -\sin\psi dx + \cos\psi dy + l_1\cos(\theta-\psi)d\theta + I_2 d\psi.
\end{aligned}
$$

The constraints one-forms above induce a constraint distribution $\mathcal{H}$ which is a subbundle of the tangent bundle. The constraint distribution is the largest potential reachable set of velocities. The constraint distribution $\mathcal{H}$ is spanned by the two vector fields

$$
\begin{aligned}
H_1 &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + \frac{\sin\psi}{l_1\cos\psi + l_2}\frac{\partial}{\partial\theta}, \\
H_2 &= -\frac{l_2}{l_2 + l_1\cos\psi}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi}.
\end{aligned}
$$

We use Gram-Schmidt and normalization to construct the corresponding $\mathbb{G}$ -orthonormal basis vector fields $\{H_{1o}, H_{2o}\}$ for $\mathcal{H}$ (see Appendix B). The single control force is defined by the one form $F_1 = d\phi$. The corresponding control vector field $\overset{\mathcal{H}}{Y}_1$ projected onto the constraint distribution $\mathcal{H}$ can be found in Appendix B. The governing equations of motion given the single input control set $\{Y_1\}$ is

$$
\begin{aligned}
\dot{q}^i(t) &= v^1(t)H^i_{1o}(\theta(t),\psi(t)) + v^2(t)H^i_{2o}(\theta(t),\psi(t)) \\
\dot{v}^k(t) &= -\widehat{\Gamma}^k_{lj}(\psi(t))v^l(t)v^j(t) + u^1(t)\overset{\mathcal{H}}{Y}_1{}^k(\psi(t))
\end{aligned}
$$

where $i = 1,\ldots,4,\ j,k,l = 1,2$ and the nonzero generalized Christoffel symbols $\widehat{\Gamma}^k_{lj}$ can be found in Appendix B.

3.3.3 Snakeboard

In this section we introduce the geometric model of the snakeboard (Figure 3.3).

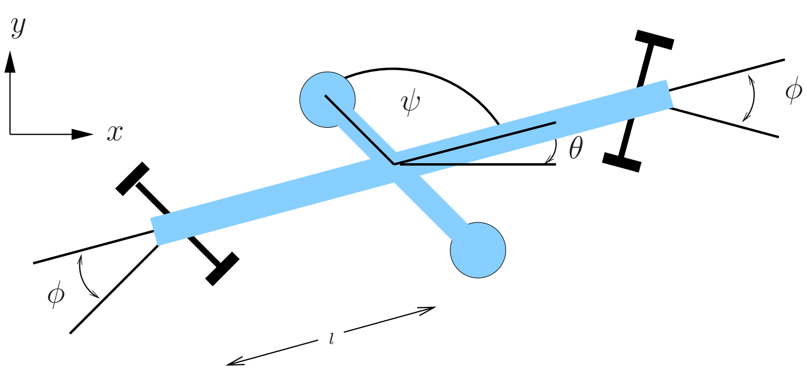

Figure 3.3. A schematic of the snakeboard.

The configuration manifold for the snakeboard is $SE(2) \times \mathbb{S} \times \mathbb{S}$ with local coordinates $(x, y, \theta, \psi, \phi)$. The Riemannian metric is given by

$$\mathbb{G} = \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & l^2 m & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & J_w \end{pmatrix},$$

where $m > 0$ is the total mass of the snakeboard, $J_r > 0$ is the moment of inertia of the rotor mounted on top of the body's center of mass, and $J_w > 0$ is the moment of inertia of the wheel axles. The constraint one-forms are given by

$$\begin{aligned}
\alpha_1 &= \sin(\phi - \theta)\, dx + \cos(\phi - \theta)\, dy + l\cos(\phi)\, d\theta, \\
\alpha_2 &= -\sin(\phi + \theta)\, dx + \cos(\phi + \theta)\, dy - l\cos(\phi)\, d\theta.
\end{aligned}$$

We use Gram-Schmidt and normalization to construct the corresponding $\mathbb{G}$ - orthonormal basis vector fields $\{H_{1o}, H_{2o}, H_{3o}\}$ for $\mathcal{H}$ (see Appendix C). The two control forces are pure torques $F^1 = d\psi$ and $F^2 = d\phi$. The corresponding control vector fields $\overset{\mathcal{H}}{Y}_1$ and $\overset{\mathcal{H}}{Y}_2$ projected onto the constraint distribution $\mathcal{H}$ can be found in Appendix C. The governing equations of motion given the two input control set $\{\overset{\mathcal{H}}{Y}_1, \overset{\mathcal{H}}{Y}_2\}$ is

$$\begin{aligned}
\dot{q}^i(t) &= v^1(t)H^i_{1o}(\theta(t), \phi(t)) + v^2(t)H^i_{2o}(\theta(t), \phi(t)) + v^3(t)H^i_{3o}(\theta(t), \phi(t)) \\
\dot{v}^k(t) &= -\widehat{\Gamma}^k_{lj}(\phi(t))v^l(t)v^j(t) + u^1(t)\overset{\mathcal{H}}{Y}_1{}^k(\phi(t)) + u^2(t)\overset{\mathcal{H}}{Y}_2{}^k
\end{aligned}$$

where $i = 1, \ldots, 5$, $j, k, l = 1, \ldots, 3$ and the nonzero generalized Christoffel symbols $\widehat{\Gamma}^k_{lj}$ can be found in Appendix C.

3.3.4 Three Link Manipulator

In this section we introduce the geometric model of the three link manipulator (Figure 3.4).

We consider the underactuated horizontal three link manipulator presented in [3]. We assume that the potential is zero thus no gravity is applied on the joints. The third joint is passive and is not equipped with an actuator. The passive joint is a revolute joint around a vertical axis. The first and second joint are

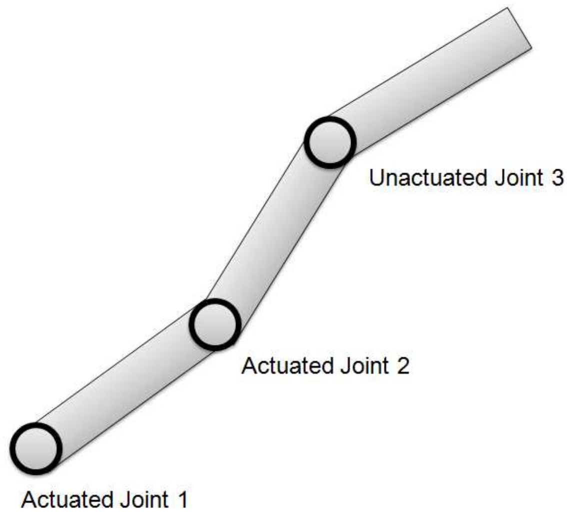

Figure 3.4. A schematic of the three link manipulator.

actuated and control the position of the passive joint in the plane. The passive joint can rotate freely and is indirectly driven by the dynamic coupling between the actuate joints and passive joint. To simplify the model, the dynamics of the first and second joint are neglected. The dynamics can then be modeled with regard to only the free link [3].

The configuration manifold for the system is the Lie group $SE(2)$. Let us use coordinates (x, y, θ) for the free link where (x, y) describes the position of the origin of the body frame $\{b_1, b_2\}$. The origin is at the third joint and the b_1-axis coincides with the center of mass of the link. Finally, the θ coordinate describes the orientation of the body frame $\{b_1, b_2\}$ with respect to the inertial frame $\{e_1, e_2\}$. In these coordinates, the Riemannian metric is given by

$$\mathbb{G} = \begin{pmatrix} m & 0 & -Lm\sin\theta \\ 0 & m & Lm\cos\theta \\ -Lm\sin\theta & Lm\cos\theta & I_c + L^2 m \end{pmatrix},$$

where m is the mass of the third link, L is the distance between the third joint and the center of mass of the third link and I_c is the moment of inertia of the third link about the center of mass. The inputs for this system consist of two translational forces applied to the third joint. The control inputs are given by

$$\begin{aligned} F^1 &= dx, \\ F^2 &= dy. \end{aligned}$$

The corresponding control vector fields Y_1 and Y_2 can be found in Appendix D. The governing equations of motion given the two input control set $\{Y_1, Y_2\}$ is

$$\begin{aligned} \dot{q}^i(t) &= v^i(t) \\ \dot{v}^k(t) &= -\Gamma^k_{ij}(\theta(t))v^i(t)v^j(t) + u^1(t)Y_1^k(\theta(t)) + u^2(t)Y_2^k(\theta(t)) \end{aligned}$$

where $i, j, k = 1, \ldots, 3$ and the nonzero Christoffel symbols Γ^k_{ij} can be found in Appendix D.

CHAPTER 4

AFFINE FOLIATION FOR UNDERACTUATED MECHANICAL SYSTEMS

This chapter contains a refinement of the basic geometric framework for mechanical control systems. Specifically, we account for the additional structure resulting from the underactuated nature of this class of mechanical control systems. Here we introduce an alternative geometric framework that models an underactuated mechanical system evolving on an affine foliation of the tangent bundle. The affine foliation of the tangent bundle is constructed from the input distribution $\mathcal{Y}$ and the Riemannian metric $\mathbb{G}$ included in the basic problem formulation. Though Riemannian geometry is a classic technique in modeling underactuated mechanical control systems, affine foliations and affine subbundles are not. In general, we think of an underactuated mechanical control system as moving from leaf to leaf in the affine foliation. Each leaf in the affine foliation is parameterized by a family of one-forms referred to as the *affine parameters*. We will show that the affine parameters represent the *unactuated* velocity states. Each leaf in the affine foliation can also be associated with an affine subbundle. The linear part of the affine subbundle is parameterized by a second family of one-forms referred to as the *linear parameters*. We will show that the linear parameters represent the *actuated* velocity states. Preliminary work can be found in two conference papers [50], [51].

Our alternative framework provides several important insights into the motion of an underactuated mechanical control system.

1. The affine and linear parameters naturally decompose the equations of motion into the actuated and unactuated dynamics.

2. The decomposition of the equations of motion for the underactuated system gives rise to an intrinsic quadratic structure that couples the actuated and unactuated dynamics.

3. The set of reachable states for an underactuated mechanical control system depends on the basic properties of the intrinsic quadratic structure.

4.1 Classic Geometric Model

We begin with the set $\Sigma = \{M, \mathbb{G}, V, \mathcal{F}, U\}$ that denotes a simple mechanical control system. Recall that we model the control forces as a codistribution $\mathcal{F}$ generated by the set of one-forms

$$\{F^a \mid a = 1, \ldots, m\}.$$

We assume that the control system is underactuated $m < n$ and the control forces are linearly independent. We can use this set of one-forms along with the Riemannian metric $\mathbb{G}$ to construct a set of dual vector fields $Y_a = \mathbb{G}^\sharp(F^a)$ for $a = 1, \ldots, m$. The set of dual vector fields generates a distribution $\mathcal{Y}$ defined by

$$\mathcal{Y}_q \equiv \mathcal{Y} \cap T_qM = \mathrm{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}$$

called the input distribution. Given the local coordinate function

$$\phi_\alpha(q) = (x^1(q), \ldots, x^n(q))$$

in the neighborhood $U_\alpha \subset M$ containing q, we take the family of n vector fields denoted by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ on U_α to be the natural basis for T_qM when evaluated at each $q \in U_\alpha$. The local expression for the one-forms that generate the codistribution $\mathcal{F}$ is

$$F^a(q) = F^a_i(q)dx^i$$

where F^a_i is the ith component of the ath one-form with respect to the dual one-forms $dx^1, \ldots, dx^n$ that form the dual basis for T^*_qM at each $q \in U_\alpha$. The local expression for the dual vector fields that generate the input distribution is

$$Y_a(q) = F^a_k(q)\mathbb{G}^{jk}(q)\frac{\partial}{\partial x^j}$$

where $F^a_k(q)\mathbb{G}^{jk}(q)$ is the jth component of the ath vector field with respect to the family of vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.

4.2 Affine Foliation Formulation

4.2.1 Orthonormal Frame

By definition, the set of vector fields

$$\{\mathbb{G}^\sharp(F^1), \ldots, \mathbb{G}^\sharp(F^m)\}$$

are linearly independent and form a basis for $\mathcal{Y}_q$ at each $q \in M$. We can use the Riemannian metric $\mathbb{G}$ and the set of vector fields $Y_a = \mathbb{G}^\sharp(F^a)$ to construct another set of vector fields $\widehat{Y}_a$ such that $\mathbb{G}(\widehat{Y}_a, \widehat{Y}_b) = 0$ if $a \neq b$ and $\mathbb{G}(\widehat{Y}_a, \widehat{Y}_a) = 1$ for all $a = 1, \ldots, m$. In other words, we use the set of vector fields $Y_a = \mathbb{G}^\sharp(F^a)$ to produce a $\mathbb{G}$-orthonormal basis for each subspace $\mathcal{Y}_q \subset T_qM$. The construction requires two steps. First, if we follow the Gram-Schmidt process given by

$$
\begin{aligned}
\widetilde{Y}_1 &= Y_1 \\
\widetilde{Y}_2 &= Y_2 - \frac{\mathbb{G}(Y_2, \widetilde{Y}_1)}{\mathbb{G}(\widetilde{Y}_1, \widetilde{Y}_1)}\widetilde{Y}_1 \\
\widetilde{Y}_3 &= Y_3 - \frac{\mathbb{G}(Y_3, \widetilde{Y}_1)}{\mathbb{G}(\widetilde{Y}_1, \widetilde{Y}_1)}\widetilde{Y}_1 - \frac{\mathbb{G}(Y_3, \widetilde{Y}_2)}{\mathbb{G}(\widetilde{Y}_2, \widetilde{Y}_2)}\widetilde{Y}_2 \\
\vdots &= \vdots \\
\widetilde{Y}_m &= Y_m - \frac{\mathbb{G}(Y_m, \widetilde{Y}_1)}{\mathbb{G}(\widetilde{Y}_1, \widetilde{Y}_1)}\widetilde{Y}_1 - \frac{\mathbb{G}(Y_m, \widetilde{Y}_2)}{\mathbb{G}(\widetilde{Y}_2, \widetilde{Y}_2)}\widetilde{Y}_2 - \cdots - \frac{\mathbb{G}(Y_m, \widetilde{Y}_{m-1})}{\mathbb{G}(\widetilde{Y}_{m-1}, \widetilde{Y}_{m-1})}\widetilde{Y}_{m-1}
\end{aligned}
$$

then $\{\widetilde{Y}_1, \ldots, \widetilde{Y}_m\}$ is an orthogonal basis for $\mathcal{Y}_q \subset T_qM$. Second, we construct an orthonormal basis from the orthogonal basis $\{\widetilde{Y}_1, \ldots, \widetilde{Y}_m\}$ by simply normalizing each $\widetilde{Y}_a$ for $a = 1, \ldots, m$ with respect to the Riemannian metric $\mathbb{G}$. The elements of the orthonormal basis are computed using

$$
\begin{aligned}
\widehat{Y}_1 &= \frac{1}{\|\widetilde{Y}_1\|_{\mathbb{G}}^2}\widetilde{Y}_1 \\
\vdots &= \vdots \\
\widehat{Y}_m &= \frac{1}{\|\widetilde{Y}_m\|_{\mathbb{G}}^2}\widetilde{Y}_m.
\end{aligned}
$$

Remark 4.2.1. *Let us simplify our notation by dropping the* $\widehat{\cdot}$ *and assume that the set of vector fields* $\{Y_1, \ldots, Y_m\}$ *when evaluated at each* $q \in M$ *forms a* $\mathbb{G}$*-orthonormal basis for* $\mathcal{Y}_q \subset T_qM$*.*

Let $O(M)$ denote the set of $\mathbb{G}$-orthonormal frames on a Riemannian manifold

$(M, \mathbb{G})$. If $B \in O(M)$ is a $\mathbb{G}$**-orthonormal frame** then

$$B_q = \text{span}_{\mathbb{R}}\{B_1(q), \ldots, B_n(q)\} = T_qM$$

where $\mathbb{G}(B_i(q), B_j(q)) = 0$ if $i \neq j$ otherwise $\mathbb{G}(B_i(q), B_j(q)) = 1$ for each $q \in M$. Let the set of $\mathbb{G}$-orthonormal vector fields $\{Y_1, \ldots, Y_m\}$ be the first m elements of the $\mathbb{G}$-orthonormal frame $B = \{B_1, \ldots, B_n\}$. We know that

$$\text{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\} = \mathcal{Y}_q$$

is a m-dimensional subspace of T_qM for each $q \in M$. It is also true that the remaining $n-m$ elements of the $\mathbb{G}$-orthonormal frame B defines a $n-m$-dimensional subspace

$$\mathcal{Y}_q^{\perp} = \text{span}_{\mathbb{R}}\{B_{m+1}(q), \ldots, B_n(q)\} \subset B_q = T_qM$$

that is the $\mathbb{G}$**-orthogonal complement** to $\mathcal{Y}_q$. Let us denote the remaining $n-m$ elements of the $\mathbb{G}$-orthonormal frame B by $\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}\}$. We are able to split

$$T_qM = \mathcal{Y}_q \oplus \mathcal{Y}_q^{\perp}$$

at each $q \in M$ where

$$\mathcal{Y}_q = \text{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}$$

and

$$\mathcal{Y}_q^{\perp} = \text{span}_{\mathbb{R}}\{Y_1^{\perp}(q), \ldots, Y_{n-m}^{\perp}(q)\}.$$

4.2.2 Affine and Linear Parameters

Now we construct a family of one-forms

$$s^b : T_qM \to \mathbb{R}$$

such that

$$s^b(Y_a(q)) = 0$$

for all $q \in M$, $a = 1, \ldots, m$ $b = 1, \ldots, n-m$ and $s^b(\cdot) \neq 0$. Such a one-form is given by

$$v_q \mapsto \mathbb{G}(Y_b^{\perp}, v_q)$$

where $v_q \in TM$ and $Y_b^{\perp} \in \mathcal{Y}^{\perp}$. Given the natural coordinates

$$((x^1, \ldots, x^n), (v^1, \ldots, v^n))$$

on TM, the local components are given by

$$s_i^b = (Y_b^{\perp})^j \mathbb{G}_{ij}$$

with respect to the dual one-forms $\{dx^1, \ldots, dx^n\}$. Here is the formal definition.

Definition 4.2.2 (Affine Parameters)**.** *Given the family of $\mathbb{G}$-orthonormal vector fields $\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}\}$ on M, the **affine parameters** $\mathrm{s} = \{s^1, \ldots, s^{n-m}\}$ is the smooth assignment of the family of one-forms $s^b(\cdot) = \mathbb{G}(Y_b^{\perp}, \cdot)$ on T_qM for each $q \in M$ and $b = 1, \ldots, n-m$.*

Now we construct a second family of one-forms

$$w^a : T_qM \to \mathbb{R}$$

such that

$$w^a(Y_b^\perp(q)) = 0$$

for all $q \in M$, $a = 1, \ldots, m$ $b = 1, \ldots, n-m$ and $w^a(\cdot) \neq 0$. Such a one-form is given by

$$v_q \mapsto \mathbb{G}(Y_a, v_q)$$

where $v_q \in TM$ and $Y_a \in \mathcal{Y}$. Given the natural coordinates

$$((x^1, \ldots, x^n), (v^1, \ldots, v^n))$$

on TM, the local components are given by

$$w_i^a = (Y_a)^j \mathbb{G}_{ij}$$

with respect to the dual one-forms $\{dx^1, \ldots, dx^n\}$.Here is the formal definition.

Definition 4.2.3 (Linear Parameters)**.** *Given the family of $\mathbb{G}$-orthonormal vector fields $\{Y_1, \ldots, Y_m\}$ on M, the **linear parameters** $\mathbf{w} = \{w^1, \ldots, w^m\}$ is the smooth assignment of the family of one-forms $w^a(\cdot) = \mathbb{G}(Y_a, \cdot)$ on T_qM for each $q \in M$ and $a = 1, \ldots, m$.*

4.2.3 Affine Foliation of Tangent Bundle

Let us examine the set $(M, \mathbb{G}, V, \mathcal{Y}, \mathcal{Y}^{\perp}, \{w^1, \ldots, w^m\}, \{s^1, \ldots, s^{n-m}\})$. Given the $\mathbb{G}$-orthonormal frame

$$\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}, Y_1, \ldots, Y_m\}$$

on M, we make three important observations. First, the $\mathbb{G}$-orthonormal frame can be used to define an affine subbundle. Specifically, we define the an affine subbundle $A \subset TM$ on M with the property that for each $q \in M$ we have

$$A_q \equiv A \cap T_qM = \{Y_1^{\perp}(q)\} + \cdots + \{Y_{n-m}^{\perp}(q)\} + \mathrm{span}_{\mathbb{R}}\{Y_1(q), \ldots, Y_m(q)\}.$$

The vector fields $\{Y_1, \ldots, Y_m\}$ are the linear generators of A. Recall that the linear generators of an affine subbundle also generate a distribution $L(A)$ defined by asking that $L(A)_q$ is the linear part of the affine subspace A_q. In this case, the distribution $L(A)$ is the input distribution $\mathcal{Y}$.

Second, the affine parameters $\{s^1, \ldots, s^{n-m}\}$ constructed from the elements $\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}\}$ of the $\mathbb{G}$-orthonormal frame naturally induces an affine foliation $\mathcal{A}$ of TM parameterized by $\mathbf{s} \in \mathbb{R}^{n-m}$ where $\mathbf{s} = (s^1(v_q), \ldots, s^{n-m}(v_q))$. An affine leaf $\mathcal{A}_{\mathbf{s}}$ of the affine foliation $\mathcal{A}$ is defined by

$$\mathcal{A}_{\mathbf{s}}(q) = \{v_q \in TM \mid \mathbb{G}(Y_b^{\perp}, v_q) = s^b, \mathbf{s} \in \mathbb{R}^{n-m}\}.$$

Each affine leaf is an affine subbundle $\mathcal{A}_{\mathbf{s}}$ of TM. Note that when $\mathbf{s} = 0$, $\mathcal{A}_{\mathbf{0}} = \mathcal{D}$ and $\mathcal{A}_{\mathbf{0}}(q) = \mathcal{D}_q$ where $\mathcal{D}$ is an immersed submanifold of TM and $\mathcal{D}_q$ is a linear subspace of T_qM. Thus, the distribution $\mathcal{D}$ is a single leaf of the affine foliation.

Third,the $\mathbb{G}$-orthonormal frame

$$\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}, Y_1, \ldots, Y_m\}$$

provides an **orthogonal decomposition of the tangent bundle** TM at each $q \in M$ where

$$T_qM = \mathcal{Y}_q \oplus \mathcal{Y}_q^{\perp}.$$

Using the linear and affine parameters $\{w^1, \ldots, w^m\}$ and $\{s^1, \ldots, s^{n-m}\}$, every $v_q \in TM$ can be expressed as a sum

$$v_q = w^a(v_q)Y_a + s^b(v_q)Y_q^{\perp}$$

where $a = 1, \ldots, m$ and $b = 1, \ldots, n-m$. The $\mathbb{G}$-orthonormal frame

$$\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}, Y_1, \ldots, Y_m\}$$

also provides an **orthogonal decomposition of the affine subbundle** A_q assigned to each affine leaf $\mathcal{A}_{\mathbf{s}}(q)$. We can decompose each $v_q \in A_q \subset T_qM$ into affine components $s^b(v_q)$ for $b = 1, \ldots, n-m$ and linear components $w^a(v_q)$ for $a = 1, \ldots, m$.

4.2.4 Characterization of Affine and Linear Parameters

Now we derive a measure of the change in the affine parameters $s^b : TM \to \mathbb{R}$ and the linear parameters $w^a : TM \to \mathbb{R}$ along trajectories of three classes of mechanical system.

1. Unactuated Mechanical System

2. Underactuated Mechanical System with No Potential

3. Underactuated Mechanical System with Gravitational Potential

4.2.4.1 Unactuated Mechanical Systems

The first class we consider is an unactuated mechanical system whose Lagrangian is $L_{\mathbb{G}} = \frac{1}{2}\mathbb{G}(v_q, v_q)$. Recall that trajectories of a simple mechanical system whose Lagrangian is $L_{\mathbb{G}}$ are geodesics γ satisfying the expression

$$\nabla_{\gamma'(t)}\gamma'(t) = 0$$

where ∇ is the Levi-Civita connection and $\gamma'(t)$ is the tangent vector field to the curve $\gamma(t)$. Given the $\mathbb{G}$-orthonormal frame

$$\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}, Y_1, \ldots, Y_m\}$$

that provides an orthogonal decomposition of T_qM for each $q \in M$, we may express the tangent vector field as the sum

$$\gamma'(t) = w^a(t)Y_a(\gamma(t)) + s^b(t)Y_b^{\perp}(\gamma(t))$$

where $w^a(t) = \mathbb{G}(Y_a(\gamma(t)), \gamma'(t))_{\gamma(t)}$ and $s^b(t) = \mathbb{G}(Y_b^{\perp}(\gamma(t)), \gamma'(t))_{\gamma(t)}$ for $a = 1, \ldots, m$ and $b = 1, \ldots, n-m$.

Proposition 4.2.4 (Characterization of Affine Parameters Along Geodesics)**.** *Let the affine parameters* $\mathbf{s} = \{s^1, \ldots, s^{n-m}\}$ *be the smooth assignment of the family of one-forms on* T_qM *for each* $q \in M$*. The following holds along a geodesic* γ *that* *satisfies* $\nabla_{\gamma'(t)}\gamma'(t) = 0$*:*

$$\begin{aligned} \frac{d}{dt}s^b(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_b^\perp) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_b^\perp) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_b^\perp) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_b^\perp) \end{aligned}$$

where $a, p \in \{1, \ldots, m\}$*,* $b, k, r \in \{1, \ldots, n-m\}$*.*

Proof. It follows from the definition of the affine parameters $\mathbf{s} = \{s^1, \ldots, s^{n-m}\}$ that

$$\frac{d}{dt}s^b(t) = \frac{d}{dt}\mathbb{G}(Y_b^\perp, \gamma'(t)) \tag{4.1}$$

where $\{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $\mathcal{Y}^\perp$. Let us begin by expanding the right-hand-side of Equation (4.1) by taking advantage of the compatibility associated with the Levi-Civita connection. This gives us

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t)) + \mathbb{G}(Y_b^\perp, \nabla_{\gamma'(t)}\gamma'(t)). \tag{4.2}$$

It follows from the definition of a geodesic that the second term on the right-hand-side of Equation (4.2) vanishes. This gives us the expression

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t)). \tag{4.3}$$

Recall that the tangent vector field $\gamma'(t)$ can be written as the sum

$$\gamma'(t) = w^a(t)Y_a(\gamma(t)) + s^r(t)Y_r^\perp(\gamma(t)) \tag{4.4}$$

where $a = 1, \ldots, m$ and $r = 1, \ldots, n-m$. We substitute Equation (4.4) into

Equation (4.3) to get

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))+s^r(t)Y_r^\perp(\gamma(t))}Y_b^\perp, w^p(t)Y_p(\gamma(t)) + s^k(t)Y_k^\perp(\gamma(t))) \quad (4.5)$$

where $a, p = 1, \ldots, m$ and $b, r, k = 1, \ldots, n-m$. We use the bilinearity of the Riemannian metric $\mathbb{G}$ to split the right-hand-side of Equation (4.5) into

$$\begin{aligned} \frac{d}{dt}s^b(t) &= \mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))+s^r(t)Y_r^\perp(\gamma(t))}Y_b^\perp, w^p(t)Y_p(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))+s^r(t)Y_r^\perp(\gamma(t))}Y_b^\perp, s^k(t)Y_k^\perp(\gamma(t))). \end{aligned} \quad (4.6)$$

Again, we expand Equation (4.6) to simplify the task of interpretation. We use the $\mathbb{R}$-linearity associated with the first argument of an affine connection to get

$$\begin{aligned} \frac{d}{dt}s^b(t) &= \mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))}Y_b^\perp, w^p(t)Y_p(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{s^r(t)Y_r^\perp(\gamma(t))}Y_b^\perp, w^p(t)Y_p(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))}Y_b^\perp, s^k(t)Y_k^\perp(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{s^r(t)Y_r^\perp(\gamma(t))}Y_b^\perp, s^k(t)Y_k^\perp(\gamma(t))). \end{aligned} \quad (4.7)$$

We use the bilinearity of $\mathbb{G}$ and the $C^\infty(M)$-linearity of the first argument of an affine connection to pull the linear and affine parameters **w** and **s** out of the inner products in Equation (4.7) to get

$$\begin{aligned} \frac{d}{dt}s^b(t) &= w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a(\gamma(t))}Y_b^\perp, Y_p(\gamma(t))) \\ &\quad +s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp(\gamma(t))}Y_b^\perp, Y_p(\gamma(t))) \\ &\quad +w^a(t)s^k(t)\mathbb{G}(\nabla_{Y_a(\gamma(t))}Y_b^\perp, Y_k^\perp(\gamma(t))) \\ &\quad +s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp(\gamma(t))}Y_b^\perp, Y_k^\perp(\gamma(t))). \end{aligned} \quad (4.8)$$

Let us assume that each vector field $\{Y_1, \ldots, Y_m, Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ is evaluated at the point $\gamma(t)$ so that we may simplify our notation. This gives us the simplified expression

$$\begin{aligned} \frac{d}{dt}s^b(t) &= w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_b^\perp, Y_p) \\ &\quad + s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_b^\perp, Y_p) \\ &\quad + w^a(t)s^k(t)\mathbb{G}(\nabla_{Y_a}Y_b^\perp, Y_k^\perp) \\ &\quad + s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_b^\perp, Y_k^\perp). \end{aligned} \tag{4.9}$$

Now we observe that given vector fields X, Y, Z on a Riemannian manifold M and the Levi-Civita affine connection ∇ the following is true:

$$X\mathbb{G}(Y, Z) = \mathbb{G}(\nabla_X Y, Z) + \mathbb{G}(Y, \nabla_X Z). \tag{4.10}$$

Using Equation (4.10), we place the terms

$$\mathbb{G}(\nabla_{Y_a}Y_b^\perp, Y_p)$$

,

$$\mathbb{G}(\nabla_{Y_r^\perp}Y_b^\perp, Y_p)$$

,

$$\mathbb{G}(\nabla_{Y_a}Y_b^\perp, Y_k^\perp)$$

and

$$\mathbb{G}(\nabla_{Y_r^\perp}Y_b^\perp, Y_k^\perp)$$

in Equation (4.9) into the following four expressions:

$$\begin{aligned}
Y_a \mathbb{G}(Y_b^\perp, Y_p) &= \mathbb{G}(\nabla_{Y_a} Y_b^\perp, Y_p) + \mathbb{G}(Y_b^\perp, \nabla_{Y_a} Y_p) \\
Y_r^\perp \mathbb{G}(Y_b^\perp, Y_p) &= \mathbb{G}(\nabla_{Y_r^\perp} Y_b^\perp, Y_p) + \mathbb{G}(Y_b^\perp, \nabla_{Y_r^\perp} Y_p) \\
Y_a \mathbb{G}(Y_b^\perp, Y_k^\perp) &= \mathbb{G}(\nabla_{Y_a} Y_b^\perp, Y_k^\perp) + \mathbb{G}(Y_b^\perp, \nabla_{Y_a} Y_k^\perp) \\
Y_r^\perp \mathbb{G}(Y_b^\perp, Y_k^\perp) &= \mathbb{G}(\nabla_{Y_r^\perp} Y_b^\perp, Y_k^\perp) + \mathbb{G}(Y_b^\perp, \nabla_{Y_r^\perp} Y_k^\perp).
\end{aligned} \tag{4.11}$$

Recall that the vector fields $\{Y_1, \ldots, Y_m, Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ are $\mathbb{G}$-orthonormal. This implies that the left-hand-side of each expression in Equation (4.11) is equal to zero for all indices. We are left with the following four equalities:

$$\begin{aligned}
\mathbb{G}(\nabla_{Y_a} Y_b^\perp, Y_p) &= -\mathbb{G}(\nabla_{Y_a} Y_p, Y_b^\perp) \\
\mathbb{G}(\nabla_{Y_r^\perp} Y_b^\perp, Y_p) &= -\mathbb{G}(\nabla_{Y_r^\perp} Y_p, Y_b^\perp) \\
\mathbb{G}(\nabla_{Y_a} Y_b^\perp, Y_k^\perp) &= -\mathbb{G}(\nabla_{Y_a} Y_k^\perp, Y_b^\perp) \\
\mathbb{G}(\nabla_{Y_r^\perp} Y_b^\perp, Y_k^\perp) &= -\mathbb{G}(\nabla_{Y_r^\perp} Y_k^\perp, Y_b^\perp).
\end{aligned} \tag{4.12}$$

Now substitute the relations established in Equation (4.12) into Equation (4.9) to get

$$\begin{aligned}
\frac{d}{dt} s^b(t) &= -w^a(t) w^p(t) \mathbb{G}(\nabla_{Y_a} Y_p, Y_b^\perp) \\
&\quad - s^r(t) w^p(t) \mathbb{G}(\nabla_{Y_r^\perp} Y_p, Y_b^\perp) \\
&\quad - w^a(t) s^k(t) \mathbb{G}(\nabla_{Y_a} Y_k^\perp, Y_b^\perp) \\
&\quad - s^r(t) s^k(t) \mathbb{G}(\nabla_{Y_r^\perp} Y_k^\perp, Y_b^\perp).
\end{aligned} \tag{4.13}$$

This completes the proof.

$\square$

Remark 4.2.5. *This result will be useful when proving Theorem 4.2.8 and Theorem 4.2.14 which characterize the affine parameters along trajectories of under-actuated simple mechanical control systems in the absence and presence of the gravitational potential force.*

Proposition 4.2.6 (Characterization of Linear Parameters Along Geodesics)**.** *Let the linear parameters* $\mathbf{w} = \{w^1, \ldots, w^m\}$ *be the smooth assignment of the family of one-forms on* T_qM *for each* $q \in M$*. The following holds along a geodesic* γ *that satisfies* $\nabla_{\gamma'(t)}\gamma'(t) = 0$*:*

$$\begin{aligned}\frac{d}{dt}w^l(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_l) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^{\perp}, Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_p, Y_l) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_k^{\perp}, Y_l)\end{aligned}$$

where $a, p, l \in \{1, \ldots, m\}$*,* $k, r \in \{1, \ldots, n-m\}$*.*

Proof. It follows from the definition of the linear parameters $\mathbf{w} = \{w^1, \ldots, w^m\}$ that

$$\frac{d}{dt}w^l(t) = \frac{d}{dt}\mathbb{G}(Y_l, \gamma'(t)) \tag{4.14}$$

where $\{Y_1, \ldots, Y_m\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $\mathcal{Y}$. Let us begin by expanding the right-hand-side of Equation (4.14) by taking advantage of the compatibility associated with the Levi-Civita connection. This gives us

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) + \mathbb{G}(Y_l, \nabla_{\gamma'(t)}\gamma'(t)). \tag{4.15}$$

It follows from the definition of a geodesic that the second term on the right-hand-side of Equation (4.15) vanishes. This gives us the expression

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)). \tag{4.16}$$

Recall that the tangent vector field $\gamma'(t)$ can be written as the sum

$$\gamma'(t) = w^a(t)Y_a(\gamma(t)) + s^r(t)Y_r^\perp(\gamma(t)) \tag{4.17}$$

where $a = 1, \ldots, m$ and $r = 1, \ldots, n-m$. We substitute Equation (4.17) into Equation (4.16) to get

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))+s^r(t)Y_r^\perp(\gamma(t))}Y_l, w^p(t)Y_p(\gamma(t)) + s^k(t)Y_k^\perp(\gamma(t))) \tag{4.18}$$

where $a, p, l = 1, \ldots, m$ and $r, k = 1, \ldots, n-m$. We use the bilinearity of the Riemannian metric $\mathbb{G}$ to split the right-hand-side of Equation (4.18) into

$$\begin{aligned}\frac{d}{dt}w^l(t) &= \mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))+s^r(t)Y_r^\perp(\gamma(t))}Y_l, w^p(t)Y_p(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))+s^r(t)Y_r^\perp(\gamma(t))}Y_l, s^k(t)Y_k^\perp(\gamma(t))).\end{aligned} \tag{4.19}$$

Again, we expand Equation (4.19) to simplify the task of interpretation. We use the $\mathbb{R}$-linearity associated with the first argument of an affine connection to get

$$\begin{aligned}\frac{d}{dt}w^l(t) &= \mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))}Y_l, w^p(t)Y_p(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{s^r(t)Y_r^\perp(\gamma(t))}Y_l, w^p(t)Y_p(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{w^a(t)Y_a(\gamma(t))}Y_l, s^k(t)Y_k^\perp(\gamma(t))) \\ &\quad +\mathbb{G}(\nabla_{s^r(t)Y_r^\perp(\gamma(t))}Y_l, s^k(t)Y_k^\perp(\gamma(t))).\end{aligned} \tag{4.20}$$

We use the bilinearity of $\mathbb{G}$ and the $C^\infty(M)$-linearity of the first argument of an affine connection to pull the linear and affine parameters $\mathbf{w}$ and $\mathbf{s}$ out of the inner products in Equation (4.20) to get

$$\begin{aligned}\frac{d}{dt}w^l(t) &= w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a(\gamma(t))}Y_l, Y_p(\gamma(t))) \\ &\quad + s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp(\gamma(t))}Y_l, Y_p(\gamma(t))) \\ &\quad + w^a(t)s^k(t)\mathbb{G}(\nabla_{Y_a(\gamma(t))}Y_l, Y_k^\perp(\gamma(t))) \\ &\quad + s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp(\gamma(t))}Y_l, Y_k^\perp(\gamma(t))).\end{aligned} \tag{4.21}$$

Let us assume that each vector field $\{Y_1, \ldots, Y_m, Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ is evaluated at the point $\gamma(t)$ so that we may simplify our notation. This gives us the simplified expression

$$\begin{aligned}\frac{d}{dt}w^b(t) &= w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_l, Y_p) \\ &\quad + s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_l, Y_p) \\ &\quad + w^a(t)s^k(t)\mathbb{G}(\nabla_{Y_a}Y_l, Y_k^\perp) \\ &\quad + s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_l, Y_k^\perp).\end{aligned} \tag{4.22}$$

Now we observe that given vector fields X, Y, Z on a Riemannian manifold M and the Levi-Civita affine connection ∇ the following is true:

$$X\mathbb{G}(Y, Z) = \mathbb{G}(\nabla_X Y, Z) + \mathbb{G}(Y, \nabla_X Z). \tag{4.23}$$

Using Equation (4.23), we place the terms

$$\mathbb{G}(\nabla_{Y_a}Y_l, Y_p)$$

,

$$\mathbb{G}(\nabla_{Y_r^\perp} Y_l, Y_p)$$

,

$$\mathbb{G}(\nabla_{Y_a} Y_l, Y_k^\perp)$$

and

$$\mathbb{G}(\nabla_{Y_r^\perp} Y_l, Y_k^\perp)$$

in Equation (4.22) into the following four expressions:

$$\begin{aligned}
Y_a \mathbb{G}(Y_l, Y_p) &= \mathbb{G}(\nabla_{Y_a} Y_l, Y_p) + \mathbb{G}(Y_l, \nabla_{Y_a} Y_p) \\
Y_r^\perp \mathbb{G}(Y_l, Y_p) &= \mathbb{G}(\nabla_{Y_r^\perp} Y_l, Y_p) + \mathbb{G}(Y_l, \nabla_{Y_r^\perp} Y_p) \\
Y_a \mathbb{G}(Y_l, Y_k^\perp) &= \mathbb{G}(\nabla_{Y_a} Y_l, Y_k^\perp) + \mathbb{G}(Y_l, \nabla_{Y_a} Y_k^\perp) \\
Y_r^\perp \mathbb{G}(Y_l, Y_k^\perp) &= \mathbb{G}(\nabla_{Y_r^\perp} Y_l, Y_k^\perp) + \mathbb{G}(Y_l, \nabla_{Y_r^\perp} Y_k^\perp).
\end{aligned} \tag{4.24}$$

Recall that the vector fields $\{Y_1, \ldots, Y_m, Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ are $\mathbb{G}$-orthonormal. This implies that the left-hand-side of each expression in Equation (4.24) is equal to zero for all indices. We are left with the following four equalities:

$$\begin{aligned}
\mathbb{G}(\nabla_{Y_a} Y_l, Y_p) &= -\mathbb{G}(\nabla_{Y_a} Y_p, Y_l) \\
\mathbb{G}(\nabla_{Y_r^\perp} Y_l, Y_p) &= -\mathbb{G}(\nabla_{Y_r^\perp} Y_p, Y_l) \\
\mathbb{G}(\nabla_{Y_a} Y_l, Y_k^\perp) &= -\mathbb{G}(\nabla_{Y_a} Y_k^\perp, Y_l) \\
\mathbb{G}(\nabla_{Y_r^\perp} Y_l, Y_k^\perp) &= -\mathbb{G}(\nabla_{Y_r^\perp} Y_k^\perp, Y_l).
\end{aligned} \tag{4.25}$$

Now substitute the relations established in Equation (4.25) into Equation

(4.22) to get

$$
\begin{aligned}
\frac{d}{dt}w^l(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_l) \\
&\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_l) \\
&\quad -w^a(t)s^k(t)\mathbb{G}(\nabla_{Y_a}Y_k^\perp, Y_l) \\
&\quad -s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_l).
\end{aligned} \tag{4.26}
$$

This completes the proof. □

Remark 4.2.7. *This result will be useful when proving Theorem 4.2.11 and Theorem 4.2.16 which characterize the linear parameters along trajectories of underactuated simple mechanical control systems in the absence and presence of the gravitational potential force.*

4.2.4.2 Underactuated Mechanical Systems with No Gravitational Potential

Proposition 4.2.8 (Characterization of Affine Parameters Along $\Sigma_{L_{\mathbb{G}}}$-Trajectories)**.** *Let the affine parameters $\mathbf{s} = \{s^1, \ldots, s^{n-m}\}$ be the smooth assignment of the family of one-forms on T_qM for each $q \in M$. The following holds along trajectories* $\mathrm{Ctraj}(\Sigma_{L_{\mathbb{G}}}) = (\gamma, u)$ *that satisfies $\nabla_{\gamma'(t)}\gamma'(t) = u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))$:*

$$
\begin{aligned}
\frac{d}{dt}s^b(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_b^\perp) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_b^\perp) \\
&\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_b^\perp) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_b^\perp)
\end{aligned} \tag{4.27}
$$

where $a, p \in \{1, \ldots, m\}$, $b, k, r \in \{1, \ldots, n-m\}$.

Proof. It follows from the definition of the affine parameters $\mathbf{s} = \{s^1, \ldots, s^{n-m}\}$

that

$$\frac{d}{dt}s^b(t) = \frac{d}{dt}\mathbb{G}(Y_b^{\perp}, \gamma'(t)) \tag{4.28}$$

where $\{Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $\mathcal{Y}^{\perp}$. Let us begin by expanding the right-hand-side of Equation (4.28) by taking advantage of the compatibility associated with the Levi-Civita connection. This gives us

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_b^{\perp}, \gamma'(t)) + \mathbb{G}(Y_b^{\perp}, \nabla_{\gamma'(t)}\gamma'(t)). \tag{4.29}$$

It follows from the definition of a simple mechanical control system with the Lagrangian $L_{\mathbb{G}} = \frac{1}{2}\mathbb{G}(v_q, v_q)$ on TM that trajectories $\mathrm{Ctraj}(\Sigma_{L_{\mathbb{G}}}) = (\gamma, u)$ satisfy

$$\nabla_{\gamma'(t)}\gamma'(t) = u^a(t)\mathbb{G}^{\sharp}(F^a(\gamma(t))) \tag{4.30}$$

where $F^1, \ldots, F^m$ are the control one-forms. We substitute the relation given in Equation (4.30) into the second term on the right-hand-side of Equation (4.29) to get

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_b^{\perp}, \gamma'(t)) + \mathbb{G}(Y_b^{\perp}, u^a(t)\mathbb{G}^{\sharp}(F^a(\gamma(t)))). \tag{4.31}$$

Let us examine the term $\mathbb{G}(Y_b^{\perp}, u^a(t)\mathbb{G}^{\sharp}(F^a(\gamma(t))))$ in the right-hand-side of Equation (4.31). First, due to the bilinearity of $\mathbb{G}$ we can pull the $\mathbf{u}$ term out of the inner product to get $u^a(t)\mathbb{G}(Y_b^{\perp}, \mathbb{G}^{\sharp}(F^a(\gamma(t))))$. Second, the vector fields $\mathbb{G}^{\sharp}(F^1(\gamma(t))), \ldots, \mathbb{G}^{\sharp}(F^m(\gamma(t)))$ when evaluated at a point $\gamma(t)$ take values in $\mathcal{Y}_{\gamma(t)}$. Recall that the family of vector fields $\{Y_1, \ldots, Y_m\}$ when evaluated at each $\gamma(t)$ form a $\mathbb{G}$-orthonormal basis for $\mathcal{Y}_{\gamma(t)}$. Therefore, we can express $\mathbb{G}^{\sharp}(F^a(\gamma(t)))$ as a linear combination of the family of $\mathbb{G}$-orthonormal vector fields $\{Y_1, \ldots, Y_m\}$

for each $a = 1, \ldots, m$. The linear combination is given by

$$\mathbb{G}^\sharp(F^a(\gamma(t))) = \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_1)Y_1 + \cdots + \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_m)Y_m. \quad (4.32)$$

We can substitute the relation given by Equation (4.32) into the term

$$u^a(t)\mathbb{G}(Y_b^\perp, \mathbb{G}^\sharp(F^a(\gamma(t))))$$

to get

$$u^a(t)\mathbb{G}(Y_b^\perp, \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_1)Y_1 + \cdots + \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_m)Y_m) \quad (4.33)$$

Using the bilinearity of $\mathbb{G}$, we expand Equation (4.33) to get

$$\begin{aligned} u^a(t)(\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_1)\mathbb{G}(Y_b^\perp, Y_1) + \cdots \\ + \mathbb{G}^\sharp(F^a(\gamma(t))), Y_m)\mathbb{G}(Y_b^\perp, Y_m)) \end{aligned} \quad (4.34)$$

It follows from the construction of the $\mathbb{G}$-orthonormal frame that each term

$$\mathbb{G}(Y_b^\perp, Y_1), \ldots, \mathbb{G}(Y_b^\perp, Y_m)$$

in Equation (4.34) vanishes for all $b = 1, \ldots, n-m$. This implies that the second term on the right-hand-side of Equation (4.31) also vanishes. We are left with the expression

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t))$$

which is equivalent to Equation (4.3) in the proof of Theorem 4.2.4. The remainder of this proof precedes from Equation (4.3) in the proof of Theorem 4.2.4.

$\square$

Remark 4.2.9. *This result will be useful when proving Theorem 4.2.14 which characterizes the affine parameters along trajectories of underactuated simple mechanical control systems in the presence of the gravitational potential force.*

Remark 4.2.10. *Note the absence of the control parameter u in Equation* (4.27). *This expression represents the **unactuated** dynamics of the underactuated simple mechanical control system with the Lagrangian $L_{\mathbb{G}}$. The right-hand-side of Equation* (4.27) *is quadratic in the affine and linear parameters. The quadratic structure couples the unactuated dynamics to the actuated dynamics. The affine parameters are the unactuated velocity states.*

Proposition 4.2.11 (Characterization of Linear Parameters Along $\Sigma_{L_{\mathbb{G}}}$-Trajectories)**.** *Let the linear parameters $\mathbf{w} = \{w^1, \ldots, w^m\}$ be the smooth assignment of the family of one-forms on T_qM for each $q \in M$. The following holds along trajectories* $\mathrm{Ctraj}(\Sigma_{L_{\mathbb{G}}}) = (\gamma, u)$ *that satisfies $\nabla_{\gamma'(t)}\gamma'(t) = u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))$:*

$$\begin{aligned} \frac{d}{dt}w^l(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_l) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_l) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_l) \\ &\quad +u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_l) \end{aligned} \tag{4.35}$$

where $a, p, l \in \{1, \ldots, m\}$, $k, r \in \{1, \ldots, n-m\}$.

Proof. It follows from the definition of the linear parameters $\mathbf{w} = \{w^1, \ldots, w^m\}$ that

$$\frac{d}{dt}w^l(t) = \frac{d}{dt}\mathbb{G}(Y_l, \gamma'(t)) \tag{4.36}$$

where $\{Y_1, \ldots, Y_m\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $\mathcal{Y}$. Let us begin by expanding the right-hand-side of Equation (4.36) by taking advantage of the compatibility associated with the Levi-Civita connection. This gives us

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) + \mathbb{G}(Y_l, \nabla_{\gamma'(t)}\gamma'(t)). \tag{4.37}$$

It follows from the definition of a simple mechanical control system with the Lagrangian $L_{\mathbb{G}} = \frac{1}{2}\mathbb{G}(v_q, v_q)$ on TM that trajectories $\mathrm{Ctraj}(\Sigma_{L_{\mathbb{G}}}) = (\gamma, u)$ satisfy

$$\nabla_{\gamma'(t)}\gamma'(t) = u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t))) \tag{4.38}$$

where $F^1, \ldots, F^m$ are the control one-forms. We substitute the relation given in Equation (4.38) into the second term on the right-hand-side of Equation (4.37) to get

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) + \mathbb{G}(Y_l, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))). \tag{4.39}$$

Let us examine the term $\mathbb{G}(Y_l, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t))))$ in the right-hand-side of Equation (4.39). First, due to the bilinearity of $\mathbb{G}$ we can pull the $\mathbf{u}$ term out of the inner product to get $u^a(t)\mathbb{G}(Y_l, \mathbb{G}^\sharp(F^a(\gamma(t))))$. Second, the vector fields $\mathbb{G}^\sharp(F^1(\gamma(t))), \ldots, \mathbb{G}^\sharp(F^m(\gamma(t)))$ when evaluated at a point $\gamma(t)$ take values in $\mathcal{Y}_{\gamma(t)}$. Recall that the family of vector fields $\{Y_1, \ldots, Y_m\}$ when evaluated at each $\gamma(t)$ form a $\mathbb{G}$-orthonormal basis for $\mathcal{Y}_{\gamma(t)}$. Therefore, we can express $\mathbb{G}^\sharp(F^a(\gamma(t)))$ as a linear combination of the family of $\mathbb{G}$-orthonormal vector fields $\{Y_1, \ldots, Y_m\}$ for each $a = 1, \ldots, m$. The linear combination is given by

$$\mathbb{G}^\sharp(F^a(\gamma(t))) = \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_1)Y_1 + \cdots + \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_m)Y_m. \tag{4.40}$$

We can substitute the relation given by Equation (4.40) into the term

$$u^a(t)\mathbb{G}(Y_l, \mathbb{G}^\sharp(F^a(\gamma(t))))$$

to get

$$u^a(t)\mathbb{G}(Y_l, \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_1)Y_1 + \cdots + \mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_m)Y_m) \tag{4.41}$$

Using the bilinearity of $\mathbb{G}$, we expand Equation (4.41) to get

$$\begin{aligned} u^a(t)(\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_1)\mathbb{G}(Y_l, Y_1) + \cdots \\ +\mathbb{G}^\sharp(F^a(\gamma(t))), Y_m)\mathbb{G}(Y_l, Y_m)) \end{aligned} \tag{4.42}$$

It follows from the construction of the $\mathbb{G}$-orthonormal frame that the terms

$$\mathbb{G}(Y_l, Y_1), \ldots, \mathbb{G}(Y_l, Y_m)$$

in Equation (4.42) are equal to 1 when the indices are equivalent, otherwise the term vanishes. This is equivalent to the term

$$u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_l)$$

Now we substitute the term given by Equation (4.43) into the second term on the right-hand-side of Equation (4.37) to get

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) + u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_l). \tag{4.43}$$

The expansion of the first term on the right-hand-side of Equation (4.43) was shown in the proof of Theorem 4.2.6 to be

$$\begin{aligned}\mathbb{G}(\nabla_{\gamma'(t)}Y_l,\gamma'(t)) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p,Y_l) \\ &\quad -w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^{\perp},Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_p,Y_l) \\ &\quad -s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_k^{\perp},Y_l).\end{aligned} \tag{4.44}$$

Now we substitute the relation given in Equation (4.47) for the first term on the right-hand-side of Equation (4.43) to get

$$\begin{aligned}\frac{d}{dt}w^l(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p,Y_l) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^{\perp},Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_p,Y_l) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_k^{\perp},Y_l) \\ &\quad +u^a(t)\mathbb{G}(\mathbb{G}^{\sharp}(F^a(\gamma(t))),Y_l)\end{aligned} \tag{4.45}$$

This completes the proof. □

Remark 4.2.12. *This result will be useful when proving Theorem 4.2.16 which characterizes the linear parameters along trajectories of underactuated simple mechanical control systems in the presence of the gravitational potential force.*

Remark 4.2.13. *Note the explicit occurrence of the control parameter u in Equation* (4.35)*. This expression represents the* ***actuated*** *dynamics of the underactuated simple mechanical control system with the Lagrangian $L_{\mathbb{G}}$. The linear parameters are the actuated velocity states.*

4.2.4.3 Underactuated Mechanical Systems with Gravitational Potential

Proposition 4.2.14 (Characterization of Affine Parameters Along Σ-Trajectories)**.** *Let the affine parameters* $\mathbf{s} = \{s^1, \ldots, s^{n-m}\}$ *be the smooth assignment of the family of one-forms on* T_qM *for each* $q \in M$*. The following holds along trajectories* $\mathrm{Ctraj}(\Sigma) = (\gamma, u)$ *that satisfies* $\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))$*:*

$$\begin{aligned}\frac{d}{dt}s^b(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_b^\perp) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_b^\perp) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_b^\perp) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_b^\perp) \\ &\quad -\mathbb{G}(\operatorname{grad} V(\gamma(t)), Y_b^\perp) \end{aligned} \tag{4.46}$$

where $a, p \in \{1, \ldots, m\}$*,* $b, k, r \in \{1, \ldots, n-m\}$*.*

Proof. It follows from the definition of the affine parameters $\mathbf{s} = \{s^1, \ldots, s^{n-m}\}$ that

$$\frac{d}{dt}s^b(t) = \frac{d}{dt}\mathbb{G}(Y_b^\perp, \gamma'(t)) \tag{4.47}$$

where $\{Y_1^\perp, \ldots, Y_{n-m}^\perp\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $\mathcal{Y}^\perp$. Let us begin by expanding the right-hand-side of Equation (4.47) by taking advantage of the compatibility associated with the Levi-Civita connection. This gives us

$$\frac{d}{dt}s^b(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t)) + \mathbb{G}(Y_b^\perp, \nabla_{\gamma'(t)}\gamma'(t)). \tag{4.48}$$

It follows from the definition of a simple mechanical control system

$$\{M, \mathbb{G}, V, \mathcal{F}, U\}$$

that trajectories $\mathrm{Ctraj}(\Sigma) = (\gamma, u)$ satisfy

$$\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t))) \tag{4.49}$$

where $F^1, \ldots, F^m$ are the control one-forms and $\operatorname{grad} V(\gamma(t)) = \mathbb{G}^\sharp(dV)(\gamma(t))$ is the gravitational potential vector field. We substitute the relation given in Equation (4.49) into the second term on the right-hand-side of Equation (4.48) to get

$$\begin{aligned} \frac{d}{dt}s^b(t) &= \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t)) \\ &\quad +\mathbb{G}(Y_b^\perp, -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))). \end{aligned} \tag{4.50}$$

We can use the bilinearity of $\mathbb{G}$ to expand the right-hand-side of Equation (4.50) to get

$$\begin{aligned} \frac{d}{dt}s^b(t) &= \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t)) \\ &\quad +\mathbb{G}(Y_b^\perp, -\operatorname{grad} V(\gamma(t))) \\ &\quad +\mathbb{G}(Y_b^\perp, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))). \end{aligned} \tag{4.51}$$

The first term in the right-hand-side of Equation (4.51) was shown in the proof of Theorem 4.2.4 to be

$$\begin{aligned} \mathbb{G}(\nabla_{\gamma'(t)}Y_b^\perp, \gamma'(t)) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_b^\perp) \\ &\quad -w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_b^\perp) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_b^\perp) \\ &\quad -s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_b^\perp). \end{aligned} \tag{4.52}$$

The third term in the right-hand-side of Equation (4.51) was shown in the proof of Theorem 4.2.8 to vanish for each $b = 1, \ldots, n-m$. Now we substitute the relationship given by Equation (4.52) into Equation (4.51) and set $\mathbb{G}(Y_b^{\perp}, u^a(t)\mathbb{G}^{\sharp}(F^a(\gamma(t)))) = 0$ to get the expression

$$\begin{aligned}
\frac{d}{dt}s^b(t) \quad = \quad & -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_b^{\perp}) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^{\perp}, Y_b^{\perp}) \\
& -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_p, Y_b^{\perp}) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^{\perp}}Y_k^{\perp}, Y_b^{\perp}) \\
& -\mathbb{G}(\operatorname{grad} V(\gamma(t)), Y_b^{\perp}).
\end{aligned} \tag{4.53}$$

This completes the proof. □

Remark 4.2.15. *Note the absence of the control parameter u in Equation* (4.46)*. This expression represents the* ***unactuated*** *dynamics of the underactuated simple mechanical control system in the presence of the gravitational potential force. The right-hand-side of Equation* (4.46) *consists of the quadratic in the affine and linear parameters and the gravitational potential force. In the language of control-affine systems, they can be combined to form part of the drift term. Again, the quadratic structure couples the unactuated dynamics to the actuated dynamics and the affine parameters are the unactuated velocity states.*

Proposition 4.2.16 (Characterization of Linear Parameters Along Σ-Trajectories)**.** *Let the linear parameters $\mathbf{w} = \{w^1, \ldots, w^m\}$ be the smooth assignment of the family of one-forms on T_qM for each $q \in M$. The following holds along trajectories*

$\mathrm{Ctraj}(\Sigma) = (\gamma, u)$ *that satisfies* $\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))$*:*

$$\begin{aligned} \frac{d}{dt}w^l(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_l) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_l) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_l) \\ &\quad -\mathbb{G}(\operatorname{grad} V(\gamma(t)), Y_l) + u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_l) \end{aligned} \tag{4.54}$$

where $a, p, l \in \{1, \ldots, m\}$*,* $k, r \in \{1, \ldots, n-m\}$*.*

Proof. It follows from the definition of the linear parameters $\mathbf{w} = \{w^1, \ldots, w^m\}$ that

$$\frac{d}{dt}w^l(t) = \frac{d}{dt}\mathbb{G}(Y_l, \gamma'(t)) \tag{4.55}$$

where $\{Y_1, \ldots, Y_m\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $\mathcal{Y}$. Let us begin by expanding the right-hand-side of Equation (4.55) by taking advantage of the compatibility associated with the Levi-Civita connection. This gives us

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) + \mathbb{G}(Y_l, \nabla_{\gamma'(t)}\gamma'(t)). \tag{4.56}$$

It follows from the definition of a simple mechanical control system

$$\{M, \mathbb{G}, V, \mathcal{F}, U\}$$

that trajectories $\mathrm{Ctraj}(\Sigma) = (\gamma, u)$ satisfy

$$\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t))) \tag{4.57}$$

where $F^1, \ldots, F^m$ are the control one-forms and $\operatorname{grad} V(\gamma(t)) = \mathbb{G}^\sharp(dV)(\gamma(t))$

is the gravitational potential vector field. We substitute the relation given in Equation (4.57) into the second term on the right-hand-side of Equation (4.56) to get

$$\frac{d}{dt}w^l(t) = \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) + \mathbb{G}(Y_l, -\operatorname{grad}V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))). \quad (4.58)$$

We can use the bilinearity of $\mathbb{G}$ to expand the right-hand-side of Equation (4.58) to get

$$\begin{aligned} \frac{d}{dt}w^l(t) &= \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) \qquad (4.59) \\ &\quad +\mathbb{G}(Y_l, -\operatorname{grad}V(\gamma(t))) \\ &\quad +\mathbb{G}(Y_l, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))). \end{aligned}$$

The first term in the right-hand-side of Equation (4.59) was shown in the proof of Theorem 4.2.6 to be

$$\begin{aligned} \mathbb{G}(\nabla_{\gamma'(t)}Y_l, \gamma'(t)) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_l) \qquad (4.60) \\ &\quad -w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_l) \\ &\quad -s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_l) \end{aligned}$$

The third term in the right-hand-side of Equation (4.59) was shown in the proof of Theorem 4.2.11 to be equal to

$$\mathbb{G}(Y_l, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))) = u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_l). \qquad (4.61)$$

Now we substitute the relations given by Equation (4.60) and Equation (4.61) into Equation (4.59) to get the expression

$$\begin{aligned}\frac{d}{dt}w^l(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{Y_a}Y_p, Y_l) - w^a(t)s^r(t)\mathbb{G}(\nabla_{Y_a}Y_r^\perp, Y_l) \\ &\quad -s^r(t)w^p(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_p, Y_l) - s^r(t)s^k(t)\mathbb{G}(\nabla_{Y_r^\perp}Y_k^\perp, Y_l) \\ &\quad -\mathbb{G}(\operatorname{grad} V(\gamma(t)), Y_l) + u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), Y_l). \end{aligned} \tag{4.62}$$

This completes the proof. $\square$

Remark 4.2.17. *Note the explicit occurrence of the control parameter u in Equation* (4.54)*. The linear parameters are the actuated velocity states. This expression represents the* ***actuated*** *dynamics of the underactuated simple mechanical control system in the presence of the gravitational potential force.*

4.2.5 Intrinsic Vector-Valued Quadratic Forms

Here we associate a vector-valued quadratic form to an underactuated simple mechanical control system. Recall that the set of $\mathbb{G}$-orthonormal vector fields $\{Y_1, \ldots, Y_m\}$ generates the input distribution $\mathcal{Y}$ at each $q \in M$. If $v_q \in Y_q$ then the linear parameters w evaluated at v_q are the components of v_q with respect to the $\mathbb{G}$-orthonormal basis $Y_1(q), \ldots, Y_m(q)$ for $\mathcal{Y}_q$. We define $Q_q : \mathcal{Y}_q \subset T_qM \to \mathcal{Y}_q^\perp$ as the $\mathcal{Y}_q^\perp$-valued quadratic map on $\mathcal{Y}_q$ given by

$$\mathcal{Y}_q \ni v_q \mapsto \mathbb{G}(Y_a(q), v_q)\mathbb{G}(Y_p(q), v_q)\mathbb{G}((\nabla_{Y_a}Y_p)(q), Y_b^\perp(q))Y_b^\perp(q) \in \mathcal{Y}_q^\perp$$

where $\{Y_1^\perp(q), \ldots, Y_{n-m}^\perp(q)\}$ is the $\mathbb{G}$-orthonormal basis for $\mathcal{Y}_q^\perp$. Given the $\mathbb{G}$-orthonormal basis $\{Y_1^\perp(q), \ldots, Y_{n-m}^\perp(q)\}$ for $\mathcal{Y}_q^\perp$, the local components for $Q_q(v_q)$

with respect to each basis vector $Y_b^\perp(q)$ for $b = 1, \ldots, n-m$ are

$$w^a w^p \mathbb{G}_{ij}(\nabla_{Y_a} Y_p)^i (Y_b^\perp)^j$$

where $a, p = 1, \ldots, m$, $i, j = 1, \ldots, n$. Let $((x^1, \ldots, x^n), (v^1, \ldots, v^n))$ be the natural coordinates on TM where $(v^1, \ldots, v^n)$ are the coefficients of a tangent vector given the usual basis $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$. The local expression for $Q_q(v_q)$ with respect to the natural coordinates is

$$\mathbb{G}_{ij} Y_a^i v^j \mathbb{G}_{ij} Y_p^j \mathbb{G}_{\alpha\beta} \left(Y_a^i \frac{\partial Y_p^\alpha}{\partial x^i} (Y_b^\perp)^\beta + \Gamma_{ij}^\alpha Y_a^i Y_p^j (Y_b^\perp)^\beta \right) (Y_b^\perp)^k \frac{\partial}{\partial x^k}$$

where

$$\begin{aligned} (Q_q)_{ap}^b &= \mathbb{G}(\nabla_{Y_a} Y_p, Y_b^\perp) \\ &= \mathbb{G}_{\alpha\beta} \left(Y_a^i \frac{\partial Y_p^\alpha}{\partial x^i} (Y_b^\perp)^\beta + \Gamma_{ij}^\alpha Y_a^i Y_p^j (Y_b^\perp)^\beta \right) \end{aligned} \tag{4.63}$$

and $\alpha, \beta, i, j, k = 1, \ldots, n$, $a, p = 1, \ldots, m$, $b = 1, \ldots, n-m$.

Remark 4.2.18. *The vector-valued quadratic form will play a critical role in our analysis and control of underactuated simple mechanical control systems. Specifically, we will use the definiteness of the vector-valued quadratic form to determine possible motion in the affine foliation of the tangent bundle. The analysis will lead to a constructive algorithm for motion planning that utilizes the intrinsic quadratic structure.*

4.2.6 Control-Affine System

Now we assign a control-affine system to the underactuated simple mechanical control system with Lagrangian $L_{\mathbb{G}}$ using the local representation of the kinematics, actuated dynamics and unactuated dynamics. Given the chart (ϕ_α, U_α) for M with coordinates $(x^1, \ldots, x^n)$ for $q \in M$ and the $\mathbb{G}$-orthonormal frame $\{Y_1^\perp, \ldots, Y_{n-m}^\perp, Y_1, \ldots, Y_m\}$ on M, the local representation for the kinematic equations are

$$\dot{x}^i(t) \quad = \quad w^a(t)Y_a^i + s^b(t)(Y_b^\perp)^i \tag{4.64}$$

where $w^a(t)$ and $s^b(t)$ are the linear and affine parameters along trajectories $\mathrm{Ctraj}(\Sigma) = (\gamma, u)$ that satisfy

$$\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t))).$$

The local representation for the actuated dynamic equations are

$$\begin{aligned}
\dot{w}^l(t) \quad = \quad & -w^a(t)w^p(t)\mathbb{G}_{\alpha\beta}\left(Y_a^i\frac{\partial Y_p^\alpha}{\partial x^i}Y_l^\beta + \Gamma_{ij}^\alpha Y_a^i Y_p^j Y_l^\beta\right) \\
& -w^a(t)s^r(t)\mathbb{G}_{\alpha\beta}\left(Y_a^i\frac{\partial (Y_r^\perp)^\alpha}{\partial x^i}Y_l^\beta + \Gamma_{ij}^\alpha Y_a^i (Y_r^\perp)^j Y_l^\beta\right) \\
& -s^r(t)w^p(t)\mathbb{G}_{\alpha\beta}\left((Y_r^\perp)^i\frac{\partial Y_p^\alpha}{\partial x^i}Y_l^\beta + \Gamma_{ij}^\alpha (Y_r^\perp)^i Y_p^j Y_l^\beta\right) \\
& -s^r(t)s^k(t)\mathbb{G}_{\alpha\beta}\left((Y_r^\perp)^i\frac{\partial (Y_k^\perp)^\alpha}{\partial x^i}Y_l^\beta + \Gamma_{ij}^\alpha (Y_r^\perp)^i (Y_k^\perp)^j Y_l^\beta\right) \\
& -\mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j}Y_l^\beta \\
& +u^a(t)\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^a Y_l^\beta\right)
\end{aligned} \tag{4.65}$$

where $a, p, l = 1, \ldots, m,\ k, r = 1, \ldots, n-m,\ \alpha, \beta, i, j = 1, \ldots, n$. The local representation for the unactuated dynamic equations are

$$\begin{aligned}
\dot{s}^b(t) &= -w^a(t)w^p(t)\mathbb{G}_{\alpha\beta}\left(Y_a^i \frac{\partial Y_p^\alpha}{\partial x^i}(Y_b^\perp)^\beta + \Gamma_{ij}^\alpha Y_a^i Y_p^j (Y_b^\perp)^\beta\right) \\
&\quad -w^a(t)s^r(t)\mathbb{G}_{\alpha\beta}\left(Y_a^i \frac{\partial (Y_r^\perp)^\alpha}{\partial x^i}(Y_b^\perp)^\beta + \Gamma_{ij}^\alpha Y_a^i (Y_r^\perp)^j (Y_b^\perp)^\beta\right) \\
&\quad -s^r(t)w^p(t)\mathbb{G}_{\alpha\beta}\left((Y_r^\perp)^i \frac{\partial Y_p^\alpha}{\partial x^i}(Y_b^\perp)^\beta + \Gamma_{ij}^\alpha (Y_r^\perp)^i Y_p^j (Y_b^\perp)^\beta\right) \\
&\quad -s^r(t)s^k(t)\mathbb{G}_{\alpha\beta}\left((Y_r^\perp)^i \frac{\partial (Y_k^\perp)^\alpha}{\partial x^i}(Y_b^\perp)^\beta + \Gamma_{ij}^\alpha (Y_r^\perp)^i (Y_k^\perp)^j (Y_b^\perp)^\beta\right) \\
&\quad -\mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j}(Y_b^\perp)^\beta
\end{aligned} \tag{4.66}$$

where $a, p = 1, \ldots, m,\ k, r, b = 1, \ldots, n-m,\ i, j, \alpha, \beta = 1, \ldots, n$.

Remark 4.2.19. *These expression can be imposing in their most explicit form. However, it is important to note that they are implementable in a symbolic programming language. These expression are also in a form that can be numerically integrated for simulations.*

Here $u : I \to U \subset \mathbb{R}^m$ are the controls or inputs taking values in the control set U. The state manifold is TM with local coordinates

$$((x^1, \ldots, x^n), (w^1, \ldots, w^m), (s^1, \ldots, s^{n-m}))$$

which represent the configuration, actuated velocity and unactuated velocity states. The local components for the drift vector field f_0 on TM when evaluated at $v_q \in TM$ is the $2n$-tuple where the first n components are

$$\big((w^a(t)Y_a^1 + s^b(t)(Y_b^\perp)^1), \ldots, (w^a(t)Y_a^n + s^b(t)(Y_b^\perp)^n)\big)$$

and the next m components are

$$\left(\left(\dot{w}^1 - u^a(t)\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F^a_j Y^\beta_1\right)\right), \ldots, \left(\dot{w}^m - u^a(t)\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F^a_j Y^\beta_1\right)\right)\right)$$

and the last $n-m$ components are

$$\left(\dot{s}^1(t), \ldots, \dot{s}^{n-m(}t)\right)$$

with respect to the basis of tangent vectors

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, Y^{\mathrm{vlft}}_1, \ldots, Y^{\mathrm{vlft}}_m, (Y^\perp_1)^{\mathrm{vlft}}, \ldots, (Y^\perp_{n-m})^{\mathrm{vlft}}\}$$

for $T_{v_q}TM$. The local components of control vector fields or input vector fields $f_1, \ldots, f_m$ are

$$\begin{gathered}\left(\underbrace{(0, \ldots, 0)}_{n}, \underbrace{\left(\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F^a_j Y^\beta_1\right), \ldots, 0\right)}_{m}, \underbrace{(0, \ldots, 0)}_{n-m}\right)\\ \vdots \\ \left(\underbrace{(0, \ldots, 0)}_{n}, \underbrace{\left(0, \ldots, \left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F^a_j Y^\beta_m\right)\right)}_{m}, \underbrace{(0, \ldots, 0)}_{n-m}\right)\end{gathered}$$

with respect to the basis of tangent vectors

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, Y^{\mathrm{vlft}}_1, \ldots, Y^{\mathrm{vlft}}_m, (Y^\perp_1)^{\mathrm{vlft}}, \ldots, (Y^\perp_{n-m})^{\mathrm{vlft}}\}$$

for $T_{v_q}TM$.

Let us take the natural chart $(TU_\alpha, T\phi_\alpha)$ on TM along with the associated family of vector fields

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\}$$

that when evaluated at point $v_q \in TU_\alpha$ generate the natural basis for $T_{v_q}TU_\alpha$. We provide an alternative representation of the drift and control vector fields with respect to the natural basis for $T_{v_q}TU_\alpha$. The local components for the drift vector field f_0 on TM when evaluated at $v_q \in TM$ is the $2n$-tuple where the first n components are

$$\big((w^a(t)Y_a^1 + s^b(t)(Y_b^\perp)^1), \ldots, (w^a(t)Y_a^n + s^b(t)(Y_b^\perp)^n)\big)$$

and the last n components are

$$\begin{aligned}&\left(\dot{s}^b(t)(Y_b^\perp)^1 + \left(\dot{w}^a(t) - u^a(t)\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^a Y_1^\beta\right)\right)Y_a^1\right), \ldots, \\ &\qquad \left(\dot{s}^b(t)(Y_b^\perp)^n + \left(\dot{w}^a(t) - u^a(t)\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^a Y_1^\beta\right)\right)Y_a^n\right)\end{aligned}$$

with respect to the basis of tangent vectors

$$\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\}$$

for $T_{v_q}TM$ where

$$w^a(t) = \mathbb{G}_{ij}Y_a^i v^j$$

and

$$s^b(t) = \mathbb{G}_{ij}(Y_b^\perp)^i v^j.$$

The local components of control vector fields or input vector fields $f_1, \ldots, f_m$ is the $2n$-tuple

$$\left(\underbrace{(0,\ldots,0)}_{n},\underbrace{\left(\left(\mathbb{G}_{\alpha 1}\mathbb{G}^{\alpha j}F_j^aY_1^1\right),\ldots,\left(\mathbb{G}_{\alpha n}\mathbb{G}^{\alpha j}F_j^aY_1^n\right)\right)}_{n}\right)$$
$$\vdots$$
$$\left(\underbrace{(0,\ldots,0)}_{n},\underbrace{\left(\left(\mathbb{G}_{\alpha 1}\mathbb{G}^{\alpha j}F_j^aY_m^1\right),\ldots,\left(\mathbb{G}_{\alpha n}\mathbb{G}^{\alpha j}F_j^aY_m^n\right)\right)}_{n}\right)$$

with respect to the basis of tangent vectors

$$\{\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n},\frac{\partial}{\partial v^1},\ldots,\frac{\partial}{\partial v^n}\}$$

for $T_{v_q}TM$.

4.3 Constrained Affine Foliation

Often times, the most interesting geometries for underactuated mechanical systems arise when linear velocity constraints exist. Recall that a linear velocity constraint is a distribution $\mathcal{H}$ on the configuration manifold M. A smooth curve $\gamma : I \to M$ is consistent with the linear velocity constraint $\mathcal{H}$ on M if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all $t \in I$. Here we present the formulation of an affine foliation for underactuated mechanical systems with linear velocity constraints.

We begin by constructing a **constrained $\mathbb{G}$-orthonormal frame** for the constraint distribution $\mathcal{H}$ with rank K. Recall that the set of vector fields

$$\{\mathbb{G}^\sharp(F^1),\ldots,\mathbb{G}^\sharp(F^m)\}$$

are linearly independent and form a basis for $\mathcal{Y}_q$ at each $q \in M$. Let the set of vector fields

$$\{H_1, \ldots, H_K\}$$

generate the constraint distribution $\mathcal{H}$. Our first step is to project $P_{\mathcal{H}} : TM \to \mathcal{H}$ the set of vector fields $\{\mathbb{G}^\sharp(F^1), \ldots, \mathbb{G}^\sharp(F^m)\}$ onto $\{H_1, \ldots, H_K\}$. This process is given by

$$\begin{aligned}
\overset{\mathcal{H}}{Y}_1 &= \frac{\mathbb{G}(\mathbb{G}^\sharp(F^1), H_1)}{\mathbb{G}(H_1, H_1)} H_1 + \cdots + \frac{\mathbb{G}(\mathbb{G}^\sharp(F^1), H_K)}{\mathbb{G}(H_K, H_K)} H_K \\
\vdots &= \vdots \\
\overset{\mathcal{H}}{Y}_m &= \frac{\mathbb{G}(\mathbb{G}^\sharp(F^m), H_1)}{\mathbb{G}(H_1, H_1)} H_1 + \cdots + \frac{\mathbb{G}(\mathbb{G}^\sharp(F^m), H_K)}{\mathbb{G}(H_K, H_K)} H_K.
\end{aligned}$$

We assume that $m < K$. Then the set of vector fields

$$\{\overset{\mathcal{H}}{Y}_1, \ldots, \overset{\mathcal{H}}{Y}_m\}$$

generates a distribution $P_{\mathcal{H}}(\mathcal{Y}) \subset \mathcal{H}$. We need to construct a $\mathbb{G}$-orthonormal frame where the first m elements generate $P_{\mathcal{H}}(\mathcal{Y})$. This process can be completed by following the procedure given in Section 4.2.1. Let us refer to the resulting constrained $\mathbb{G}$-orthonormal frame for $\mathcal{H}$ as the set of vector fields

$$\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_K\}$$

where the first m elements generate $P_{\mathcal{H}}(\mathcal{Y})$ and last $K - m$ elements generate the $\mathbb{G}$-orthogonal complement to $P_{\mathcal{H}}(\mathcal{Y})$ with respect to $\mathcal{H}$.

We define the **constrained affine parameters** to be the mapping

$$\hat{s}^{b-m} : \mathcal{H}_q \to \mathbb{R}$$

such that

$$\hat{s}^{b-m}(\overset{\mathcal{H}}{X}_a) = 0$$

for all $q \in M$, $a = 1, \ldots, m$, $b = m+1, \ldots, K$. The local components are given by

$$\hat{s}_i^{b-m} = \overset{\mathcal{H}}{X}{}_b^j \mathbb{G}_{ij}$$

with respect to the dual one-forms $\{dx^1, \ldots, dx^n\}$.

We define the **constrained linear parameters** to be the mapping

$$\hat{w}^a : \mathcal{H}_q \to \mathbb{R}$$

such that

$$\hat{w}^a(\overset{\mathcal{H}}{X}_b) = 0$$

for all $q \in M$, $a = 1, \ldots, m$, $b = m+1, \ldots, K$. The local components are given by

$$\hat{w}_i^a = \overset{\mathcal{H}}{X}{}_a^j \mathbb{G}_{ij}$$

with respect to the dual one-forms $\{dx^1, \ldots, dx^n\}$.

Now we derive a measure of the change in the constrained affine parameters $\hat{s}^b : TM \to \mathbb{R}$ and the constrained linear parameters $\hat{w}^a : TM \to \mathbb{R}$ along trajectories of an underactuated mechanical system with linear velocity constraints and gravitational potential. Recall that trajectories $\mathrm{Ctraj}(\Sigma_{\mathcal{H}}) = (\gamma, u)$ satisfy

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t)))) + u^a(t) P_{\mathcal{H}}(Y_a(\gamma(t)))$$

where $\overset{\mathcal{H}}{\nabla}$ is the constrained affine connection associated with $\mathcal{H}$, $P_{\mathcal{H}}$ is the $\mathbb{G}$-

orthogonal projection mapping $TM \mapsto \mathcal{H}$ and $\gamma'(t) \in \mathcal{H}$ is the tangent vector field to the curve $\gamma(t)$. Given the constrained $\mathbb{G}$-orthonormal frame $\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_K\}$ that provides an orthogonal decomposition of $\mathcal{H}_q$ for each $q \in M$, we may express the tangent vector field as the sum

$$\gamma'(t) = \hat{w}^a(t)\overset{\mathcal{H}}{X}_a(\gamma(t)) + \hat{s}^{b-m}(t)\overset{\mathcal{H}}{X}_b(\gamma(t))$$

where $a = 1, \ldots, m$ and $b = m+1, \ldots, K$.

Proposition 4.3.1 (Characterization of Constrained Affine Parameters)**.** *Let the constrained affine parameters* $\hat{\mathbf{s}} = \{\hat{s}^1, \ldots, \hat{s}^{K-m}\}$ *be the smooth assignment of the family of one-forms on* $\mathcal{H}_q$ *for each* $q \in M$*. The following holds along trajectories*

$$\mathrm{Ctraj}(\Sigma_{\mathcal{H}}) = (\gamma, u)$$

that satisfy $\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^\sharp(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t)))$*:*

$$\begin{aligned}\frac{d}{dt}s^{b-m}(t) \quad = \quad & -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) - \hat{w}^a(t)\hat{s}^{r-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_r, \overset{\mathcal{H}}{X}_b) \\ & -\hat{s}^{r-m}(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) - \hat{s}^{r-m}(t)\hat{s}^{k-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_k, \overset{\mathcal{H}}{X}_b) \\ & -\mathbb{G}(\operatorname{grad} V(\gamma(t)), \overset{\mathcal{H}}{X}_b)\end{aligned} \tag{4.67}$$

where $a, p = 1, \ldots, m$*,* $b, k, r = m+1, \ldots, K$*.*

Proof. It follows from the definition of the constrained affine parameters $\hat{\mathbf{s}} = \{\hat{s}^1, \ldots, \hat{s}^{K-m}\}$ that

$$\frac{d}{dt}\hat{s}^{b-m}(t) = \frac{d}{dt}\mathbb{G}(\overset{\mathcal{H}}{X}_b, \gamma'(t)) \tag{4.68}$$

where $\{\overset{\mathcal{H}}{X}_m, \ldots, \overset{\mathcal{H}}{X}_K\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the

$\mathbb{G}$-orthogonal complement to $P_{\mathcal{H}}(\mathcal{Y})$ with respect to $\mathcal{H}$. Let us begin by expanding the right-hand-side of Equation (4.68) by taking advantage of the compatibility associated with the constrained connection. This gives us

$$\frac{d}{dt}\hat{s}^{b-m}(t) = \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_b, \gamma'(t)) + \mathbb{G}(\overset{\mathcal{H}}{X}_b, \overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t)). \tag{4.69}$$

It follows from the definition of a mechanical control system with linear velocity constraints

$$\{M, \mathbb{G}, V, \mathcal{H}, \mathcal{F}, U\}$$

that trajectories $\mathrm{Ctraj}(\Sigma_{\mathcal{H}}) = (\gamma, u)$ satisfy

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t))) \tag{4.70}$$

where $F^1, \ldots, F^m$ are the control one-forms and $\operatorname{grad} V(\gamma(t)) = \mathbb{G}^{\sharp}(dV)(\gamma(t))$ is the gravitational potential vector field. We substitute the relation given in Equation (4.70) into the second term on the right-hand-side of Equation (4.69) to get

$$\begin{aligned} \frac{d}{dt}\hat{s}^{b-m}(t) &= \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_b, \gamma'(t)) \\ &\quad + \mathbb{G}(\overset{\mathcal{H}}{X}_b, P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t)))). \end{aligned} \tag{4.71}$$

We can use the bilinearity of $\mathbb{G}$ to expand the right-hand-side of Equation (4.71) to get

$$\begin{aligned}\frac{d}{dt}\hat{s}^{b-m}(t) &= \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_b, \gamma'(t)) \\ &\quad +\mathbb{G}(\overset{\mathcal{H}}{X}_b, P_{\mathcal{H}}(\mathbb{G}^\sharp(-\operatorname{grad} V(\gamma(t))))) \\ &\quad +\mathbb{G}(\overset{\mathcal{H}}{X}_b, u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t)))).\end{aligned} \tag{4.72}$$

Using the proof of Theorem 4.2.4, we know that the first term in the right-hand-side of Equation (4.72) can be written

$$\begin{aligned}\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_b, \gamma'(t)) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) \\ &\quad -\hat{w}^a(t)\hat{s}^{r-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_r, \overset{\mathcal{H}}{X}_b) \\ &\quad -\hat{s}^{r-m}(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) \\ &\quad -\hat{s}^{r-m}(t)\hat{s}^{k-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_k, \overset{\mathcal{H}}{X}_b).\end{aligned} \tag{4.73}$$

It follows from the proof of Theorem 4.2.8 that the third term in the right-hand-side of Equation (4.72) will vanish for each $b = m, \ldots, K$. Now we substitute the relationship given by Equation (4.73) into Equation (4.72) and set

$$\mathbb{G}(\overset{\mathcal{H}}{X}_b, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))) = 0$$

to get the expression

$$\begin{aligned}\frac{d}{dt}s^b(t) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) - \hat{w}^a(t)\hat{s}^{r-m}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_r, \overset{\mathcal{H}}{X}_b) \\ &\quad -\hat{s}^{r-m}(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) - \hat{s}^{r-m}(t)\hat{s}^{k-m}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_k, \overset{\mathcal{H}}{X}_b) \\ &\quad -\mathbb{G}(\operatorname{grad} V(\gamma(t)), \overset{\mathcal{H}}{X}_b).\end{aligned} \tag{4.74}$$

This completes the proof. $\square$

Remark 4.3.2. *Note the absence of the control parameter u in Equation (4.67). This expression represents the* ***unactuated*** *dynamics of the underactuated mechanical system in the presence of the gravitational potential force and linear velocity constraints.*

Proposition 4.3.3 (Characterization of Constrained Linear Parameters)**.** *Let the linear parameters $\hat{\mathbf{w}} = \{\hat{w}^1, \ldots, \hat{w}^m\}$ be the smooth assignment of the family of one-forms on $\mathcal{H}_q$ for each $q \in M$. The following holds along trajectories*

$$\mathrm{Ctraj}(\Sigma_{\mathcal{H}}) = (\gamma, u)$$

that satisfy $\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^\sharp(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t)))$:

$$\begin{aligned}
\frac{d}{dt}\hat{w}^l(t) \quad = \quad & -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_l) - \hat{w}^a(t)\hat{s}^{r-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_r, \overset{\mathcal{H}}{X}_l) \\
& -\hat{s}^{r-m}(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_l) - \hat{s}^{r-m}(t)\hat{s}^{k-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_k, \overset{\mathcal{H}}{X}_l) \\
& -\mathbb{G}(\operatorname{grad} V(\gamma(t)), \overset{\mathcal{H}}{X}_l) + u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), \overset{\mathcal{H}}{X}_l) \qquad (4.75)
\end{aligned}$$

where $a, p, l = 1, \ldots, m,\ k, r = m+1, \ldots, K$.

Proof. It follows from the definition of the linear parameters $\mathbf{w} = \{w^1, \ldots, w^m\}$ that

$$\frac{d}{dt}\hat{w}^l(t) = \frac{d}{dt}\mathbb{G}(\overset{\mathcal{H}}{X}_l, \gamma'(t)) \qquad (4.76)$$

where $\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_m\}$ is the family of $\mathbb{G}$-orthonormal vector fields that generate the distribution $P_{\mathcal{H}}(\mathcal{Y})$. Let us begin by expanding the right-hand-side of Equation (4.76) by taking advantage of the compatibility associated with the constrained connection. This gives us

$$\frac{d}{dt}\hat{w}^l(t) = \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_l, \gamma'(t)) + \mathbb{G}(\overset{\mathcal{H}}{X}_l, \overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t)). \tag{4.77}$$

It follows from the definition of a mechanical control system with linear velocity constraints

$$\{M, \mathbb{G}, V, \mathcal{H}, \mathcal{F}, U\}$$

that trajectories $\mathrm{Ctraj}(\Sigma_{\mathcal{H}}) = (\gamma, u)$ satisfy

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t))) \tag{4.78}$$

where $F^1, \ldots, F^m$ are the control one-forms and $\operatorname{grad} V(\gamma(t)) = \mathbb{G}^{\sharp}(dV)(\gamma(t))$ is the gravitational potential vector field. We substitute the relation given in Equation (4.78) into the second term on the right-hand-side of Equation (4.77) to get

$$\frac{d}{dt}\hat{w}^l(t) = \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_l, \gamma'(t)) + \mathbb{G}(\overset{\mathcal{H}}{X}_l, P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t)))) + u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t)))). \tag{4.79}$$

We can use the bilinearity of $\mathbb{G}$ to expand the right-hand-side of Equation (4.79) to get

$$\begin{aligned}\frac{d}{dt}\hat{w}^l(t) = \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_l, \gamma'(t))& \\ &+\mathbb{G}(\overset{\mathcal{H}}{X}_l, P_{\mathcal{H}}(\mathbb{G}^{\sharp}(-\operatorname{grad} V(\gamma(t))))) \\ &+\mathbb{G}(\overset{\mathcal{H}}{X}_l, u^a(t)P_{\mathcal{H}}(Y_a(\gamma(t))))).\end{aligned}$$

Using the proof of Theorem 4.2.6, we know that the first term in the right-hand-side of Equation (4.80) can be written

$$\begin{aligned}\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\overset{\mathcal{H}}{X}_l, \gamma'(t)) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_l) \\ &\quad -\hat{w}^a(t)\hat{s}^{r-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_r, \overset{\mathcal{H}}{X}_l) \\ &\quad -\hat{s}^{r-m}(t)\hat{w}^p(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_l) \\ &\quad -\hat{s}^{r-m}(t)\hat{s}^{k-m}(t)\mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_k, \overset{\mathcal{H}}{X}_l)\end{aligned} \tag{4.80}$$

It follows from the proof of Theorem 4.2.11 that the third term in the right-hand-side of Equation (4.80) will be equal to

$$\mathbb{G}(\overset{\mathcal{H}}{X}_l, u^a(t)\mathbb{G}^\sharp(F^a(\gamma(t)))) = u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), \overset{\mathcal{H}}{X}_l). \tag{4.81}$$

Now we substitute the relations given by Equation (4.80) and Equation (4.81) into Equation (4.80) to get the expression

$$\begin{aligned}\frac{d}{dt}\hat{w}^l(t) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_l) - \hat{w}^a(t)\hat{s}^{r-m}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_r, \overset{\mathcal{H}}{X}_l) \\ &\quad -\hat{s}^{r-m}(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_l) - \hat{s}^{r-m}(t)\hat{s}^{k-m}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_r}\overset{\mathcal{H}}{X}_k, \overset{\mathcal{H}}{X}_l) \\ &\quad -\mathbb{G}(\operatorname{grad} V(\gamma(t)), \overset{\mathcal{H}}{X}_l) + u^a(t)\mathbb{G}(\mathbb{G}^\sharp(F^a(\gamma(t))), \overset{\mathcal{H}}{X}_l).\end{aligned} \tag{4.82}$$

This completes the proof. $\square$

Remark 4.3.4. *Note the explicit occurrence of the control parameter u in Equation (4.75). The linear parameters are the actuated velocity states. This expression represents the* ***actuated*** *dynamics of the underactuated mechanical system in the presence of the gravitational potential force and linear velocity constraints.*

Now we associate a vector-valued quadratic form to an underactuated mechanical system with linear velocity constraints. We define

$$\overset{\mathcal{H}}{Q}_q : P_{\mathcal{H}}(\mathcal{Y}_q) \subset T_qM \to \mathcal{H}/P_{\mathcal{H}}(\mathcal{Y}_q)$$

as the $\mathcal{H}/P_{\mathcal{H}}(\mathcal{Y}_q)$-valued quadratic map on $P_{\mathcal{H}}(\mathcal{Y}_q)$ given by

$$P_{\mathcal{H}}(\mathcal{Y}_q) \ni v_q \mapsto \mathbb{G}(\overset{\mathcal{H}}{X}_a(q), v_q)\mathbb{G}(\overset{\mathcal{H}}{X}_p(q), v_q)\mathbb{G}((\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a} \overset{\mathcal{H}}{X}_p)(q), \overset{\mathcal{H}}{X}_b(q))\overset{\mathcal{H}}{X}_b(q) \in \mathcal{H}/P_{\mathcal{H}}(\mathcal{Y}_q)$$

where $\{\overset{\mathcal{H}}{X}_m(q), \ldots, \overset{\mathcal{H}}{X}_K(q)\}$ is the $\mathbb{G}$-orthonormal basis for $\mathcal{H}/P_{\mathcal{H}}(\mathcal{Y}_q)$. Given the $\mathbb{G}$-orthonormal basis $\{\overset{\mathcal{H}}{X}_m(q), \ldots, \overset{\mathcal{H}}{X}_K(q)\}$ for $\mathcal{H}/P_{\mathcal{H}}(\mathcal{Y}_q)$, the local components for $\overset{Q}{\mathcal{H}}_q(v_q)$ with respect to each basis vector $\overset{\mathcal{H}}{X}_b(q)$ for $b = m+1, \ldots, K$ are

$$\hat{w}^a \hat{w}^p \mathbb{G}_{ij} (\nabla_{\overset{\mathcal{H}}{X}_a} \overset{\mathcal{H}}{X}_p)^i (\overset{\mathcal{H}}{X}_b)^j$$

where $a, p = 1, \ldots, m,\ i, j = 1, \ldots, n$. Let $((x^1, \ldots, x^n),\ (v^1, \ldots, v^n))$ be the natural coordinates on TM where $(v^1, \ldots, v^n)$ are the coefficients of a tangent vector given the usual basis $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$. The local expression for $\overset{\mathcal{H}}{Q}_q(v_q)$ with respect to the natural coordinates is

$$\mathbb{G}_{ij} \overset{\mathcal{H}}{X}{}^i_a v^j \mathbb{G}_{ij} \overset{\mathcal{H}}{X}{}^i_p v^j \mathbb{G}_{\alpha\beta} \left(\overset{\mathcal{H}}{X}{}^i_a \frac{\partial \overset{\mathcal{H}}{X}{}^\alpha_p}{\partial x^i} (\overset{\mathcal{H}}{X}_b)^\beta + \overset{\mathcal{H}}{\Gamma}{}^\alpha_{ij} \overset{\mathcal{H}}{X}{}^i_a \overset{\mathcal{H}}{X}{}^j_p (\overset{\mathcal{H}}{X}_b)^\beta \right) (\overset{\mathcal{H}}{X}_b)^k \frac{\partial}{\partial x^k}$$

where

$$\begin{aligned} (\overset{\mathcal{H}}{Q}_q)^{b-m}_{ap} &= \mathbb{G}(\overset{\mathcal{H}}{\nabla}_{\overset{\mathcal{H}}{X}_a} \overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_b) \\ &= \mathbb{G}_{\alpha\beta} \left(\overset{\mathcal{H}}{X}{}^i_a \frac{\partial \overset{\mathcal{H}}{X}{}^\alpha_p}{\partial x^i} (\overset{\mathcal{H}}{X}_b)^\beta + \overset{\mathcal{H}}{\Gamma}{}^\alpha_{ij} \overset{\mathcal{H}}{X}{}^i_a \overset{\mathcal{H}}{X}{}^j_p (\overset{\mathcal{H}}{X}_b)^\beta \right) \end{aligned} \tag{4.83}$$

and $i, j, k = 1, \ldots, n,\ a, p = 1, \ldots, m,\ b = m, \ldots, K$.

Remark 4.3.5. *The vector-valued quadratic form will play a critical role in our analysis and control of underactuated mechanical systems with linear velocity constraints. Specifically, we will use the definiteness of the vector-valued quadratic form to determine possible motion in the affine foliation of the tangent bundle. The analysis will lead to an iterative algorithm for motion planning that utilizes the intrinsic quadratic structure.*

4.4 Examples

In this section we construct the affine foliation formulation for our motivating examples. The classic geometric model for each of these systems can be found in Section 3.3.

4.4.1 Planar Rigid Body

Let us consider the planar rigid body with control set $\{Y_1, Y_2\}$. The $\mathbb{G}$-orthonormal frame is the set of vector fields $\{X_1, X_2, X_3\}$ given by

$$\begin{pmatrix} \sqrt{\frac{1}{m}}\cos(\theta) \\ -\frac{\sin(\theta)}{m\sqrt{\frac{h^2}{J}+\frac{1}{m}}} \\ -h\sin(\theta)\sqrt{\frac{1}{h^2m+J}} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{m}}\sin(\theta) \\ \frac{\cos(\theta)}{m\sqrt{\frac{h^2}{J}+\frac{1}{m}}} \\ h\cos(\theta)\sqrt{\frac{1}{h^2m+J}} \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{h}{J\sqrt{\frac{h^2}{J}+\frac{1}{m}}} \\ \sqrt{\frac{1}{h^2m+J}} \end{pmatrix}$$

with respect to the natural frame

$$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\}.$$

The linear parameters are

$$
\begin{aligned}
w^1 &= \frac{\cos(\theta)}{\sqrt{\frac{1}{m}}}dx + \frac{\sin(\theta)}{\sqrt{\frac{1}{m}}}dy \\
w^2 &= -\frac{\sin(\theta)}{\sqrt{\frac{h^2}{J}+\frac{1}{m}}}dx + \frac{\cos(\theta)}{\sqrt{\frac{h^2}{J}+\frac{1}{m}}}dy + -\frac{h}{\sqrt{\frac{h^2}{J}+\frac{1}{m}}}d\theta
\end{aligned}
$$

and the affine parameter is

$$
s = -hm\sin(\theta)\sqrt{\frac{1}{h^2m+J}}dx + hm\cos(\theta)\sqrt{\frac{1}{h^2m+J}}dy + J\sqrt{\frac{1}{h^2m+J}}d\theta
$$

with respect to the dual basis $\{dx, dy, d\theta\}$. The actuated dynamics are

$$
\begin{aligned}
\frac{d}{dt}w^1(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_1) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_1) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_1) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_1) \\
&\quad +u^1(t)\mathbb{G}(\mathbb{G}^\sharp(F^1(\gamma(t))), X_1) + u^2(t)\mathbb{G}(\mathbb{G}^\sharp(F^2(\gamma(t))), X_1) \\
\frac{d}{dt}w^2(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_2) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_2) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_2) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_2) \\
&\quad +u^1(t)\mathbb{G}(\mathbb{G}^\sharp(F^1(\gamma(t))), X_2) + u^2(t)\mathbb{G}(\mathbb{G}^\sharp(F^2(\gamma(t))), X_2) \quad (4.84)
\end{aligned}
$$

where $a, p = 1, 2$ and the nonzero coefficients $\mathbb{G}(\nabla_{X_i}X_j, X_k)$ can be found in Appendix A. The unactuated dynamics are

$$
\begin{aligned}
\frac{d}{dt}s(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_3) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_3) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_3) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_3)
\end{aligned}
$$

where $a, p = 1, 2$ and the nonzero coefficients $\mathbb{G}(\nabla_{X_i} X_j, X_k)$ can be found in Appendix A. The entries of the quadratic form are

$$Q_{ap} = \mathbb{G}(\nabla_{X_a} X_p, X_3).$$

Let us consider the planar rigid body with control set $\{Y_1, Y_3\}$. The $\mathbb{G}$-orthonormal frame is the set of vector fields $\{X_1, X_2, X_3\}$ given by

$$\begin{pmatrix} \sqrt{\frac{1}{m}}\cos(\theta) \\ 0 \\ -\sqrt{\frac{1}{m}}\sin(\theta) \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{m}}\sin(\theta) \\ 0 \\ \sqrt{\frac{1}{m}}\cos(\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\frac{1}{J}} \\ 0 \end{pmatrix}$$

with respect to the natural frame

$$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\}.$$

The linear parameters are

$$\begin{aligned} w^1 &= \frac{\cos(\theta)}{\sqrt{\frac{1}{m}}}dx + \frac{\sin(\theta)}{\sqrt{\frac{1}{m}}}dy \\ w^2 &= \frac{1}{\sqrt{\frac{1}{J}}}d\theta \end{aligned}$$

and the affine parameter is

$$s = -\frac{\sin(\theta)}{\sqrt{\frac{1}{m}}}dx + \frac{\cos(\theta)}{\sqrt{\frac{1}{m}}}dy$$

with respect to the dual basis $\{dx, dy, d\theta\}$. The actuated dynamics are

$$\begin{aligned}
\frac{d}{dt}w^1(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_1) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_1) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_1) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_1) \\
&\quad +u^1(t)\mathbb{G}(\mathbb{G}^\sharp(F^1(\gamma(t))), X_1) + u^3(t)\mathbb{G}(\mathbb{G}^\sharp(F^3(\gamma(t))), X_1) \\
\frac{d}{dt}w^2(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_2) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_2) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_2) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_2) \\
&\quad u^1(t)\mathbb{G}(\mathbb{G}^\sharp(F^1(\gamma(t))), X_2) + u^3(t)\mathbb{G}(\mathbb{G}^\sharp(F^3(\gamma(t))), X_2) \qquad (4.85)
\end{aligned}$$

where $a, p = 1, 2$ and the nonzero coefficients $\mathbb{G}(\nabla_{X_i}X_j, X_k)$ can be found in Appendix A. The unactuated dynamics are

$$\begin{aligned}
\frac{d}{dt}s(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_3) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_3) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_3) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_3)
\end{aligned}$$

where $a, p = 1, 2$ and the nonzero coefficients $\mathbb{G}(\nabla_{X_i}X_j, X_k)$ can be found in Appendix A. The entries of the quadratic form are

$$Q_{ap} = \mathbb{G}(\nabla_{X_a}X_p, X_3).$$

4.4.2 Roller Racer

Let us consider the roller racer with the single control $\{Y_1\}$. The constrained $\mathbb{G}$-orthonormal frame with respect to the natural frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}\}$ can be found in Appendix B. The components of the constrained linear parameter are

$$\begin{aligned}
\hat{w}_1 &= \frac{2m\cos(\theta)\sin(\psi)(I_1L_2 - I_2L_1\cos(\psi))}{C(\psi)\left(I_2\cos(2\psi)\left(L_1^2m - I_1\right) + I_1\left(I_2 + 2L_2^2m\right) + I_2L_1^2m\right)} \\
\hat{w}_2 &= \frac{2m\sin(\theta)\sin(\psi)(I_1L_2 - I_2L_1\cos(\psi))}{C(\psi)\left(I_2\cos(2\psi)\left(L_1^2m - I_1\right) + I_1\left(I_2 + 2L_2^2m\right) + I_2L_1^2m\right)} \\
\hat{w}_3 &= -\frac{2m(L_1\cos(\psi) + L_2)(I_1L_2 - I_2L_1\cos(\psi))}{C(\psi)\left(I_2\cos(2\psi)\left(L_1^2m - I_1\right) + I_1\left(I_2 + 2L_2^2m\right) + I_2L_1^2m\right)} \\
\hat{w}_4 &= \frac{I_2\left(\cos(2\psi)\left(L_1^2m - I_1\right) + I_1 + L_1^2m + 2L_1L_2m\cos(\psi)\right)}{C(\psi)\left(I_2\cos(2\psi)\left(L_1^2m - I_1\right) + I_1\left(I_2 + 2L_2^2m\right) + I_2L_1^2m\right)}
\end{aligned}$$

with respect to the dual basis $\{dx, dy, d\theta, d\psi\}$. The term $C(\psi)$ can be found in Appendix B. The components of the constrained affine parameter with respect to the dual basis $\{dx, dy, d\theta, d\psi\}$ are

$$\begin{aligned}
\hat{s}_1 &= \frac{m\cos(\theta)}{K(\psi)} \\
\hat{s}_2 &= \frac{m\sin(\theta)}{K(\psi)} \\
\hat{s}_3 &= \frac{(I_1 + I_2)\sin(\psi)}{(L_1\cos(\psi) + L_2)K(\psi)} \\
\hat{s}_4 &= \frac{I_2\sin(\psi)}{(L_1\cos(\psi) + L_2)K(\psi)}
\end{aligned}$$

where

$$K(\psi) = \sqrt{\frac{(I_1 + I_2)\sin^2(\psi)}{(L_1\cos(\psi) + L_2)^2} + m}.$$

The control vector field projected onto the $\mathbb{G}$-orthonormal frame are

$$\overset{\mathcal{H}}{Y}_1 = C(\psi)\overset{\mathcal{H}}{X}_1$$

The actuated dynamics are

$$
\begin{aligned}
\frac{d}{dt}\hat{w}(t) \quad &= \quad -\hat{w}(t)\hat{w}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_1}\overset{\mathcal{H}}{X}_1, \overset{\mathcal{H}}{X}_1) - \hat{w}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_1}\overset{\mathcal{H}}{X}_2, \overset{\mathcal{H}}{X}_1) \\
&\quad\quad -\hat{s}(t)\hat{w}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_2}\overset{\mathcal{H}}{X}_1, \overset{\mathcal{H}}{X}_1) - \hat{s}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_2}\overset{\mathcal{H}}{X}_2, \overset{\mathcal{H}}{X}_1) \\
&\quad\quad +u^1(t)C(\psi)
\end{aligned}
$$

where the nonzero coefficients $\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_i}\overset{\mathcal{H}}{X}_j, \overset{\mathcal{H}}{X}_k)$ can be found in Appendix B. The unactuated dynamics are

$$
\begin{aligned}
\frac{d}{dt}\hat{s}(t) \quad &= \quad -\hat{w}(t)\hat{w}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_1}\overset{\mathcal{H}}{X}_1, \overset{\mathcal{H}}{X}_2) - \hat{w}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_1}\overset{\mathcal{H}}{X}_2, \overset{\mathcal{H}}{X}_2) \\
&\quad\quad -\hat{s}(t)\hat{w}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_2}\overset{\mathcal{H}}{X}_1, \overset{\mathcal{H}}{X}_2) - \hat{s}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_2}\overset{\mathcal{H}}{X}_2, \overset{\mathcal{H}}{X}_2)
\end{aligned}
$$

where the nonzero coefficients $\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_i}\overset{\mathcal{H}}{X}_j, \overset{\mathcal{H}}{X}_k)$ can be found in Appendix B. The single entry of the quadratic form are

$$
Q_{11} \quad = \quad \mathbb{G}(\nabla_{X_1}X_1, X_2).
$$

4.4.3 Snakeboard

Let us consider the snakeboard with the set of control vector fields $\{Y_1, Y_2\}$. The constrained $\mathbb{G}$-orthonormal frame is the set of vector fields $\{\overset{\mathcal{H}}{X}_1, \overset{\mathcal{H}}{X}_2, \overset{\mathcal{H}}{X}_3\}$ given by

$$\begin{pmatrix} \frac{\sqrt{2}J_r\cos(\theta)\sin(\phi)\cos(\phi)}{lm\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ \frac{\sqrt{2}J_r\sin(\theta)\sin(\phi)\cos(\phi)}{lm\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ -\frac{\sqrt{2}J_r\sin^2(\phi)}{l^2m\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ \frac{\sqrt{2}}{\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{J_w}} \end{pmatrix}, \begin{pmatrix} \frac{l\cos(\theta)\cos(\phi)}{\sqrt{l^2m}} \\ \frac{l\sin(\theta)\cos(\phi)}{\sqrt{l^2m}} \\ -\frac{\sin(\phi)}{\sqrt{l^2m}} \\ 0 \\ 0 \end{pmatrix}$$

with respect to the natural frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \phi}\}$.

The components of the constrained linear parameters are

$$\begin{aligned} \hat{w}_1^1 &= \frac{\sqrt{2}J_r\cos(\theta)\sin(\phi)\cos(\phi)}{l\sqrt{\frac{J_r(J_r\cos(2\phi)-J_r+2l^2m)}{l^2m}}} \\ \hat{w}_2^1 &= \frac{\sqrt{2}J_r\sin(\theta)\sin(\phi)\cos(\phi)}{l\sqrt{\frac{J_r(J_r\cos(2\phi)-J_r+2l^2m)}{l^2m}}} \\ \hat{w}_3^1 &= \frac{\sqrt{2}J_r\cos^2(\phi)}{\sqrt{\frac{J_r(J_r\cos(2\phi)-J_r+2l^2m)}{l^2m}}} \\ \hat{w}_4^1 &= \frac{\sqrt{\frac{J_r(J_r\cos(2\phi)-J_r+2l^2m)}{l^2m}}}{\sqrt{2}} \\ \hat{w}_5^1 &= 0 \end{aligned}$$

and

$$\begin{aligned}
\hat{w}_1^2 &= 0 \\
\hat{w}_2^2 &= 0 \\
\hat{w}_3^2 &= 0 \\
\hat{w}_4^2 &= 0 \\
\hat{w}_5^2 &= \sqrt{J_w}.
\end{aligned}$$

with respect to the dual basis $\{dx, dy, d\theta, d\psi, d\phi\}$. The components of the constrained affine parameter with respect to the dual basis $\{dx, dy, d\theta, d\psi, d\phi\}$ are

$$\begin{aligned}
\hat{s}_1 &= \frac{lm\cos(\theta)\cos(\phi)}{\sqrt{l^2m}} \\
\hat{s}_2 &= \frac{lm\sin(\theta)\cos(\phi)}{\sqrt{l^2m}} \\
\hat{s}_3 &= -\sqrt{l^2m}\sin(\phi) \\
\hat{s}_4 &= -\frac{J_r\sin(\phi)}{\sqrt{l^2m}} \\
\hat{s}_5 &= 0.
\end{aligned}$$

The control vector fields projected onto the $\mathbb{G}$-orthonormal frame is

$$\overset{\mathcal{H}}{Y}_1 = \frac{\sqrt{2}}{\sqrt{\frac{J_r(J_r\cos(2\phi)-J_r+2l^2m)}{l^2m}}}\overset{\mathcal{H}}{X}_1$$

and

$$\overset{\mathcal{H}}{Y}_2 = \frac{1}{\sqrt{J_w}}\overset{\mathcal{H}}{X}_2.$$

The actuated dynamics are

$$\begin{aligned}
\frac{d}{dt}\hat{w}^1(t) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_1) - \hat{w}^a(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_3, \overset{\mathcal{H}}{X}_1) \\
&\quad -\hat{s}(t)\hat{w}^a(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_a, \overset{\mathcal{H}}{X}_1) - \hat{s}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_3, \overset{\mathcal{H}}{X}_1) \\
&\quad +u^1(t)\frac{\sqrt{2}}{\sqrt{\frac{J_r(J_r\cos(2\phi)-J_r+2l^2m)}{l^2m}}} \\
\frac{d}{dt}\hat{w}^2(t) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_2) - \hat{w}^a(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_3, \overset{\mathcal{H}}{X}_2) \\
&\quad -\hat{s}(t)\hat{w}^a(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_a, \overset{\mathcal{H}}{X}_2) - \hat{s}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_3, \overset{\mathcal{H}}{X}_2) \\
&\quad +u^2(t)\frac{1}{\sqrt{J_w}}
\end{aligned}$$

where the nonzero coefficients $\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_i}\overset{\mathcal{H}}{X}_j, \overset{\mathcal{H}}{X}_k)$ can be found in Appendix C. The unactuated dynamics are

$$\begin{aligned}
\frac{d}{dt}\hat{s}(t) &= -\hat{w}^a(t)\hat{w}^p(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_p, \overset{\mathcal{H}}{X}_3) - \hat{w}^a(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_a}\overset{\mathcal{H}}{X}_3, \overset{\mathcal{H}}{X}_3) \\
&\quad -\hat{s}(t)\hat{w}^a(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_a, \overset{\mathcal{H}}{X}_3) - \hat{s}(t)\hat{s}(t)\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_3, \overset{\mathcal{H}}{X}_3)
\end{aligned}$$

where the nonzero coefficients $\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_i}\overset{\mathcal{H}}{X}_j, \overset{\mathcal{H}}{X}_k)$ can be found in Appendix C. The single entry of the quadratic form are

$$Q_{ap} = \mathbb{G}(\nabla_{X_a}X_p, X_3).$$

4.4.4 Three Link Manipulator

Let us consider the three link manipulator with the control set $\{Y_1, Y_2\}$. The $\mathbb{G}$-orthonormal frame is the set of vector fields $\{X_1, X_2, X_3\}$ given by

$$\begin{pmatrix} \frac{\sqrt{\frac{2I_c - L^2 m\cos(2\theta) + L^2 m}{I_c m}}}{\sqrt{2}} \\ -\frac{\sqrt{2}L^2\sin(\theta)\cos(\theta)}{I_c\sqrt{\frac{2I_c - L^2 m\cos(2\theta)+L^2 m}{I_c m}}} \\ \frac{\sqrt{2}L\sin(\theta)}{I_c\sqrt{\frac{2I_c - L^2 m\cos(2\theta)+L^2 m}{I_c m}}} \end{pmatrix}, \begin{pmatrix} 0 \\ 2\sqrt{\frac{I_c + L^2 m}{4I_c m - 2L^2 m^2\cos(2\theta) + 2L^2 m^2}} \\ -\frac{\sqrt{2}Lm\cos(\theta)\sqrt{\frac{I_c + L^2 m}{m(2I_c - L^2 m\cos(2\theta) + L^2 m)}}}{I_c + L^2 m} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{I_c + L^2 m}} \end{pmatrix}$$

with respect to the natural frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}\}$. The linear parameters are

$$\begin{aligned} w^1 &= \frac{\sqrt{2}}{\sqrt{\frac{2I_c - L^2 m\cos(2\theta)+L^2 m}{I_c m}}}dx \\ w^2 &= \frac{L^2 m^2\sin(2\theta)\sqrt{\frac{I_c + L^2 m}{4I_c m - 2L^2 m^2\cos(2\theta)+2L^2 m^2}}}{I_c + L^2 m}dx \\ &\quad + \frac{1}{2\sqrt{\frac{I_c + L^2 m}{4I_c m - 2L^2 m^2\cos(2\theta)+2L^2 m^2}}}dy \end{aligned}$$

and the affine parameter is

$$s = -Lm\sin(\theta)\sqrt{\frac{1}{I_c + L^2 m}}dx + Lm\cos(\theta)\sqrt{\frac{1}{I_c + L^2 m}}dy + \frac{1}{\sqrt{\frac{1}{I_c + L^2 m}}}d\theta$$

with respect to the dual basis $\{dx, dy, d\theta\}$. The actuated dynamics are

$$
\begin{aligned}
\frac{d}{dt}w^1(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_1) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_1) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_1) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_1) \\
&\quad +u^1(t)\mathbb{G}(\mathbb{G}^\sharp(F^1(\gamma(t))), X_1) + u^2(t)\mathbb{G}(\mathbb{G}^\sharp(F^2(\gamma(t))), X_1) \\
\frac{d}{dt}w^2(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_2) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_2) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_2) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_2) \\
&\quad +u^1(t)\mathbb{G}(\mathbb{G}^\sharp(F^1(\gamma(t))), X_2) + u^2(t)\mathbb{G}(\mathbb{G}^\sharp(F^2(\gamma(t))), X_2) \quad (4.86)
\end{aligned}
$$

where $a, p = 1, 2$ and the nonzero coefficients $\mathbb{G}(\nabla_{X_i}X_j, X_k)$ can be found in Appendix D. The unactuated dynamics are

$$
\begin{aligned}
\frac{d}{dt}s(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_3) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_3) \\
&\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_3) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_3)
\end{aligned}
$$

where $a, p = 1, 2$ and the nonzero coefficients $\mathbb{G}(\nabla_{X_i}X_j, X_k)$ can be found in Appendix D. The entries of the quadratic form are

$$
Q_{ap} = \mathbb{G}(\nabla_{X_a}X_p, X_3).
$$

CHAPTER 5

PARTITIONING CONNECTIONS FOR UNDERACTUATED MECHANICAL SYSTEMS

A common starting point for treatments of underactuated mechanical systems is that there exists a set of coordinates $(q^1, \ldots, q^n)$ such that the local expression for the governing equations of motion are

$$\mathbb{G}_{ak}(q)\ddot{q}^k + \mathbb{G}_{ak}\Gamma^k_{ij}\dot{q}^i\dot{q}^j = -\frac{\partial V}{\partial q^a} + u^a, \quad a = 1, \ldots, m, \tag{5.1}$$

$$\mathbb{G}_{\alpha k}(q)\ddot{q}^k + \mathbb{G}_{\alpha k}\Gamma^k_{ij}\dot{q}^i\dot{q}^j = -\frac{\partial V}{\partial q^\alpha}, \quad \alpha = m+1, \ldots, n. \tag{5.2}$$

This local expression implies that only the first m degrees of freedom are actuated. Equation (5.1) represents the actuated dynamics while Equation (5.2) represents the unactuated dynamics. A known limitation of this formulation for underactuated mechanical systems is that it requires that the input codistribution be integrable [10]. It is not always physically valid to assume that the input codistribution is integrable for a general underactuated mechanical system. For example, the forced planar rigid body and various constrained systems considered in this thesis do not satisfy this assumption.

This thesis contains an alternative formulation for underactuated mechanical systems that utilizes partitioning connections. We introduce two linear connections that provide a coordinate invariant representation that partitions the actuated and unactuated dynamics. Our formulation does not require that the input codistribution be integrable, therefore can be viewed as a generalization of the partitioning used in existing literature on underactuated mechanical systems [59], [57], [53]. We show that feedback linearization of the actuated dynamics gives rise to a control-affine system whose drift vector field is the geodesic spray of the unactuated connection associated with unactuated dynamics. We call this control-affine system the *geometric normal form* for underactuated mechanical systems. The geometric normal form is the starting point for our reachability analysis and motion algorithms for mechanical systems underactuated by one. Similar to the affine foliation formalism, the unactuated connection gives rise to an intrinsic vector-valued symmetric bilinear (quadratic) form. Again, a significant advantage of the partitioning connections is that the formulation is still valid for the extended class of underactuated mechanical systems with linear velocity constraints.

Here we introduce two connections that partition the actuated and unactuated dynamic equations. Recall the $\mathbb{G}$-orthonormal frame

$$\{Y_1, \ldots, Y_m, Y_1^{\perp}, \ldots, Y_{n-m}^{\perp}\}$$

on M is the set of vector fields constructed using the input distribution $\mathcal{Y}$ and the Riemannian metric $\mathbb{G}$ included in the basic problem formulation of an underactuated simple mechanical control system (see Chapter 4). Let us use the $\mathbb{G}$-orthonormal frame to construct the Poincare representation of the equations of motion

$$\nabla_{\gamma'(t)}\gamma'(t) = -\operatorname{grad} V(\gamma(t)) + u^a(t)Y_a(\gamma(t)).$$

For notational simplicity, let us associated the first m elements of the $\mathbb{G}$ -orthonormal frame with

$$\{X_1, \dots, X_m\}$$

and the remaining $n - m$ elements of the $\mathbb{G}$-orthonormal frame with

$$\{X_{m+1}, \dots, X_n\}.$$

The generalized Christoffel symbols for ∇ with respect to $\{X_1, \dots, X_n\}$ is the n^3 functions

$$\widehat{\Gamma}^k_{ij} : M \to \mathbb{R}$$

for $i, j, k \in \{1, \dots, n\}$ such that

$$\nabla_{X_i} X_j = \widehat{\Gamma}^k_{ij} X_k.$$

Then the left-hand-side of the equations of motion for an underactuated simple mechanical control system is defined by

$$\nabla_{\gamma'(t)} \gamma'(t) \;\; = \;\; \left(\dot{v}^k(t) + \widehat{\Gamma}^k_{ij}(\gamma(t)) v^i(t) v^j(t) \right) X_k(\gamma(t))$$

where $v^k(t) = \mathbb{G}(\gamma'(t), X_k(\gamma(t)))$ is the velocity component with respect to the $\mathbb{G}$-orthonormal frame. Now we make the notational assignment of $v^k(t) = w^k(t)$ for $k = 1, \dots, m$ and $v^k(t) = s^{k-m}(t)$ for $k = m+1, \dots, n$. This gives us the local coordinates $((x^1, \dots, x^n), (w^1, \dots, w^m, s^1, \dots, s^{n-m}))$ on TM where $w^\nu = \mathbb{G}(v_q, X_\nu)$ for $\nu = 1, \dots, m$ and $s^{\mu-m} = \mathbb{G}(v_q, X_\mu)$ for $\mu = m+1, \dots, n$. This naturally induces the coordinate frame

$$\left\{\left\{\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right\},\left\{\frac{\partial}{\partial w^1},\ldots,\frac{\partial}{\partial w^m},\frac{\partial}{\partial s^1},\ldots,\frac{\partial}{\partial s^{n-m}}\right\}\right\}$$

for $T_{v_q}TM$. Futhermore, the vertical lift of the $\mathbb{G}$-orthonormal frame with elements X_a is the tangent vector to the curve in the fiber defined by

$$X_a^{\mathrm{vlft}} = \frac{d}{dt}\left(v_q + tX_a\right)$$

where $X_\nu^{\mathrm{vlft}} = \frac{\partial}{\partial w^\nu}$ and $X_\mu^{\mathrm{vlft}} = \frac{\partial}{\partial s^{\mu-m}}$. This gives rise to the following local representation of the equations of motion

$$\begin{aligned}
\dot{x}^i &= w^a X_a^i + s^{b-m} X_b^i \\
\dot{w}^\nu &= -\widehat{\Gamma}_{ij}^\nu v^i v^j \\
&\quad -\mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j} X_\nu^\beta \\
&\quad +u^a\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j} F_j^a X_\nu^\beta\right) \\
\dot{s}^{\mu-m} &= -\widehat{\Gamma}_{ij}^\mu v^i v^j \\
&\quad -\mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j} X_\mu^\beta
\end{aligned} \tag{5.3}$$

where $\nu, a = 1,\ldots,m$, $b,\mu = m+1,\ldots,n$ and $i,j,\alpha,\beta = 1,\ldots,n$. An alternative representation of the system of first-order equations on TM would be

$$\begin{aligned}
\Psi' &= \left(w^a X_a^i + s^b X_b^i\right)\frac{\partial}{\partial x^i} \\
&\quad \left(-\widehat{\Gamma}_{ij}^\nu v^i v^j - \mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j} X_\nu^\beta + u^a\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j} F_j^a X_\nu^\beta\right)\right)\frac{\partial}{\partial w^\nu} \\
&\quad \left(-\widehat{\Gamma}_{ij}^\mu v^i v^j - \mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j} X_\mu^\beta\right)\frac{\partial}{\partial s^{\mu-m}}.
\end{aligned}$$

5.1 Actuated Connection

Here we introduce a new linear connection that will be associated with the actuated dynamics. Let us begin by defining the projection mapping $P_{\mathcal{Y}} : TM \to \mathcal{Y}$ where

$$P_{\mathcal{Y}}(v_q) = \mathbb{G}(X_a, v_q)X_a, \quad a = 1, \ldots, m$$

given the $\mathbb{G}$-orthonormal frame $\{X_1, \ldots, X_n\}$ constructed from the input distribution $\mathcal{Y}$ and the Riemannian metric $\mathbb{G}$ found in the basic problem formulation of an underactuated simple mechanical control system.

Definition 5.1.1 (Actuated Connection)**.** *Let $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, V, U\}$ be an underactuated simple mechanical control system. The* ***actuated connection*** *is the linear connection $\overset{\mathcal{Y}}{\nabla}$ on M given by*

$$\overset{\mathcal{Y}}{\nabla}_X Y \quad = \quad P_{\mathcal{Y}}(\nabla_X Y)$$

for all $X, Y \in \Gamma(TM)$.

One important differential geometric property of the actuated connection $\overset{\mathcal{Y}}{\nabla}$ is that

$$\overset{\mathcal{Y}}{\nabla}_X Y \in \Gamma(\mathcal{Y})$$

which is equivalent to the statement that $\overset{\mathcal{Y}}{\nabla}$ restricts to $\mathcal{Y}$. With the notion of an actuated connection, we state the following result concerning the actuated dynamics.

Proposition 5.1.2 (Actuated Dynamics)**.** *Let $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, V, U\}$ be an under-*

actuated simple mechanical control system with the $\mathbb{G}$-orthonormal frame

$$\{X_1, \ldots, X_n\}$$

where $\{X_1, \ldots, X_m\}$ generates $\mathcal{Y}$. The following holds along the curve $\gamma(t)$ satisfying

$$\nabla_{\gamma'(t)}\gamma'(t) = \left(\dot{v}^k(t) + \widehat{\Gamma}^k_{ij}(\gamma(t))v^i(t)v^j(t)\right) X_k(\gamma(t)) :$$

$$\overset{\mathcal{Y}}{\nabla}_{\gamma'(t)}\gamma'(t) \quad = \quad \left(\dot{w}^\nu + v^i v^j \widehat{\Gamma}^\nu_{ij}\right) X_\nu$$

and

$$\dot{w}^\nu \quad = \quad -v^i v^j \widehat{\Gamma}^\nu_{ij} - \mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j} X^\beta_\nu + u^a \mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j} F^a_j X^\beta_\nu$$

where $\nu, a = 1, \ldots, m$, $\mu = m+1, \ldots, n$ and $i, j, k, \alpha, \beta = 1, \ldots, n$.

Remark 5.1.3. *Note the explicit appearance of the control parameter u. This represents the actuated dynamics.*

The vector field $Z_\mathcal{Y}$ along the velocity curve $\gamma'(t)$ on TM that satisfies

$$\overset{\mathcal{Y}}{\nabla}_{\gamma'(t)}\gamma'(t) = 0$$

is the geodesic spray for $\overset{\mathcal{Y}}{\nabla}$. The vector field $Z_\mathcal{Y} \in \Gamma(TTM)$ has the property that the integral curves of $Z_\mathcal{Y}$, when projected to M by π_{TM}, are geodesics for $\overset{\mathcal{Y}}{\nabla}$. The local expression for the geodesic spray $Z_\mathcal{Y}$ along the velocity curve $\gamma'(t)$ is

$$Z_\mathcal{Y} = v^i(t)X_i - v^i v^j \widehat{\Gamma}^\nu_{ij}\frac{\partial}{\partial w^\nu}.$$

5.2 Unactuated Connection

Here we introduce a second linear connection that will be associated with the unactuated dynamics. Let us begin by defining the projection mapping $P_{\mathcal{Y}^\perp} : TM \to \mathcal{Y}^\perp$ where

$$P_{\mathcal{Y}^\perp}(v_q) = \mathbb{G}(X_b, v_q)X_b, \quad b = m+1, \ldots, n$$

given the $\mathbb{G}$-orthonormal frame $\{X_1, \ldots, X_n\}$ constructed from the input distribution $\mathcal{Y}$ and the Riemannian metric $\mathbb{G}$ found in the basic problem formulation of an underactuated simple mechanical control system.

Definition 5.2.1 (Unactuated Connection). *Let $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, V, U\}$ be an underactuated simple mechanical control system with the $\mathbb{G}$-orthonormal frame*

$$\{X_1, \ldots, X_n\}.$$

*The **unactuated connection** is the linear connection $\overset{\mathcal{Y}^\perp}{\nabla}$ on M given by*

$$\overset{\mathcal{Y}^\perp}{\nabla}_X Y \quad = \quad P_{\mathcal{Y}^\perp}(\nabla_X Y)$$

for all $X, Y \in \Gamma(TM)$.

One important differential geometric property of the unactuated connection $\overset{\mathcal{Y}^\perp}{\nabla}$ is that

$$\overset{\mathcal{Y}^\perp}{\nabla}_X Y \in \Gamma(\mathcal{Y}^\perp)$$

which is equivalent to the statement that $\overset{\mathcal{Y}^\perp}{\nabla}$ restricts to $\mathcal{Y}^\perp$. With the notion of an actuated connection, we state the following result concerning the actuated dynamics.

Proposition 5.2.2 (Unactuated Dynamics)**.** *Let $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, V, U\}$ be an underactuated simple mechanical control system with the $\mathbb{G}$-orthonormal frame*

$$\{X_1, \dots, X_n\}.$$

The following holds along the curve $\gamma(t)$ satisfying

$$\nabla_{\gamma'(t)}\gamma'(t) = \left(\dot{v}^k(t) + \widehat{\Gamma}^k_{ij}(\gamma(t))v^i(t)v^j(t)\right) X_k(\gamma(t)) :$$

$$\overset{\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)}\gamma'(t) \quad = \quad \left(\dot{s}^{\mu-m} - v^i v^j \widehat{\Gamma}^\mu_{ij}\right) X_\mu$$

and

$$\dot{s}^{\mu-m} \quad = \quad -v^i v^j \widehat{\Gamma}^\mu_{ij} - \mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j} X^\beta_\mu$$

where $\nu, a = 1, \dots, m$, $\mu, r = m+1, \dots, n$ and $i, j, k, \alpha, \beta = 1, \dots, n$.

Remark 5.2.3. *Note the absence of the control parameter u. This represents the unactuated dynamics.*

The vector field $Z_{\mathcal{Y}^\perp}$ along the velocity curve $\gamma'(t)$ on TM that satisfies $\overset{\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)}\gamma'(t) = 0$ is the geodesic spray for $\overset{\mathcal{Y}^\perp}{\nabla}$. The vector field $Z_{\mathcal{Y}^\perp} \in \Gamma(TTM)$ has the property that the integral curves of $Z_{\mathcal{Y}^\perp}$, when projected to M by π_{TM}, are geodesics for $\overset{\mathcal{Y}^\perp}{\nabla}$. The local expression for the geodesic spray $Z_{\mathcal{Y}^\perp}$ along the velocity curve $\gamma'(t)$ is

$$Z_{\mathcal{Y}^\perp} = v^i(t)X_i - v^i v^j \widehat{\Gamma}^\mu_{ij} \frac{\partial}{\partial s^{\mu-m}}.$$

5.3 Representation of Underactuated Simple Mechanical Systems

Let us consider an underactuated simple mechanical control system $\Sigma_{L_{\mathbb{G}}}$ whose Lagrangian is $L_{\mathbb{G}} = \frac{1}{2}\mathbb{G}(v_q, v_q)$. Given the projection mapping $P_{\mathcal{Y}}$, the actuated connection $\overset{\mathcal{Y}}{\nabla}$ and the unactuated connection $\overset{\mathcal{Y}^\perp}{\nabla}$, an alternative coordinate invariant representation of the equations of motion is

$$\overset{\mathcal{Y}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{\mathcal{Y}}\left(u^a(t)Y_a(\gamma(t))\right)) \tag{5.4}$$

$$\overset{\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)}\gamma'(t) = 0. \tag{5.5}$$

This expression partitions the actuated and unactuated dynamics. Let $VP_{\mathcal{Y}^\perp} : TTM \to V\mathcal{Y}^\perp$ be the projection mapping naturally induced by $P_{\mathcal{Y}^\perp}$ on TM. We can construct a control-affine system with the state manifold TM that corresponds to the system of second-order equations on M given by Equation (5.4) and Equation (5.5). Using the notion of vertical lift, the coordinate invariant representation of the first-order equations on TM can be written as

$$\Psi'(t) = Z_{\mathcal{Y}}(\gamma'(t)) + VP_{\mathcal{Y}^\perp}(Z_{\mathcal{Y}^\perp}(\gamma'(t))) + u^a(t)(\mathbb{G}^\sharp(F^a(\gamma(t)))^{\mathrm{vlft}}$$

where $Z_{\mathcal{Y}}$ is the geodesic spray associated with the actuated connection $\overset{\mathcal{Y}}{\nabla}$ and $Z_{\mathcal{Y}^\perp}$ is the geodesic spray associated with the unactuated connection $\overset{\mathcal{Y}^\perp}{\nabla}$. The local representation for this system of first-order equations is

$$
\begin{aligned}
\dot{q}^i &= w^a X_a^i + s^{r-m} X_r^i \\
\dot{w}^\nu &= -v^i v^j \widehat{\Gamma}_{ij}^\nu + u^a \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^a X_\nu^\beta \\
\dot{s}^{\mu-m} &= -v^i v^j \widehat{\Gamma}_{ij}^\mu
\end{aligned}
$$

where $\nu, a = 1, \ldots, m$, $\mu, r = m+1, \ldots, n$ and $i, j, k, \alpha, \beta = 1, \ldots, n$.

5.4 Partial Feedback Linearization

In general, feedback linearization or feedback transformation consists of a change in coordinates and a state-dependent affine change in controls. The feedback linearization transforms a control-affine system into another control-affine system. The basic idea being that feedback linearization can transform the nonlinear system into a linear system by a change in coordinates and control vector fields. The systems for which this technique can be applied are relatively uncommon. An alternative approach is **partial feedback linearization** where a control law is introduced that linearizes part of the full nonlinear system. Here we introduce a control law that achieves partial linearization of the equations of motion for an underactuated mechanical control system $\Sigma_{L_\mathbb{G}}$ whose Lagrangian is $L_\mathbb{G} = \frac{1}{2}\mathbb{G}(v_q, v_q)$. We begin with the coordinate invariant equations of motion

$$
\begin{aligned}
\overset{\mathcal{Y}}{\nabla}_{\gamma'(t)} \gamma'(t) &= P_\mathcal{Y}\left(u^a(t) Y_a(\gamma(t))\right) \\
\overset{\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)} \gamma'(t) &= 0.
\end{aligned}
$$

An alternative local representation takes the form

$$\dot{\mathbf{q}} = \begin{pmatrix} w^a X_a^1 + s^{r-m} X_r^1 \\ \vdots \\ w^a X_a^n + s^{r-m} X_r^n \end{pmatrix} \tag{5.6}$$

$$\dot{\mathbf{w}} = \begin{pmatrix} -v^i v^j \widehat{\Gamma}_{ij}^1 \\ \vdots \\ -v^i v^j \widehat{\Gamma}_{ij}^m \end{pmatrix} + \begin{pmatrix} u^a \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^a X_1^\beta \\ \vdots \\ u^a \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^a X_m^\beta \end{pmatrix} \tag{5.7}$$

$$\dot{\mathbf{s}} = \begin{pmatrix} -v^i v^j \widehat{\Gamma}_{ij}^{m+1} \\ \vdots \\ -v^i v^j \widehat{\Gamma}_{ij}^n \end{pmatrix} \tag{5.8}$$

where $\nu, a = 1, \ldots, m$, $\mu, r = m+1, \ldots, n$ and $i, j, k, \alpha, \beta = 1, \ldots, n$. We wish to construct a control law $\mathbf{u}$ that linearizes Equation (5.7). Let us begin by modifying the second term on the right-hand-side of Equation (5.7). This term can be expanded to

$$\begin{pmatrix} u^1 \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^1 X_1^\beta + \cdots + u^m \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^m X_1^\beta \\ \vdots \\ u^1 \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^1 X_m^\beta + \cdots + u^m \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^m X_m^\beta \end{pmatrix}. \tag{5.9}$$

We can express the right-hand-side of Equation (5.9) as the product of an $m \times m$ matrix $\mathbf{g}$ defined by

$$\mathbf{g} = \begin{pmatrix} \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^1 X_1^\beta & \cdots & \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^m X_1^\beta \\ \vdots & \ddots & \vdots \\ \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^1 X_m^\beta & \cdots & \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F_j^m X_m^\beta \end{pmatrix} \tag{5.10}$$

and the $m \times 1$ input vector

$$\mathbf{u} = \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix}. \tag{5.11}$$

Let us set the first term in Equation (5.7) equal to

$$\mathbf{f} = \begin{pmatrix} -v^i v^j \widehat{\Gamma}^1_{ij} \\ \vdots \\ -v^i v^j \widehat{\Gamma}^m_{ij} \end{pmatrix}. \tag{5.12}$$

We can express Equation (5.7) in terms of $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{u}$ to get

$$\dot{\mathbf{w}} = \mathbf{f} + \mathbf{g}\mathbf{u}. \tag{5.13}$$

Now we introduce the control law that will linearize Equation (5.13). The control law is

$$\mathbf{u} = \mathbf{g}^{-1} (\tilde{\mathbf{u}} - \mathbf{f}) \tag{5.14}$$

where

$$\tilde{\mathbf{u}} = \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^m \end{pmatrix}. \tag{5.15}$$

Since we assume that the input distribution $\mathcal{Y}$ generated by $X_1, \ldots, X_m$ has constant rank m then $\mathbf{g}^{-1}$ exists. Now substitute Equation (5.14) into Equation (5.13) to get

$$\dot{\mathbf{w}} \;=\; \tilde{\mathbf{u}}. \tag{5.16}$$

which is linear. The local representation of our new control-affine system is

$$\dot{\mathbf{q}} \;=\; \begin{pmatrix} w^a X_a^1 + s^{r-m} X_r^1 \\ \vdots \\ w^a X_a^n + s^{r-m} X_r^n \end{pmatrix} \tag{5.17}$$

$$\dot{\mathbf{w}} \;=\; \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^m \end{pmatrix} \tag{5.18}$$

$$\dot{\mathbf{s}} \;=\; \begin{pmatrix} -v^i v^j \widehat{\Gamma}_{ij}^{m+1} \\ \vdots \\ -v^i v^j \widehat{\Gamma}_{ij}^{n} \end{pmatrix} \tag{5.19}$$

where $a = 1, \ldots, m$, $r = m+1, \ldots, n$ and $i, j = 1, \ldots, n$.

5.5 Geometric Normal Form

An alternative representation of the system of first-order Equations (5.17), (5.18) and (5.19) on TM is

$$\Psi' \;=\; \left(w^a X_a^i + s^{r-m} X_r^i\right) \frac{\partial}{\partial x^i} - \left(v^i v^j \widehat{\Gamma}_{ij}^{\mu}\right) \frac{\partial}{\partial s^{\mu-m}} + \tilde{u}^{\nu} \frac{\partial}{\partial w^{\nu}}. \tag{5.20}$$

Using the notion of vertical lift, the coordinate invariant representation of the first-order Equation (5.20) on TM can be written as

$$\Psi'(t) = Z_{\mathcal{Y}^\perp}(\gamma'(t)) + \tilde{u}^\nu(t)(X_\nu(\gamma(t)))^{\mathrm{vlft}}$$

where $Z_{\mathcal{Y}^\perp}$ is the geodesic spray associated with the unactuated connection $\overset{\mathcal{Y}^\perp}{\nabla}$. We assign the control-affine system $\{M, \mathcal{C} = \{f_0, f_1, \ldots, f_m\}, U\}$ such that

1. $M = TM$ (abuse of notation)
2. $f_0 = Z_{\mathcal{Y}^\perp}$
3. $f_\nu = X_\nu^{\mathrm{vlft}},\ \nu = 1, \ldots, m$ and
4. $U = U$ (abuse of notation).

We call this control-affine system the **geometric normal form** for underactuated mechanical systems.

5.6 Intrinsic Symmetric Bilinear Form

Definition 5.6.1. *We define the* ***generalized symmetric Christoffel symbols*** *for ∇ with respect to the basis of $\mathbb{G}$-orthonormal vector fields $\{X_1, \ldots, X_n\}$ on M as the n^3 functions $\tilde{\Gamma}_{ij}^k : M \to \mathbb{R}$ defined by*

$$\begin{aligned} \widetilde{\Gamma}_{ij}^k X_k &= \frac{1}{2}\left(\widehat{\Gamma}_{ij}^k + \widehat{\Gamma}_{ji}^k\right) X_k \\ &= \frac{1}{2}\mathbb{G}\left(\langle X_i : X_j\rangle, X_k\right) X_k. \end{aligned}$$

Proposition 5.6.2. *Let $\Sigma = \{M, \mathbb{G}, \mathcal{Y}, V, U\}$ be an underactuated simple mechanical control system with the $\mathbb{G}$-orthonormal frame $\{X_1, \ldots, X_n\}$. The following holds along the curve $\gamma(t)$ satisfying*

$$\nabla_{\gamma'(t)}\gamma'(t) = \left(\dot{v}^k(t) + \widehat{\Gamma}^k_{ij}(\gamma(t))v^i(t)v^j(t)\right) X_k(\gamma(t)) :$$

$$\begin{aligned}
\dot{w}^\nu &= -v^i v^j \mathbb{G}(\langle X_i : X_j\rangle, X_\nu) \\
&\quad -\mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j}X^\beta_\nu \\
&\quad +u^a\left(\mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F^a_j X^\beta_\nu\right) \\
\dot{s}^\mu &= -v^i v^j \mathbb{G}(\langle X_i : X_j\rangle, X_\mu) \\
&\quad -\mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j}X^\beta_\mu
\end{aligned} \tag{5.21}$$

where $\nu, a = 1, \ldots, m$, $\mu = m+1, \ldots, n$ and $i, j, k, \alpha, \beta = 1, \ldots, n$.

Proof. We substitute Definition 5.6.1 into Equation (5.3). ☐

We observe that Equation (5.21) is quadratic in the parameter $w(t)$. Now we relate an intrinsic vector-valued symmetric bilinear form to the measure derived in Proposition 5.6.2.

Definition 5.6.3. *Let $\Sigma_{L_\mathbb{G}} = (M, \mathbb{G}, \mathcal{Y}, U)$ be an underactuated mechanical control system whose Lagrangian is $L_\mathbb{G}$. Let $\mathcal{Y}$ be the input distribution generated by the $\mathbb{G}$-orthonormal frame $\{X_1, \ldots, X_m\}$ and $\mathcal{Y}^\perp$ be the $\mathbb{G}$-orthogonal distribution generated by $\{X_{m+1}, \ldots, X_n\}$. We define the* ***intrinsic vector-valued symmetric bilinear form*** *to be $B_q : \mathcal{Y}_q \times \mathcal{Y}_q \to \mathcal{Y}^\perp_q$ given in coordinates by*

$$B^{b-m}_{ap} w^a w^p \;=\; \frac{1}{2}\mathbb{G}(\langle X_a : X_p\rangle, X_b) w^a w^p,$$

where $a, p = 1, \ldots, m, b = m + 1, \ldots, n$.

Remark 5.6.4. *If* $\Sigma_{L_{\mathbb{G}}}$ *is underactuated by one control then* $N - m = 1$ *and* B *is a* $\mathbb{R}$*-valued symmetric bilinear form.*

The intrinsic vector-valued symmetric bilinear form defined above is an important measure of how the actuated velocity components w influence the unactuated velocity components s.

5.7 Constrained Partitioning Connections

Once again, the most interesting geometries for underactuated mechanical systems arise when linear velocity constraints exist. Recall that a linear velocity constraint is a distribution $\mathcal{H}$ on the configuration manifold M. A smooth curve $\gamma : I \to M$ is consistent with the linear velocity constraint $\mathcal{H}$ on M if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for all $t \in I$. Let

$$\{\overset{\mathcal{H}}{Y}_1, \ldots, \overset{\mathcal{H}}{Y}_m\}$$

be the set of vector fields that generates the distribution $P_{\mathcal{H}}(\mathcal{Y}) \subset \mathcal{H}$. Recall that the constrained $\mathbb{G}$-orthonormal frame for $\mathcal{H}$ is the set of vector fields

$$\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_K\}$$

where the first m elements generate $P_{\mathcal{H}}(\mathcal{Y})$ and last $K - m$ elements generate the $\mathbb{G}$-orthogonal complement $\mathcal{H}/P_{\mathcal{H}}(\mathcal{Y})$. Now we make the notational assignment of $\hat{v}^k(t) = \hat{w}^k(t)$ for $k = 1, \ldots, m$ and $\hat{v}^k(t) = \hat{s}^{k-m}(t)$ for $k = m + 1, \ldots, K$. This gives us the local coordinates $((x^1, \ldots, x^n), (\hat{w}^1, \ldots, \hat{w}^m, \hat{s}^1, \ldots, \hat{s}^{K-m}))$ on $\mathcal{H}$ where $\hat{w}^\nu = \mathbb{G}(v_q, \overset{\mathcal{H}}{X}_\nu)$ for $\nu = 1, \ldots, m$ and $\hat{s}^{\mu-m} = \mathbb{G}(v_q, \overset{\mathcal{H}}{X}_\mu)$ for $\mu = m +$

$1, \ldots, K$. This naturally induces the coordinate frame

$$\left\{ \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}, \left\{ \frac{\partial}{\partial \hat{w}^1}, \ldots, \frac{\partial}{\partial \hat{w}^m}, \frac{\partial}{\partial \hat{s}^1}, \ldots, \frac{\partial}{\partial \hat{s}^{n-m}} \right\} \right\}$$

for $T_{v_q}\mathcal{H}$. Futhermore, the vertical lift of the $\mathbb{G}$-orthonormal frame with elements $\overset{\mathcal{H}}{X}_a$ is the tangent vector to the curve in the fiber defined by

$$\overset{\mathcal{H}}{X}{}_a^{\mathrm{vlft}} = \frac{d}{dt}\left(v_q + t\overset{\mathcal{H}}{X}_a\right)$$

where $\overset{\mathcal{H}}{X}{}_\nu^{\mathrm{vlft}} = \frac{\partial}{\partial \hat{w}^\nu}$ and $\overset{\mathcal{H}}{X}{}_\mu^{\mathrm{vlft}} = \frac{\partial}{\partial \hat{s}^{\mu-m}}$.

Here we introduce the notion of a constrained actuated connection.

Definition 5.7.1 (Constrained Actuated Connection)**.** *Let*

$$\Sigma = \{M, \mathbb{G}, V, \mathcal{H}, \mathcal{Y}, U\}$$

be an underactuated simple mechanical control system with linear velocity constraints. The ***constrained actuated connection*** *is the linear connection* $\overset{\mathcal{H},\mathcal{Y}}{\nabla}$ *on* M *given by*

$$\overset{\mathcal{H},\mathcal{Y}}{\nabla}_X Y \quad = \quad P_{P_{\mathcal{H}}(\mathcal{Y})}(\nabla_X Y)$$

for all $X, Y \in \Gamma(TM)$.

An important differential geometric property of the constrained actuated connection $\overset{\mathcal{H},\mathcal{Y}}{\nabla}$ is that

$$\overset{\mathcal{H},\mathcal{Y}}{\nabla}_X Y \in \Gamma(P_{\mathcal{H}}(\mathcal{Y}))$$

which is equivalent to the statement that $\overset{\mathcal{H},\mathcal{Y}}{\nabla}$ restricts to $P_{\mathcal{H}}(\mathcal{Y})$. With the notion of an actuated connection, we state the following result concerning the constrained actuated dynamics.

Proposition 5.7.2 (Constrained Actuated Dynamics)**.** *Let* $\Sigma = \{M, \mathbb{G}, V, \mathcal{H}, \mathcal{Y}, U\}$ *be an underactuated simple mechanical control system with linear velocity constraints and let* $\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_K\}$ *be the* $\mathbb{G}$*-orthonormal frame that generates* $\mathcal{H}$ *where* $\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_m\}$ *generates* $P_{\mathcal{H}}(\mathcal{Y})$*. The following holds along the curve* $\gamma(t)$ *satisfying*

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = \left(\dot{v}^k(t) + \overset{\mathcal{H}}{\Gamma}{}^k_{ij}(\gamma(t))v^i(t)v^j(t)\right)\overset{\mathcal{H}}{X}_k(\gamma(t)) :$$

$$\overset{\mathcal{H},\mathcal{Y}}{\nabla}_{\gamma'(t)}\gamma'(t) \quad = \quad \left(\dot{\hat{w}}^{\nu} + v^i v^j \overset{\mathcal{H}}{\Gamma}{}^{\nu}_{ij}\right)\overset{\mathcal{H}}{X}_{\nu}$$

and

$$\dot{\hat{w}}^{\nu} \quad = \quad -v^i v^j \overset{\mathcal{H}}{\Gamma}{}^{\nu}_{ij} - \mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j}\overset{\mathcal{H}}{X}{}^{\beta}_{\nu} + u^a \mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F^a_j \overset{\mathcal{H}}{X}{}^{\beta}_{\nu}$$

where $\nu, a = 1, \ldots, m,\ \mu = m+1, \ldots, K$ *and* $i, j, k, \alpha, \beta = 1, \ldots, n.$

Remark 5.7.3. *Note the explicit appearance of the control parameter* u*. This represents the actuated dynamics.*

The vector field $Z_{\mathcal{H},\mathcal{Y}}$ along the velocity curve $\gamma'(t)$ on TM that satisfies $\overset{\mathcal{H},\mathcal{Y}}{\nabla}_{\gamma'(t)}\gamma'(t) = 0$ is the geodesic spray for $\overset{\mathcal{H},\mathcal{Y}}{\nabla}$. The vector field $Z_{\mathcal{Y}} \in \Gamma(TTM)$ has the property that the integral curves of $Z_{\mathcal{Y}}$, when projected to M by π_{TM}, are geodesics for $\overset{\mathcal{H},\mathcal{Y}}{\nabla}$. The local expression for the geodesic spray $Z_{\mathcal{H},\mathcal{Y}}$ along the velocity curve $\gamma'(t)$ is

$$Z_{\mathcal{H},\mathcal{Y}} = v^i \overset{\mathcal{H}}{X}_i - -v^i v^j \overset{\mathcal{H}}{\Gamma}{}^{\nu}_{ij}\frac{\partial}{\partial \hat{w}^{\nu}}.$$

Now we introduce the notion of a constrained unactuated connection.

Definition 5.7.4 (Constrained Unactuated Connection)**.** *Let* $\Sigma = \{M, \mathbb{G}, V, \mathcal{H}, \mathcal{Y}, U\}$ *be an underactuated simple mechanical control system with linear velocity constraints. The* ***constrained unactuated connection*** *is the linear connection* $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$ *on* M *given by*

$$\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}_X Y \quad = \quad P_{P_{\mathcal{H}}(\mathcal{Y}^\perp)}(\nabla_X Y)$$

for all $X, Y \in \Gamma(TM)$*.*

One important differential geometric property of the unactuated connection $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$ is that

$$\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}_X Y \in \Gamma(P_{\mathcal{H}}(\mathcal{Y}^\perp))$$

which is equivalent to the statement that $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$ restricts to $P_{\mathcal{H}}(\mathcal{Y}^\perp)$. With the notion of an actuated connection, we state the following result concerning the constrained actuated dynamics.

Proposition 5.7.5 (Constrained Unactuated Dynamics)**.** *Let* $\Sigma = \{M, \mathbb{G}, V, \mathcal{H}, \mathcal{Y}, U\}$ *be an underactuated simple mechanical control system with linear velocity constraints and let* $\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_K\}$ *be the* $\mathbb{G}$*-orthonormal frame that generates* $\mathcal{H}$ *where* $\{\overset{\mathcal{H}}{X}_{m+1}, \ldots, \overset{\mathcal{H}}{X}_K\}$ *generates* $P_{\mathcal{H}}(\mathcal{Y}^\perp)$*. The following holds along the curve* $\gamma(t)$ *satisfying*

$$\overset{\mathcal{H}}{\nabla}_{\gamma'(t)}\gamma'(t) = \left(\dot{v}^k(t) + \overset{\mathcal{H}}{\Gamma}{}^k_{ij}(\gamma(t))v^i(t)v^j(t)\right)\overset{\mathcal{H}}{X}_k(\gamma(t)) :$$

$$\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)}\gamma'(t) \quad = \quad \left(\dot{\hat{s}}^{\mu-m} - v^i v^j \widehat{\Gamma}^\mu_{ij}\right)\overset{\mathcal{H}}{X}_\mu$$

and

$$\dot{\hat{s}}^{\mu-m} = -v^i v^j \overset{\mathcal{H}}{\Gamma}{}^{\mu}_{ij} - \mathbb{G}_{\alpha\beta}\frac{\partial V}{\partial x^j}\mathbb{G}^{\alpha j}\overset{\mathcal{H}}{X}{}^{\beta}_{\mu}$$

where $\nu, a = 1, \ldots, m,\ \mu, r = m+1, \ldots, K$ *and* $i, j, k, \alpha, \beta = 1, \ldots, n.$

Remark 5.7.6. *Note the absence of the control parameter* u*. This represents the constrained unactuated dynamics.*

The vector field $Z_{\mathcal{H},\mathcal{Y}^\perp}$ along the velocity curve $\gamma'(t)$ on TM that satisfies $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)}\gamma'(t) = 0$ is the geodesic spray for $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$. The vector field $Z_{\mathcal{H},\mathcal{Y}^\perp} \in \Gamma(TTM)$ has the property that the integral curves of $Z_{\mathcal{H},\mathcal{Y}^\perp}$, when projected to M by π_{TM}, are geodesics for $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$. The local expression for the geodesic spray $Z_{\mathcal{H},\mathcal{Y}^\perp}$ along the velocity curve $\gamma'(t)$ is

$$Z_{\mathcal{H},\mathcal{Y}^\perp} = v^i \overset{\mathcal{H}}{X}_i - -v^i v^j \overset{\mathcal{H}}{\Gamma}{}^{\mu}_{ij}\frac{\partial}{\partial \hat{s}^{\mu-m}}.$$

Let us consider an underactuated simple mechanical control system $\Sigma_{\mathcal{H}}$ with linear velocity constraints whose Lagrangian is $L_{\mathbb{G}} = \frac{1}{2}\mathbb{G}(v_q, v_q)$. Given the constrained actuated connection $\overset{\mathcal{H},\mathcal{Y}}{\nabla}$ and the constrained unactuated connection $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$, an alternative coordinate invariant representation of the constrained equations of motion is

$$\overset{\mathcal{H},\mathcal{Y}}{\nabla}_{\gamma'(t)}\gamma'(t) = P_{P_{\mathcal{H}}(\mathcal{Y})}\left(u^a(t) Y_a(\gamma(t))\right)) \qquad (5.22)$$

$$\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}_{\gamma'(t)}\gamma'(t) = 0. \qquad (5.23)$$

This expression partitions the constrained actuated and constrained unactuated dynamics. The local representation for this system in first-order equation form on

$\mathcal{H}$ is

$$\begin{aligned}
\dot{q}^i &= \hat{w}^a \overset{\mathcal{H}}{X}{}^i_a + \hat{s}^{r-m} \overset{\mathcal{H}}{X}{}^i_r \\
\dot{\hat{w}}^\nu &= -v^k v^l \overset{\mathcal{H}}{\Gamma}{}^\nu_{kl} + u^a \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F^a_j \overset{\mathcal{H}}{X}{}^\beta_\nu \\
\dot{\hat{s}}^{\mu-m} &= -v^k v^l \overset{\mathcal{H}}{\Gamma}{}^\mu_{kl}
\end{aligned}$$

where $\nu, a = 1, \ldots, m$, $\mu, r = m+1, \ldots, K$, $k, l = 1, \ldots, K$ and $i, j, \alpha, \beta = 1, \ldots, n$. An alternative local representation takes the form

$$\dot{\mathbf{q}} = \begin{pmatrix} \hat{w}^a \overset{\mathcal{H}}{X}{}^1_a + \hat{s}^{r-m} \overset{\mathcal{H}}{X}{}^1_r \\ \vdots \\ \hat{w}^a \overset{\mathcal{H}}{X}{}^K_a + \hat{s}^{r-m} \overset{\mathcal{H}}{X}{}^K_r \end{pmatrix} \tag{5.24}$$

$$\dot{\hat{w}} = \begin{pmatrix} -\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}^1_{ij} \\ \vdots \\ -\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}^m_{ij} \end{pmatrix} + \begin{pmatrix} u^a \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F^a_j \overset{\mathcal{H}}{X}{}^\beta_1 \\ \vdots \\ u^a \mathbb{G}_{\alpha\beta} \mathbb{G}^{\alpha j} F^a_j \overset{\mathcal{H}}{X}{}^\beta_m \end{pmatrix} \tag{5.25}$$

$$\dot{\hat{s}} = \begin{pmatrix} -\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}^{m+1}_{ij} \\ \vdots \\ -\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}^K_{ij} \end{pmatrix} \tag{5.26}$$

where $\nu, a = 1, \ldots, m$, $\mu, r = m+1, \ldots, K$ and $i, j, k, \alpha, \beta = 1, \ldots, n$. Following a similar procedure detailed in Section 5.4, the control law that linearizes Equation (5.25) is

$$\mathbf{u} = \hat{\mathbf{g}}^{-1} \left(\tilde{\mathbf{u}} - \hat{\mathbf{f}} \right) \tag{5.27}$$

where

$$\hat{\mathbf{g}} = \begin{pmatrix} \mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^1\overset{\mathcal{H}}{X}{}_1^\beta & \cdots & \mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^m\overset{\mathcal{H}}{X}{}_1^\beta \\ \vdots & \ddots & \vdots \\ \mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^1\overset{\mathcal{H}}{X}{}_m^\beta & \cdots & \mathbb{G}_{\alpha\beta}\mathbb{G}^{\alpha j}F_j^m\overset{\mathcal{H}}{X}{}_m^\beta \end{pmatrix} \tag{5.28}$$

and

$$\hat{\mathbf{f}} = \begin{pmatrix} -\hat{v}^i\hat{v}^j\overset{\mathcal{H}}{\Gamma}{}_{ij}^1 \\ \vdots \\ -\hat{v}^i\hat{v}^j\overset{\mathcal{H}}{\Gamma}{}_{ij}^m \end{pmatrix}. \tag{5.29}$$

Since we assume that the projection of the input distribution $P_{\mathcal{H}}(\mathcal{Y})$ generated by $\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_m$ has constant rank m then $\mathbf{g}^{-1}$ exists. We can also express Equation (5.25) in terms of $\hat{\mathbf{f}}$, $\hat{\mathbf{g}}$ and $\mathbf{u}$ to get

$$\dot{\mathbf{w}} = \mathbf{f} + \mathbf{g}\mathbf{u}. \tag{5.30}$$

Now substitute Equation (5.27) into Equation (5.30) to get

$$\dot{\hat{w}} = \tilde{\mathbf{u}}. \tag{5.31}$$

which is linear. The local representation of our new control-affine system is

$$\dot{\mathbf{q}} = \begin{pmatrix} \hat{w}^a \overset{\mathcal{H}}{X}{}_a^1 + \hat{s}^{r-m} \overset{\mathcal{H}}{X}{}_r^1 \\ \vdots \\ \hat{w}^a \overset{\mathcal{H}}{X}{}_a^K + \hat{s}^{r-m} \overset{\mathcal{H}}{X}{}_r^K \end{pmatrix} \tag{5.32}$$

$$\dot{\hat{w}} = \begin{pmatrix} \tilde{u}^1 \\ \vdots \\ \tilde{u}^m \end{pmatrix} \tag{5.33}$$

$$\dot{\hat{s}} = \begin{pmatrix} -\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}_{ij}^{m+1} \\ \vdots \\ -\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}_{ij}^{K} \end{pmatrix} \tag{5.34}$$

where $a = 1, \ldots, m$, $r = m+1, \ldots, K$ and $i, j = 1, \ldots, n$. An alternative representation of the system of first-order Equations (5.32), (5.33) and (5.34) on $\mathcal{H}$ is

$$\overset{\mathcal{H}}{\Psi}{}' = \left(\hat{w}^a \overset{\mathcal{H}}{X}{}_a^i + \hat{s}^{r-m} \overset{\mathcal{H}}{X}{}_r^i\right) \frac{\partial}{\partial x^i} - \left(\hat{v}^i \hat{v}^j \overset{\mathcal{H}}{\Gamma}{}_{ij}^{\mu}\right) \frac{\partial}{\partial \hat{s}^{\mu-m}} + \tilde{u}^{\nu} \frac{\partial}{\partial \hat{w}^{\nu}}. \tag{5.35}$$

Using the notion of vertical lift, the coordinate invariant representation of the first-order Equation (5.35) on $\mathcal{H}$ can be written as

$$\overset{\mathcal{H}}{\Psi}{}' = Z_{\mathcal{H},\mathcal{Y}^{\perp}}(\gamma'(t)) + \tilde{u}^{\nu}(t)(\overset{\mathcal{H}}{X}_{\nu}(\gamma(t)))^{\mathrm{vlft}}$$

where $Z_{\mathcal{H},\mathcal{Y}^{\perp}}$ is the geodesic spray associated with the unactuated connection $\overset{\mathcal{H},\mathcal{Y}^{\perp}}{\nabla}$. We assign the control-affine system $\{M, \mathcal{C} = \{f_0, f_1, \ldots, f_m\}, U\}$ such that

1. $M = \mathcal{H}$ (abuse of notation)

2. $f_0 = Z_{\mathcal{H},\mathcal{Y}^\perp}$

3. $f_\nu = \overset{\mathcal{H}}{X}{}_\nu^{\text{vlft}},\ \nu = 1, \ldots, m$ and

4. $U = U$ (abuse of notation).

We call this control-affine system the **constrained geometric normal form** for underactuated mechanical systems with linear velocity constraints.

Finally, we introduce an intrinsic symmetric bilinear form that can be associated with underactuated mechanical systems with linear velocity constraints.

Definition 5.7.7. *Let $\Sigma_{\mathcal{H}} = (M, \mathbb{G}, \mathcal{H}, \mathcal{Y}, U)$ be an underactuated mechanical system with linear velocity constraints whose Lagrangian is $L_{\mathbb{G}}$. Let $P_{\mathcal{H}}(\mathcal{Y}) \subset \mathcal{H}$ be the distribution generated by the $\mathbb{G}$-orthonormal frame $\{\overset{\mathcal{H}}{X}_1, \ldots, \overset{\mathcal{H}}{X}_m\}$ and $\mathcal{H}/P_{\mathcal{H}}(\mathcal{Y})$ be the distribution generated by the $\mathbb{G}$-orthonormal frame $\{\overset{\mathcal{H}}{X}_{m+1}, \ldots, \overset{\mathcal{H}}{X}_K\}$. We define the* ***constrained intrinsic vector-valued symmetric bilinear form*** *to be $\overset{\mathcal{H}}{B}_q : P_{\mathcal{H}}(\mathcal{Y}_q) \times P_{\mathcal{H}}(\mathcal{Y}_q) \to \mathcal{H}_q/P_{\mathcal{H}}(\mathcal{Y}_q)$ given in coordinates by*

$$\overset{\mathcal{H}}{B}{}^{b-m}_{ap}\hat{w}^a\hat{w}^p \quad = \quad \frac{1}{2}(\overset{\mathcal{H}}{\Gamma}{}^b_{ap} + \overset{\mathcal{H}}{\Gamma}{}^b_{pa})\hat{w}^a\hat{w}^p,$$

where $a, p \in \{1, \ldots, m\}, b \in \{m+1, \ldots, K\}$.

Remark 5.7.8. *If $\Sigma_{\mathcal{H}}$ is underactuated by one control then $K - m = 1$ and $\overset{\mathcal{H}}{B}$ is a $\mathbb{R}$-valued symmetric bilinear form.*

5.8 Examples

In this section we construct the partial feedback linearization law, geometric normal form and symmetric bilinear form for our motivating examples. The classic geometric model for each of these systems can be found in Section 3.3.

5.8.1 Planar Rigid Body

Let us consider the planar rigid body with control set $\{Y_1, Y_2\}$. The partial feedback linearization law is

$$u(t) = \begin{pmatrix} \dfrac{\frac{h^2 m w^2(t) s(t)\left(-\sqrt{\frac{h^2}{J}+\frac{1}{m}}\right)\sqrt{\frac{1}{h^2 m+J}}+J w^2(t) s(t)\sqrt{\frac{h^2}{J}+\frac{1}{m}}\sqrt{\frac{1}{h^2 m+J}}+h\left(s(t)^2-w^2(t)^2\right)}{\sqrt{\frac{1}{m}}(h^2 m+J)}+\tilde{u}^1(t)}{\sqrt{\frac{1}{m}}} \\ \dfrac{\frac{h w^1(t) w^2(t)-J w^1(t) s(t)\sqrt{\frac{h^2}{J}+\frac{1}{m}}\sqrt{\frac{1}{h^2 m+J}}}{\sqrt{\frac{1}{m}}(h^2 m+J)}+\tilde{u}^2(t)}{\sqrt{\frac{h^2}{J}+\frac{1}{m}}} \end{pmatrix}.$$

The resulting geometric normal form is

$$\begin{aligned} \dot{q}^i(t) &= w^1(t) X_1^i(q(t)) + w^2(t) X_2^i(q(t)) + s(t) X_3^i(q(t)) \\ \dot{w}^a(t) &= \tilde{u}^a(t) \\ \dot{s}(t) &= \frac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{\left(\frac{1}{m}\right)^{3/2}} w^1(t) w^2(t) \\ &\quad -\frac{h}{\sqrt{\frac{1}{m}}\left(h^2 m+J\right)} w^1(t) s(t). \end{aligned}$$

The entries of the symmetric bilinear form are

$$\begin{pmatrix} 0 & -\dfrac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{2\left(\frac{1}{m}\right)^{3/2}} \\ -\dfrac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{2\left(\frac{1}{m}\right)^{3/2}} & 0 \end{pmatrix}.$$

Let us consider the planar rigid body with control set $\{Y_1, Y_3\}$. The partial feedback linearization law is

$$u(t) = \begin{pmatrix} \dfrac{\sqrt{\frac{1}{J}} w^2(t) s(t)+\tilde{u}^1(t)}{\sqrt{\frac{1}{m}}} \\ \dfrac{\tilde{u}^2(t)}{\sqrt{\frac{1}{J}}} \end{pmatrix}.$$

The resulting geometric normal form is

$$\begin{aligned}
\dot{q}^i(t) &= w^1(t)X_1^i(q(t)) + w^2(t)X_2^i(q(t)) + s(t)X_3^i(q(t)) \\
\dot{w}^a(t) &= \tilde{u}^a(t) \\
\dot{s}(t) &= -\sqrt{\frac{1}{J}}w^1(t)w^2(t)
\end{aligned}$$

The entries of the symmetric bilinear form are

$$\begin{pmatrix} 0 & -\frac{\sqrt{\frac{1}{J}}}{2} \\ -\frac{\sqrt{\frac{1}{J}}}{2} & 0 \end{pmatrix}.$$

5.8.2 Roller Racer

Let us consider the roller racer with control set $\{Y_1\}$. The partial feedback linearization law is

$$u(t) = \left(\frac{\frac{2m\hat{w}(t)\hat{s}(t)(L_1+L_2\cos(\psi))(I_1L_2-I_2L_1\cos(\psi))}{(L_1\cos(\psi)+L_2)\left(I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m\right)K(\psi)}+\tilde{u}^1(t)}{C(\psi)} \right).$$

The resulting constrained geometric normal form is

$$\begin{aligned}
\dot{q}^i(t) &= w(t)\overset{\mathcal{H}}{X}{}_1^i(q(t)) + s(t)\overset{\mathcal{H}}{X}{}_2^i(q(t)) \\
\dot{\hat{w}}(t) &= \tilde{u}(t) \\
\dot{\hat{s}}(t) &= \overset{\mathcal{H}}{B}(\psi)\hat{w}(t)\hat{w}(t)
\end{aligned}$$

where

$$\overset{\mathcal{H}}{B}(\psi) = \frac{2m(L_1 + L_2\cos(\psi))(I_1L_2 - I_2L_1\cos(\psi))}{(L_1\cos(\psi) + L_2)\left(I_2\cos(2\psi)\left(L_1^2m - I_1\right) + I_1\left(I_2 + 2L_2^2m\right) + I_2L_1^2m\right)K(\psi)}.$$

The single entry of the symmetric bilinear form is $\overset{\mathcal{H}}{B}(\psi)$.

5.8.3 Snakeboard

Let us consider the snakeboard with control set $\{Y_1, Y_2\}$. The partial feedback linearization law is

$$u(t) = \left(\begin{array}{c} \dfrac{\frac{\sqrt{2} J_r \sqrt{\frac{1}{J_w}} \hat{w}^2(t)\hat{s}(t)\cos(\phi)\sqrt{\frac{l^2 m}{J_r\left(J_r \cos(2\phi) - J_r + 2l^2 m\right)}}}{\sqrt{l^2 m}} + \tilde{u}^1(t)}{\sqrt{2}\sqrt{\frac{l^2 m}{J_r\left(J_r \cos(2\phi) - J_r + 2l^2 m\right)}}} \\ \dfrac{\tilde{u}^2(t)}{\sqrt{\frac{1}{J_w}}} \end{array} \right).$$

The resulting constrained geometric normal form is

$$\begin{aligned} \dot{q}^i(t) &= w^1(t)\overset{\mathcal{H}}{X}{}_1^i(q(t)) + w^2(t)\overset{\mathcal{H}}{X}{}_2^i(q(t)) + s(t)\overset{\mathcal{H}}{X}{}_3^i(q(t)) \\ \dot{\hat{w}}^a(t) &= \tilde{u}^a(t) \\ \dot{\hat{s}}(t) &= -2\frac{J_r\sqrt{\frac{1}{J_w}}\cos(\phi)\sqrt{\frac{l^2 m}{2J_r^2\cos(2\phi) - 2J_r^2 + 4J_r l^2 m}}}{\sqrt{l^2 m}}\hat{w}^1(t)\hat{w}^2(t). \end{aligned}$$

The entries of the symmetric bilinear form is

$$\left(\begin{array}{cc} 0 & -\dfrac{J_r\sqrt{\frac{1}{J_w}}\cos(\phi)\sqrt{\frac{l^2 m}{2J_r^2\cos(2\phi) - 2J_r^2 + 4J_r l^2 m}}}{\sqrt{l^2 m}} \\ -\dfrac{J_r\sqrt{\frac{1}{J_w}}\cos(\phi)\sqrt{\frac{l^2 m}{2J_r^2\cos(2\phi) - 2J_r^2 + 4J_r l^2 m}}}{\sqrt{l^2 m}} & 0 \end{array} \right).$$

5.8.4 Three Link Manipulator

Let us consider the three link manipulator with control set $\{Y_1, Y_2\}$. The partial feedback linearization law is

$$u(t) = G^{-1}\left(\tilde{u}(t) - F\right)$$

where

$$G^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{\frac{2I_c - L^2 m\cos(2\theta)+L^2 m}{I_c m}}} & 0 \\ \frac{L^2 m^2 \sin(2\theta)\sqrt{\frac{I_c+L^2 m}{4I_c m - 2L^2 m^2\cos(2\theta)+2L^2 m^2}}}{I_c + L^2 m} & \frac{1}{2\sqrt{\frac{I_c+L^2 m}{4I_c m - 2L^2 m^2\cos(2\theta)+2L^2 m^2}}} \end{pmatrix}$$

and

$$F = \begin{pmatrix} -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_1) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_1) - s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_1) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_1) \\ -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_2) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_2) - s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_2) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_2) \end{pmatrix}.$$

The resulting geometric normal form is

$$\begin{aligned} \dot{q}^i(t) &= w^1(t)X_1^i(q(t)) + w^2(t)X_2^i(q(t)) + s(t)X_3^i(q(t)) \\ \dot{w}^a(t) &= \tilde{u}^a(t) \\ \dot{s}(t) &= -w^a(t)w^p(t)\mathbb{G}(\nabla_{X_a}X_p, X_3) - w^a(t)s(t)\mathbb{G}(\nabla_{X_a}X_3, X_3) \\ &\quad -s(t)w^p(t)\mathbb{G}(\nabla_{X_3}X_p, X_3) - s(t)s(t)\mathbb{G}(\nabla_{X_3}X_3, X_3) \end{aligned}$$

The entries of the symmetric bilinear form are

$$\begin{pmatrix} B_{11}(\theta) & B_{12}(\theta) \\ B_{21}(\theta) & B_{22}(\theta) \end{pmatrix}$$

where

$$B_{11}(\theta) = -B_{22}(\theta) = \frac{L^2 m\sin(2\theta)\sqrt{\frac{1}{I_c+L^2 m}}}{2I_c - L^2 m\cos(2\theta) + L^2 m}$$

and

$$B_{12}(\theta) = B_{21}(\theta) = \frac{L^2\left(\frac{1}{I_c+L^2 m}\right)^{3/2}\sqrt{\frac{I_c+L^2 m}{mC_1}}\left(L^2 m(\cos(4\theta)+3)\left(2I_c+L^2 m\right) - 4\cos(2\theta)\left(2I_c^2+2I_c L^2 m+L^4 m^2\right)\right)}{4I_c^2\left(\frac{C_1}{I_c m}\right)^{3/2}}$$

and

$$C_1 = \left(2I_c - L^2 m \cos(2\theta) + L^2 m\right).$$

CHAPTER 6

CHARACTERIZATION OF REACHABLE VELOCITIES FOR MECHANICAL SYSTEMS UNDERACTUATED BY ONE

The characterization of the set of states reachable from an initial state is a fundamental problem in control theory. Problems of this nature are commonly referred to as controllability. An initial study into the local controllability and local accessibility properties of a class of underactuated mechanical systems referred to as affine connection control systems was published by Lewis and Murray [44], [43]. The conditions for local accessibility in this work are characterised geometrically by using the symmetric product provided by Lewis [40]. These results were extended to affine connection control systems with linear velocity constraints [41], [13]. The local controllability and local accessibility of a smaller class of underactuated mechanical systems with partial feedback linearization was published by Reyhanoglu et al. [57]. It is important to note that the sufficient conditions for local controllability provided by Lewis and Murray and Reyhanoglu et al., following Sussmann [64], have several known limitations. The first limitation for these results is that the sufficient conditions are not feedback-invariant. The lack of feedback invariance can be seen even in very simple examples, where a system can fail the sufficient condition test, but still be controllable. This limitation motivated several efforts to obtain conditions for low-order controllability results for a class of underactuated mechanical systems which are not dependent on a choice of basis for the input distribution [9], [67], [31], [32]. The conditions depend on the definiteness of an intrinsic vector-valued quadratic form. A thorough review of controllability and existing results for underactuated mechanical systems can be found in Section 1.3.

The second limitation associated with the results of Sussmann [64] is that they are limited to equilibrium states of control-affine systems (*i.e.* states where the drift vector field is zero). Consequently, the existing literature on local accessibility and local controllability for underactuated mechanical systems is limited to initial states with zero velocity. The matter of determining the general structure of states reachable from a nonzero velocity state is currently unresolved [10], [21], [14]. We provide a general test for mechanical systems underactuated by one control that depends on the definiteness of an intrinsic symmetric bilinear form that determines the systems ability to reach a specified velocity from a nonzero velocity state. In other words, we provide a sufficient condition dependent on the definiteness of a symmetric bilinear form for velocity to velocity motion planning. Our results carry with it several important features.

1. Our results do not depend on the choice of basis for the input distribution.
2. Our results are valid in the nonzero velocity setting.
3. Our results can be applied to mechanical systems underactuated by one control with linear velocity constraints.

6.1 Main Results

Let us take $\Sigma_{\mathcal{H}} = \{M, \mathbb{G}, \mathcal{F}, \mathcal{H}, \mathbb{R}^m\}$ to be a mechanical systems underactuated by one control with linear velocity constraints. Recall that linear velocity constraints are defined by a distribution $\mathcal{H}$ on M with rank K. The local coordinates for $\Sigma_{\mathcal{H}}$'s configuration and velocity, $v_q \in \mathcal{H}$, will be denoted by

$$((q^1, \ldots, q^n), (w^1, \ldots, w^m, s))$$

where the w and s parameters represent the decomposition of $\Sigma_{\mathcal{H}}$'s velocity along our constrained $\mathbb{G}$-orthonormal frame $\{X_1(q(t)), \ldots, X_K(q(t))\}$. The decomposition of the local velocity curve is given by

$$v(t) = w^a(t)X_a(q(t)) + s(t)X_K(q(t))$$

for $a = 1, \ldots, m$ where $w^a(t) = \mathbb{G}(v(t), X_a(q(t)))$ and $s(t) = \mathbb{G}(v(t), X_K(q(t)))$. The local expression for a mechanical system underactuated by one in constrained geometric normal form is

$$\begin{aligned} \dot{q}^i(t) &= w^a(t)X_a^i(q(t)) + s(t)X_K^i(q(t)) && (6.1) \\ \dot{w}^a(t) &= u^a(t) && (6.2) \\ \dot{s}(t) &= -w^a(t)w^p(t)\overset{\mathcal{H}}{\Gamma}_{ap}(q(t)) - 2w^a(t)s(t)\overset{\mathcal{H}}{\Gamma}_{aK}(q(t)) && (6.3) \\ &\quad -s(t)s(t)\overset{\mathcal{H}}{\Gamma}_{KK}(q(t)) \end{aligned}$$

where $a, p = 1, \ldots, m$, $i = 1, \ldots, n$ and $\overset{\mathcal{H}}{\Gamma}$ are the constrained generalized symmetric Christoffel symbols associated with the constrained unactuated connection $\overset{\mathcal{H},\mathcal{Y}^\perp}{\nabla}$. Locally, the states are denoted by $(q, w, s) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. The following alternative local representation for Equation (6.3) will be used in our proofs:

$$\dot{s}(t) \;=\; \begin{bmatrix} \mathbf{w}(t) & s(t) \end{bmatrix} \begin{bmatrix} B(q(t)) & S(q(t)) \\ S^T(q(t)) & T(q(t)) \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ s(t) \end{bmatrix} \tag{6.4}$$

where $B(q(t)) \in \mathbb{R}^{m\times m}$, $S(q(t)) \in \mathbb{R}^m$ and $T(q(t)) \in \mathbb{R}$. It follows from Definition 5.6.3 that the entries of the $\mathbb{R}$-valued symmetric bilinear form $B(q(t))$ are

$$B_{ap}(q(t)) \;=\; \frac{1}{2}\mathbb{G}(\langle X_a(q(t)) : X_p(q(t))\rangle, X_K(q(t)))$$

where $a, p = 1, \ldots, m$. We can expand Equation (6.4) to get

$$\dot{s}(t) \;=\; B_{ap}(q(t))w^a(t)w^p(t) + 2S_a(q(t))w^a(t)s(t) + T(q(t)). \tag{6.5}$$

Remark 6.1.1. *Note that if there are no linear velocity constraints then $\mathcal{H} = TM$ and $K = n$. An unconstrained mechanical system is a special case of a constrained mechanical system.*

We are now ready to state our main results.

Theorem 6.1.2 (Velocity Reachability Indefinite)**.** *Let $\Sigma_{\mathcal{H}} = \{M, \mathbb{G}, \mathcal{F}, \mathcal{H}, \mathbb{R}^m\}$ be a mechanical system (possibly with linear velocity constraints) underactuated by one control with the initial state $(q(t_0), v(t_0)) \in \mathcal{H}$. For any constants $\epsilon > 0$, $\Delta > 0, \alpha > 0$ and any target velocity v_T, if $B(q(t_0))$ is indefinite then there exists a piecewise control law $u : [t_0, T] \subset \mathbb{R} \to \mathbb{R}^m$ such that*

(i) $\|v_T - v(T)\| < \epsilon$,

(ii) $\|q(t) - q(t_0)\| < \alpha$ *for all* $t \in [t_0, T]$,

(iii) $|T - t_0| < \Delta$.

Proof of Theorem 6.1.2. We work locally. Let us begin by decomposing $\Sigma_{\mathcal{H}}$'s velocity along our constrained $\mathbb{G}$-orthonormal frame $\{X_1(q(t)), \ldots, X_K(q(t))\}$. The decomposition of the velocity is given by

$$v(t) = w^a(t)X_a(q(t)) + s(t)X_K(q(t))$$

for $a = 1, \ldots, m$ which allows us to express the initial velocity as

$$v(t_0) = (w^1(t_0), \ldots, w^m(t_0), s(t_0))$$

and the target velocity as $v_T = (w_T^1, \ldots, w_T^m, s_T)$. Clearly, if $\|v_T - v(t_0)\| < \epsilon$ then the conditions are already satisfied. We consider the following cases:

1. $|s_T - s(t_0)| > \epsilon$,
2. $0 < |s_T - s(t_0)| \leq \epsilon$,
3. $|s_T - s(t_0)| = 0$.

Note that the first part of the proof assumes that $|s_T - s(t_0)| > \epsilon$. However, if $v(t_0)$ is such that $0 < |s_T - s(t_0)| \leq \epsilon$ then we pick a new bound $\epsilon' > 0$ such that $\epsilon' = \frac{1}{2}|s_T - s(t_0)|$. We have $\epsilon' < \epsilon$ and we simply adjust the prescribed bound ϵ by setting it equal to the new bound ϵ' and proceed. The last case, $|s_T - s(t_0)| = 0$, will be addressed at the end of the proof.

Let the components of our candidate piecewise control law $u : [t_0, T] \subset \mathbb{R} \to$

$\mathbb{R}^m$ be of the form

$$u^a(t) = \begin{cases} \frac{Av_i^a - w^a(t_0)}{t_1 - t_0}, & \text{if } t \in [t_0, t_1) \\ 0, & \text{if } t \in [t_1, t_2) \\ \frac{w_T^a - Av_i^a}{T - t_2}, & \text{if } t \in [t_2, T]. \end{cases} \tag{6.6}$$

We take $v_i \in \mathbb{R}^m$ in control law (6.6) to be the eigenvector with unit length of the symmetric bilinear form $B(q(0)) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ corresponding to the eigenvalue λ_i where $\mathrm{sgn}(\lambda_i) = \mathrm{sgn}(s_T - s(t_0))$. It follows from the indefiniteness of $B(q(t_0))$ that for any $s(t_0)$ and s_T there exists λ_i such that $\mathrm{sgn}(\lambda_i) = \mathrm{sgn}(s_T - s(t_0))$ holds.

Now we introduce a set and several constants that will be used throughout this proof and the proof of the technical lemmas. First, we take F to be the set of operators

$$\{B_{ap}(q)v_i^a v_i^p, S_a(q)v_i^a, T(q), X_1(q), \ldots, X_K(q) \mid q \in B_\alpha(q(t_0))\}.$$

Second, we let

$$\begin{aligned} P_0 &= \|X_1(q(t_0))\| + \cdots + \|X_m(q(t_0))\| + m\frac{|\lambda_i|}{2}, \\ P_1 &= 2\max(|s_T|, |s(t_0)| + \epsilon)(\|X_K(q(t_0))\| + \frac{|\lambda_i|}{2}), \end{aligned}$$

$$
\begin{aligned}
C_2 &= \max_{f \in F} \sup(\|f'(q)\|), \\
C_3^2 &= \min(\frac{\alpha}{4}, \frac{|\lambda_i|}{8C_2}), \\
M_0 &= \frac{|\lambda_i|}{2}, \\
M_1 &= 4(|S_a(q(t_0))v_i^a| + \frac{|\lambda_i|}{2}) \max(|s_T|, |s(t_0)| + \epsilon), \\
M_1' &= M_1 + |s_T - s(t_0)| + \epsilon, \\
M_1'' &= M_1 + \frac{P_0(|s_T - s(t_0)| + \epsilon)}{C_3^2}, \\
M_2 &= (|T(q(t_0))| + \frac{|\lambda_i|}{2})(2 \max(|s_T|, |s(t_0)| + \epsilon))^2, \\
M_2'' &= M_2 + \frac{P_1(|s_T - s(t_0)| + \epsilon)}{C_3^2}, \\
L_0 &= \frac{1}{2}\sqrt{\frac{(M_1')^2 + 4M_0 M_2}{M_2^2}} - \frac{M_1'}{2M_2}, \\
L_1 &= \frac{1}{2}\sqrt{\frac{(M_1'')^2 + 4M_0 M_2''}{(M_2'')^2}} - \frac{M_1''}{2M_2''}.
\end{aligned}
$$

We set the constant $A > 0$ in control law (6.6) to be

$$A = \max(\frac{1}{\delta}, \|w(t_0)\|, \|w_T\|)$$

where

$$\delta = \frac{1}{2}\min(L_0, L_1, \frac{\Delta}{3}). \tag{6.7}$$

In addition, we have the constants

$$\begin{aligned}
C_0^1 &= A(\|X_1(q(t_0))\| + \frac{|\lambda_i|}{4}) + \cdots + A(\|X_m(q(t_0))\| + \frac{|\lambda_i|}{4}) \\
&\quad + (\|X_n(q(t_0))\| + \frac{|\lambda_i|}{4})(|s(t_0)| + \epsilon), \\
C_1^1 &= A^2 5\frac{|\lambda_i|}{4} + 2A(|S_a(q(t_0))v_i^a| + \frac{|\lambda_i|}{4})(|s(t_0)| + \epsilon) \\
&\quad + (|T(q(t_0))| + \frac{|\lambda_i|}{4})(|s(t_0)| + \epsilon)^2, \\
C_3^1 &= \frac{|\lambda_i|}{8C_2},
\end{aligned}$$

$$\begin{aligned}
C_0^2 &= A(\|X_1(q(t_0))\| + \frac{|\lambda_i|}{2}) + \cdots + A(\|X_m(q(t_0))\| + \frac{|\lambda_i|}{2}) \\
&\quad + (\|X_n(q(t_0))\| + \frac{|\lambda_i|}{2}) 2\max(|s_T|, |s(t_0)| + \epsilon), \\
C_1^2 &= A^2 3\frac{|\lambda_i|}{2} + 4A(|S_a(q(t_0))v_i^a| + \frac{|\lambda_i|}{2})\max(|s_T|, |s(t_0)| + \epsilon) \\
&\quad + (|T(q(t_0))| + \frac{|\lambda_i|}{2})(2\max(|s_T|, |s(t_0)| + \epsilon))^2,
\end{aligned}$$

$$\begin{aligned}
C_0^3 &= A(\|X_1(q(t_0))\| + 3\frac{|\lambda_i|}{4}) + \cdots + A(\|X_m(q(t_0))\| + 3\frac{|\lambda_i|}{4}) \\
&\quad + (\|X_n(q(t_0))\| + 3\frac{|\lambda_i|}{4}) 2\max(|s_T|, |s(t_0)| + \epsilon), \\
C_1^3 &= A^2 7\frac{|\lambda_i|}{4} + 4A(|S_a(q(t_0))v_i^a| + 3\frac{|\lambda_i|}{4})\max(|s_T|, |s(t_0)| + \epsilon) \\
&\quad + (|T(q(t_0))| + 3\frac{|\lambda_i|}{4})(2\max(|s_T|, |s(t_0)| + \epsilon))^2, \\
C_3^3 &= \min(\frac{\alpha}{4}, \frac{|\lambda_i|}{8C_2}), \\
N_0 &= \frac{|s_T - s(t_0)| + \epsilon}{\frac{M_0}{\delta} - M_1 - M_2\delta} \\
N_1 &= \frac{C_3^2}{P_0 + P_1\delta}.
\end{aligned}$$

Now we claim that given control law (6.6), A, v_i and δ, there exists t_1, t_2, T where $t_0 < t_1 < t_2 < T$, $t_1 - t_0 < \delta$, $t_2 - t_1 < \delta$ and $T - t_2 < \delta$ such that

(i) $\|v_T - v(T)\| < \epsilon$,

(ii) $\|q(t) - q(t_0)\| < \alpha$ for all $t \in [t_0, T]$,

(iii) $|T - t_0| < \Delta$.

We take t_1 to be of the form $t_1 = t_0 + \eta_1 \delta$ where $0 < \eta_1 < 1$, $t_2 < t_2'$ such that t_2' is of the form $t_2' = t_1 + \eta_2 \delta$ where $0 < \eta_2 < 1$, T to be of the form $T = t_2 + \eta_3 \delta$ where $0 < \eta_3 < 1$. By construction, we have $t_1 - t_0 < \delta$, $t_2 - t_1 < \delta$ and $T - t_2 < \delta$ which implies $|T - t_0| < \Delta$. Futhermore, the following lemmas hold.

Lemma 6.2.1 Given the piecewise control law (6.6), A, v_i, δ, $w(t_0)$, $s(t_0)$, s_T, if

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

then

(i) $w(t_1) = Av_i$,

(ii) $|s(t) - s(t_0)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \frac{\alpha}{3}$,

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$,

(v) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < \frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$,

(vi) $|T(q(t)) - T(q(t_0))| < \frac{|\lambda_i|}{4}$,

(vii) $\|X_j(q(t)) - X_j(q(t_0))\| < \frac{|\lambda_i|}{4}$ for each $j = 1, \ldots, K$,

for all $t \in [t_0, t_1]$.

Lemma 6.3.1 Given the piecewise control law (6.6), A, v_i and δ, if $\delta < \min(L_0, L_1)$ then

$$N_0 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta}).$$

Furthermore, if conditions $(i) - (vii)$ of Lemma 6.2.1 hold for all $t \in [t_0, t_1]$ and

$$N_0 < \eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

then there exists a $t_2 < t_2'$ such that

(i) $w(t_2) = Av_i$,

(ii) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < \frac{|\lambda_i|}{2}$ for $a, p = 1, \ldots, m$ and $t \in [t_0, t_2]$,

(iii) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < \frac{|\lambda_i|}{2}$ for $a = 1, \ldots, m$ and $t \in [t_0, t_2]$,

(iv) $|T(q(t)) - T(q(t_0))| < \frac{|\lambda_i|}{2}$ for all $t \in [t_0, t_2]$,

(v) $\|X_j(q(t)) - X_j(q(t_0))\| < \frac{|\lambda_i|}{2}$ for each $j = 1, \ldots, K$,

(vi) $\|q(t) - q(t_0)\| < \frac{2\alpha}{3}$,

(vii) $s(t_2) = s_T$.

for all $t \in [t_0, t_2]$.

Lemma 6.4.1 Given the piecewise control law (6.6), A, v_i and δ, if conditions $(i) - (vii)$ of Lemma 6.3.1 hold for all $t \in [t_0, t_2]$ and

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

then

(i) $w(T) = w_T$,

(ii) $|s(T) - s(t_2)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \alpha$,

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < 3\frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$,

(v) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < 3\frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$,

(vi) $|T(q(t)) - T(q(t_0))| < 3\frac{|\lambda_i|}{4}$,

(vii) $\|X_j(q(t)) - X_j(q(t_0))\| < 3\frac{|\lambda_i|}{4}$ for each $j = 1, \ldots, K$,

for all $t \in [t_0, T]$.

Following Lemma 6.2.1, Lemma 6.3.1 and Lemma 6.4.1, we set

$$\eta_1 = \frac{1}{2}\min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta}),$$

$$\eta_2 = \frac{1}{2}\left(\min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2\delta}) + N_0\right)$$

and

$$\eta_3 = \frac{1}{2}\min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta}).$$

This gives us

(i) $w(T) = w_T$,

(ii) $|s(T) - s(t_2)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \alpha$,

(iv) $|T - t_0| < \Delta$,

for all $t \in [t_0, T]$. If $w(T) = w_T$ and $|s(T) - s(t_2)| < \epsilon$ then $\|v(T) - v_T\| < \epsilon$. This completes the proof for the case when $|s_T - s(t_0)| > \epsilon$.

Finally, we consider the case when $|s_T - s(t_0)| = 0$. The following lemma holds.

Lemma 6.5.1 Given the piecewise control law (6.6), A, v_i, δ, $w(t_0)$, $s(t_0)$, s_T, w_T, if

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

then

(i) $w(t_1) = w_T$,

(ii) $|s(t) - s(t_0)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \frac{\alpha}{3}$,

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ for $a, p = 1, \dots, m$,

(v) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < \frac{|\lambda_i|}{4}$ for $a = 1, \dots, m$,

(vi) $|T(q(t)) - T(q(t_0))| < \frac{|\lambda_i|}{4}$,

(vii) $\|X_j(q(t)) - X_j(q(t_0))\| < \frac{|\lambda_i|}{4}$ for each $j = 1, \dots, K$,

for all $t \in [t_0, t_1]$.

Following Lemma 6.5.1, we set

$$\eta_1 = \frac{1}{2}\min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta}).$$

This gives us

(i) $w(t_1) = w_T$,

(ii) $|s(t_1) - s(t_0)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \alpha$,

(iv) $|t_1 - t_0| < \Delta$.

for all $t \in [t_0, t_1]$. If $w(t_1) = w_T$ and $|s(t_1) - s(t_0)| < \epsilon$ then $\|v(t_1) - v_T\| < \epsilon$. This completes the proof for the case when $|s_T - s(t_0)| = \epsilon$.

□

Theorem 6.1.3 (Velocity Reachability Positive Definite)**.** *Let*

$$\Sigma_{\mathcal{H}} = \{M, \mathbb{G}, \mathcal{F}, \mathcal{H}, \mathbb{R}^m\}$$

be a mechanical system (possibly with linear velocity constraints) underactuated by one control with the initial state $(q(t_0), v(t_0)) \in \mathcal{H}$. *For any constants* $\epsilon > 0$, $\Delta > 0, \alpha > 0$ *and a target velocity* v_T *such that the unactuated component of* v_T *satisfies* $s_T > s(t_0)$, *if* $\overset{\mathcal{H}}{B}(q(t_0))$ *is positive definite then there exists a piecewise control law* $u : [t_0, T] \in \mathbb{R} \to \mathbb{R}^m$ *such that*

(i) $\|v_T - v(T)\| < \epsilon$,

(ii) $\|q(t) - q(t_0)\| < \alpha$ *for all* $t \in [t_0, T]$,

(iii) $0 < T < \Delta$.

Proof of Theorem 6.1.3. This proof is almost identical to the proof of Theorem 6.1.3. The difference being the choice of v_i in the candidate piecewise control law (6.6). It follows from the positive definiteness of $B(q(t_0))$ and the assumption that $s_T > s(t_0)$ that there exists λ_i such that $\operatorname{sgn}(\lambda_i) = \operatorname{sgn}(s_T - s(t_0))$ holds. Here we set v_i to be the eigenvector with unit length of the symmetric bilinear form $B(q(0)) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ corresponding to any eigenvalue $\lambda_i \neq 0$ of $B(q(t_0))$. □

Theorem 6.1.4 (Velocity Reachability Negative Definite)**.** *Let*

$$\Sigma_{\mathcal{H}} = \{M, \mathbb{G}, \mathcal{F}, \mathcal{H}, \mathbb{R}^m\}$$

be a mechanical system (possibly with linear velocity constraints) underactuated by one control with the initial state $(q(t_0), v(t_0)) \in \mathcal{H}$*. For any constants* $\epsilon > 0$, $\Delta > 0, \alpha > 0$ *and a target velocity* v_T *such that the unactuated component of* v_T *satisfies* $s_T < s(t_0)$*, if* $\overset{\mathcal{H}}{B}(q(t_0))$ *is negative definite then there exists a piecewise control law* $u : [t_0, T] \in \mathbb{R} \to \mathbb{R}^m$ *such that*

(i) $\|v_T - v(T)\| < \epsilon$,

(ii) $\|q(t) - q(t_0)\| < \alpha$ *for all* $t \in [t_0, T]$,

(iii) $0 < T < \Delta$.

Proof of Theorem 6.1.4. Similar to the positive definite result, this proof is almost identical to the proof of Theorem 6.1.3. The difference being the choice of v_i in the candidate piecewise control law (6.6). It follows from the negative definiteness of $B(q(t_0))$ and the assumption that $s_T < s(t_0)$ that there exists λ_i such that $\text{sgn}(\lambda_i) = \text{sgn}(s_T - s(t_0))$ holds. Here we set v_i to be the eigenvector with unit length of the symmetric bilinear form $B(q(0)) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ corresponding to any eigenvalue $\lambda_i \neq 0$ of $B(q(t_0))$. □

6.2 Proof of First Technical Lemma

Lemma 6.2.1 (Stage 1)**.** *Given the piecewise control law* (6.6), A, v_i, δ, $w(t_0)$, $s(t_0)$, s_T, *if*

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

then

(i) $w(t_1) = Av_i$,

(ii) $|s(t) - s(t_0)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \frac{\alpha}{3}$,

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ *for* $a, p = 1, \ldots, m$,

(v) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < \frac{|\lambda_i|}{4}$ *for* $a = 1, \ldots, m$,

(vi) $|T(q(t)) - T(q(t_0))| < \frac{|\lambda_i|}{4}$,

(vii) $\|X_j(q(t)) - X_j(q(t_0))\| < \frac{|\lambda_i|}{4}$ *for each* $j = 1, \ldots, K$,

for all $t \in [t_0, t_1]$.

Proof of Lemma 6.2.1. The control law (6.6) along with straightforward integration of Equation (6.1) yields condition (i) for any choice of t_1 such that $t_1 > t_0$ and $t_1 - t_0 < \delta$. We use proof by contradiction for conditions $(ii) - (vii)$. This part of the proof requires six similar steps. In each step, we assume that there exists a time $t_1' < t_1$ such that one of the conditions $(ii) - (vii)$ is violated while the remaining conditions hold. We show that each of the six cases lead to a contradiction.

Step One We begin by assuming

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

and that there exists a $t_1' < t_1$ such that $|s(t_1') - s(t_0)| = \epsilon$ while conditions $(iii) - (vii)$ hold. We know from the mean value theorem and Schwartz inequality that

$$|s(t_1') - s(t_0)| \leq \sup_{t\in[t_0,t_1']} |\dot{s}(t)||t_1' - t_0|$$

where

$$\begin{aligned}\sup_{t\in[t_0,t_1']}|\dot{s}(t)| &= \sup_{t\in[t_0,t_1']}|B_{ap}(q(t))w^a(t)w^p(t)\\ &\quad +2S_a(q(t))w^a(t)s(t)+T(q(t))s(t)s(t)|.\end{aligned}$$

It follows from our assumptions that

$$\sup_{t\in[t_0,t_1']}|\dot{s}(t)| < C_1^1.$$

where

$$\begin{aligned}C_1^1 &= A^2 5\frac{|\lambda_i|}{4}+2A(|S_a(q(t_0))v_i^a|+\frac{|\lambda_i|}{4})(|s(t_0)|+\epsilon)\\ &\quad +(|T(q(t_0))|+\frac{|\lambda_i|}{4})(|s(t_0)|+\epsilon)^2.\end{aligned} \tag{6.8}$$

This implies

$$|s(t_1')-s(t_0)| < C_1^1|t_1'-t_0|.$$

Since we assume that $t_1' < t_0+\eta_1\delta$ then $|t_1'-t_0| < \eta_1\delta$ and

$$C_1^1|t_1'-t_0| < C_1^1\eta_1\delta.$$

Furthermore, we assume that $\eta_1 < \frac{\epsilon}{C_1^1\delta}$ which implies

$$|s(t_1')-s(t_0)| < \epsilon.$$

This is a contradiction.

Step Two Now we assume that

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

and that there exists a $t_1' < t_1$ such that $\|q(t_1') - q(t_0)\| = \frac{\alpha}{3}$ for $a, p = 1, \ldots, m$ while conditions (ii) and $(iv) - (vii)$ hold. We know from the mean value theorem and Schwartz inequality that

$$\|q(t_1') - q(t_0)\| \leq \sup_{t\in[t_0,t_1']} \|\dot{q}(t)\| |t_1' - t_0|$$

where

$$\sup_{t\in[t_0,t_1']} \|\dot{q}(t)\| \quad = \quad \sup_{t\in[t_0,t_1']} \|w^a(t)X_a(q(t)) + s(t)X_K(q(t))\|.$$

It follows from our assumptions that

$$\sup_{t\in[t_0,t_1']} \|\dot{q}(t)\| \quad < \quad C_0^1.$$

where

$$\begin{aligned} C_0^1 \quad &= \quad A(\|X_1(q(t_0))\| + \frac{|\lambda_i|}{4}) + \cdots + A(\|X_m(q(t_0))\| + \frac{|\lambda_i|}{4}) \\ &\quad +(\|X_n(q(t_0))\| + \frac{|\lambda_i|}{4})(|s(t_0)| + \epsilon). \end{aligned}$$

This implies

$$\|q(t_1') - q(t_0)\| < C_0^1 |t_1' - t_0|.$$

Since we assume that $t_1' < t_0 + \eta_1\delta$ then $|t_1' - t_0| < \eta_1\delta$ and

$$C_0^1|t_1' - t_0| \;\;<\;\; C_0^1\eta_1\delta.$$

Furthermore, we assume that $\eta_1 < \frac{\alpha}{3C_0^1\delta}$ which implies

$$\|q(t_1') - q(t_0)\| < \frac{\alpha}{3}.$$

This is a contradiction.

Step Three Now we assume that

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

and that there exists a $t_1' < t_1$ such that $|B_{ap}(q(t_1'))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| = \frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$ while conditions (ii), (iii) and $(v) - (vii)$ hold. We know from the mean value theorem and Schwartz inequality that

$$\|q(t_1') - q(t_0)\| \leq \sup_{t\in[t_0,t_1']} \|\dot{q}(t)\||t_1' - t_0|$$

where

$$\sup_{t\in[t_0,t_1']} \|\dot{q}(t)\| \;\;=\;\; \sup_{t\in[t_0,t_1']} \|w^a(t)X_a(q(t)) + s(t)X_K(q(t))\|.$$

It follows from our assumptions that

$$\sup_{t\in[t_0,t_1']} \|\dot{q}(t)\| \;\;<\;\; C_0^1.$$

which implies

$$\|q(t_1') - q(t_0)\| < C_0^1 |t_1' - t_0|.$$

Since we assume that $t_1' < t_0 + \eta_1 \delta$ then $|t_1' - t_0| < \eta_1 \delta$ and

$$C_0^1 |t_1' - t_0| \quad < \quad C_0^1 \eta_1 \delta.$$

Furthermore, we assume that $\eta_1 < \frac{C_3^1}{C_0^1 \delta}$ which implies

$$\|q(t_1') - q(t_0)\| < C_3^1$$

where

$$C_3^1 \quad = \quad \frac{|\lambda_i|}{8C_2}$$

and

$$C_2 \quad = \quad \max_{f \in F} \sup(\|f'(q)\|).$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|B_{ap}(q(t_1'))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \|q(t_1') - q(t_0)\|$$

where

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \quad = \quad \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\left(\frac{\partial B_{ap}(q(t))v_i^a v_i^p}{\partial q^1}, \ldots, \frac{\partial B_{ap}(q(t))v_i^a v_i^p}{\partial q^n} \right)\|.$$

We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \;\leq\; C_2.$$

which implies

$$\begin{aligned} |B_{ap}(q(t_1'))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| &\leq C_2\|q(t_1') - q(t_0)\|, \\ &< C_2 C_3^1. \end{aligned}$$

This gives us

$$|B_{ap}(q(t_1'))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| \;<\; \frac{|\lambda_i|}{8}$$

for all $a, p = 1, \ldots, m$. This is a contradiction.

Step Four Now we assume that

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

and that there exists a $t_1' < t_1$ such that $|S_a(q(t_1'))v_i^a - S_a(q(t_0))v_i^a| = \frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$ while conditions $(ii)-(iv)$, (vi) and (vii) hold. We assume that $\eta_1 < \frac{C_3^1}{C_0^1\delta}$ which implies

$$\|q(t_1') - q(t_0)\| < C_3^1.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|S_a(q(t_1'))v_i^a - S_a(q(t_0))v_i^a| \le \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla S_a(q(t))v_i^a\| \|q(t_1') - q(t_0)\|$$

where

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla S_a(q(t))v_i^a\| \;=\; \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\left(\frac{\partial S_a(q(t))v_i^a}{\partial q^1}, \ldots, \frac{\partial S_a(q(t))v_i^a}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla S_a(q(t))v_i^a\| \;\le\; C_2.$$

which implies

$$\begin{aligned} |S_a(q(t_1'))v_i^a - S_a(q(t_0))v_i^a| &\le C_2\|q(t_1') - q(t_0)\|, \\ &< C_2 C_3^1. \end{aligned}$$

This gives us

$$|S_a(q(t_1'))v_i^a - S_a(q(t_0))v_i^a| \;<\; \frac{|\lambda_i|}{8}$$

for all $a = 1, \ldots, m$. This is a contradiction.

Step Five Now we assume that

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

and that there exists a $t_1' < t_1$ such that $|T(q(t_1')) - T(q(t_0))| = \frac{|\lambda_i|}{4}$ while conditions $(ii)-(v)$ (vii) hold. We assume that $\eta_1 < \frac{C_3^1}{C_0^1\delta}$ which implies

$$\|q(t_1') - q(t_0)\| < C_3^1.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|T(q(t_1')) - T(q(t_0))| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla T(q(t))\| \|q(t_1') - q(t_0)\|$$

where

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla T(q(t))\| \quad = \quad \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\left(\frac{\partial T(q(t))}{\partial q^1}, \ldots, \frac{\partial T(q(t))}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla T(q(t))\| \quad \leq \quad C_2.$$

which implies

$$\begin{aligned} |T(q(t_1')) - T(q(t_0))| \quad &\leq \quad C_2 \|q(t_1') - q(t_0)\|, \\ &< \quad C_2 C_3^1. \end{aligned}$$

This gives us

$$|T(q(t_1')) - T(q(t_0))| \quad < \quad \frac{|\lambda_i|}{8}$$

for all $a = 1, \ldots, m$. This is a contradiction.

Step Six Finally, we assume that

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

and that there exists a $t_1' < t_1$ such that $\|X_j(q(t_1')) - X_j(q(t_0))\| = \frac{|\lambda_i|}{4}$ for some $j = 1, \ldots, K$ while conditions $(ii) - (vi)$ hold. We assume that $\eta_1 < \frac{C_3^1}{C_0^1\delta}$ which implies

$$\|q(t_1') - q(t_0)\| < C_3^1.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$\|X_j(q(t_1')) - X_j(q(t_0))\| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|X_j(q(t))'\| \|q(t_1') - q(t_0)\|$$

where $X_j(q(t))'$ is the Jacobian matrix with the i, k entry $\frac{\partial X_j^i(q(t))}{\partial q^k}$ and $\|\cdot\|$ is the appropriate matrix norm. We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_0))} \|\nabla X_j(q(t))'\| \leq C_2.$$

which implies

$$\begin{aligned} \|X_j(q(t_1')) - X_j(q(t_0))\| &\leq C_2\|q(t_1') - q(t_0)\|, \\ &< C_2 C_3^1. \end{aligned}$$

This gives us

$$\|X_j(q(t_1')) - X_j(q(t_0))\| < \frac{|\lambda_i|}{8}$$

for all $j = 1, \ldots, n$. This is a contradiction.

$\square$

6.3 Proof of Second Technical Lemma

Lemma 6.3.1 (Stage 2)**.** *Given the piecewise control law* (6.6)*,* A*,* v_i *and* δ*, if conditions* $(i) - (vii)$ *of Lemma 6.2.1 hold for all* $t \in [t_0, t_1]$ *and*

$$N_0 < \eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

then there exists a $t_2 < t_2'$ *such that*

(i) $w(t_2) = Av_i$,

(ii) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < \frac{|\lambda_i|}{2}$ *for* $a, p = 1, \ldots, m$,

(iii) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < \frac{|\lambda_i|}{2}$ *for* $a = 1, \ldots, m$,

(iv) $|T(q(t)) - T(q(t_0))| < \frac{|\lambda_i|}{2}$,

(v) $\|X_j(q(t)) - X_j(q(t_0))\| < \frac{|\lambda_i|}{2}$ *for each* $j = 1, \ldots, K$,

(vi) $\|q(t) - q(t_0)\| < \frac{2\alpha}{3}$,

(vii) $s(t_2) = s_T$.

for all $t \in [t_0, t_2]$.

Proof of Lemma 6.3.1. It suffices to show that there exists t_2 where $t_2 - t_1 < \delta$ such that

(i) $w(t_2) = Av_i$,

(ii) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| < \frac{\lambda_i}{4}$ for $a, p = 1, \ldots, m$,

(iii) $|S_a(q(t))v_i^a - S_a(q(t_1))v_i^a| < \frac{\lambda_i}{4}$ for $a = 1, \ldots, m$,

(iv) $|T(q(t)) - T(q(t_1))| < \frac{\lambda_i}{4}$,

(v) $\|X_j(q(t)) - X_j(q(t_1))\| < \frac{\lambda_i}{4}$ for $j = 1, \ldots, n$,

(vi) $\|q(t) - q(t_1)\| < \frac{\alpha}{3}$,

(vii) $s(t_2) = s_T$.

for all $t \in [t_1, t_2]$. The control law (6.6) along with straightforward integration of Equation (6.1) yields condition (i) for any choice of t_2 such that $t_2 > t_1$ and $t_2 - t_1 < \delta$. Moreover, the following lemmas hold.

Lemma 6.5.2 If $\delta < \min(L_0, L_1)$ then

$$N_0 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta}).$$

We take t_2' to be of the form $t_2' = t_1 + \eta_2 \delta$ where $0 < \eta_2 < 1$. By construction, we have $t_2' - t_1 < \delta$.

Lemma 6.5.3 Given the piecewise control law (6.6), A, v_i and δ, if conditions $(i) - (vii)$ of Lemma 6.2.1 hold for all $t \in [t_0, t_1]$ and

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

then

(i) $|s(t) - s(t_1)| < 2\max(|s_T|, |s(t_0)| + \epsilon)$,

(ii) $\|q(t) - q(t_1)\| < \frac{\alpha}{3}$,

(iii) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$,

(iv) $|S_a(q(t))v_i^a - S_a(q(t_1))v_i^a| < \frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$,

(v) $|T(q(t)) - T(q(t_1))| < \frac{|\lambda_i|}{4}$,

(vi) $\|X_j(q(t)) - X_j(q(t_1))\| < \frac{|\lambda_i|}{4}$ for $j = 1, \ldots, K$,

for all $t \in [t_1, t_2']$.

Lemma 6.5.4 Given the piecewise control law (6.6), A, v_i and δ, if conditions $(i) - (vi)$ of Lemma 6.5.3 hold for all $t \in [t_1, t_2']$ and

$$N_0 < \eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

then

(i) $|B_{ap}(q(t))Av_i^a Av_i^p| > |2S_a(q(t))Av_i^a s(t) + T(q(t))s(t)s(t)|$ for all $t \in [t_1, t_2']$,

(ii) $|s(t_2') - s(t_1)| > |s_T - s(t_1)|$,

(iii) $\mathrm{sgn}(B_{ap}(q(t))Av_i^a Av_i^p) = \mathrm{sgn}(B_{ap}(q(t_1))Av_i^a Av_i^p)$ for all $t \in [t_1, t_2']$.

Lemma 6.5.5 If conditions $(i) - (iii)$ of Lemma 6.5.4 hold for all $t \in [t_1, t_2']$ then there exists $t_2 < t_2'$ such that $s(t_2) = s_T$.

Following Lemma 6.5.3 and Lemma 6.5.4, we set

$$\eta_2 = \frac{1}{2}\left(N_0 + \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})\right).$$

This gives us conditions $(ii) - (vi)$. Finally, it follows from Lemma 6.5.5 that condition (vii) holds. $\square$

6.4 Proof of Third Technical Lemma

Lemma 6.4.1 (Stage 3)**.** *Given the piecewise control law* (6.6)*,* A*,* v_i *and* δ*, if conditions* $(i)-(vii)$ *of Lemma 6.3.1 hold for all* $t \in [t_0, t_1]$ *and*

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

then

(i) $w(T) = w_T$,

(ii) $|s(T) - s(t_2)| < \epsilon$,

(iii) $\|q(t) - q(t_0)\| < \alpha$,

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < 3\frac{|\lambda_i|}{4}$ *for* $a, p = 1, \ldots, m$,

(v) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < 3\frac{|\lambda_i|}{4}$ *for* $a = 1, \ldots, m$,

(vi) $|T(q(t)) - T(q(t_0))| < 3\frac{|\lambda_i|}{4}$,

(vii) $\|X_j(q(t)) - X_j(q(t_0))\| < 3\frac{|\lambda_i|}{4}$ *for* $j = 1, \ldots, n$,

for all $t \in [t_0, T]$.

Proof of Lemma 6.4.1. It suffices to show that if conditions $(i) - (vii)$ of Lemma 6.3.1 hold for all $t \in [t_0, t_1]$ and

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

then

(i) $w(T) = w_T$,

(ii) $|s(T) - s(t_2)| < \epsilon$,

(iii) $\|q(t) - q(t_2)\| < \frac{\alpha}{3}$,

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_2))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$,

(v) $|S_a(q(t))v_i^a - S_a(q(t_2))v_i^a| < \frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$,

(vi) $|T(q(t)) - T(q(t_2))| < \frac{|\lambda_i|}{4}$,

(vii) $\|X_j(q(t)) - X_j(q(t_2))\| < \frac{|\lambda_i|}{4}$ for each $j = 1, \ldots, n$,

for all $t \in [t_2, T]$. The control law (6.6) along with straightforward integration of Equation (6.1) yields condition (i) for any choice of T such that $T > t_2$ and $T - t_2 < \delta$. We use proof by contradiction for conditions $(ii) - (vii)$. This part of the proof requires six similar steps. In each step, we assume that there exists a time $T' < T$ such that one of the conditions $(ii) - (vii)$ is violated while the remaining conditions hold. We show that each of the six cases lead to a contradiction.

Step One We begin by assuming

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

and that there exists a $T' < T$ such that $|s(T') - s(t_2)| = \epsilon$ while conditions $(iii) - (vii)$ hold. We know from the mean value theorem and Schwartz inequality that

$$|s(T') - s(t_2)| \leq \sup_{t \in [t_2, T']} |\dot{s}(t)||T' - t_2|$$

where

$$\sup_{t\in[t_2,T']} |\dot{s}(t)| = \sup_{t\in[t_2,T']} |B_{ap}(q(t))w^a(t)w^p(t) + 2S_a(q(t))w^a(t)s(t) + T(q(t))s(t)s(t)|.$$

It follows from our assumptions that

$$\sup_{t\in[t_2,T']} |\dot{s}(t)| < C_1^3$$

where

$$C_1^3 = A^2 7\frac{|\lambda_i|}{4} + 4A(|S_a(q(t_0))v_i^a| + 3\frac{|\lambda_i|}{4}) \max(|s_T|, |s(t_0)| + \epsilon) + (|T(q(t_0))| + 3\frac{|\lambda_i|}{4})(2\max(|s_T|, |s(t_0)| + \epsilon))^2.$$

This implies

$$|s(T') - s(t_2)| < C_1^3 |T' - t_2|.$$

Since we assume that $T' < t_2 + \eta_3\delta$ then $|T' - t_2| < \eta_3\delta$ and

$$C_1^3|T' - t_2| < C_1^3\eta_3\delta.$$

Furthermore, we assume that $\eta_3 < \frac{\epsilon}{C_1^3\delta}$ which implies

$$|s(T') - s(t_2)| < \epsilon.$$

This is a contradiction.

Step Two Now we assume that

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

and that there exists a $T' < T$ such that $\|q(T') - q(t_2)\| = \frac{\alpha}{3}$ for $a, p = 1, \ldots, m$ while conditions (ii) and $(iv) - (vii)$ hold. We know from the mean value theorem and Schwartz inequality that

$$\|q(T') - q(t_2)\| \leq \sup_{t\in[t_2,T']} \|\dot{q}(t)\||T' - t_2|$$

where

$$\sup_{t\in[t_2,T']} \|\dot{q}(t)\| = \sup_{t\in[t_2,T']} \|w^a(t)X_a(q(t)) + s(t)X_K(q(t))\|.$$

It follows from our assumptions that

$$\sup_{t\in[t_2,T']} \|\dot{q}(t)\| < C_0^3.$$

where

$$\begin{aligned} C_0^3 &= A(\|X_1(q(t_0))\| + 3\frac{|\lambda_i|}{4}) + \cdots + A(\|X_m(q(t_0))\| + 3\frac{|\lambda_i|}{4}) \\ &\quad + (\|X_n(q(t_0))\| + 3\frac{|\lambda_i|}{4}) 2\max(|s_T|, |s(t_0)| + \epsilon). \end{aligned}$$

This implies

$$\|q(T') - q(t_2)\| < C_0^3|T' - t_2|.$$

Since we assume that $T' < t_2 + \eta_3\delta$ then $|T' - t_2| < \eta_3\delta$ and

$$C_0^3|T' - t_2| \;\; < \;\; C_0^3\eta_3\delta.$$

Furthermore, we assume that $\eta_3 < \frac{\alpha}{3C_0^3\delta}$ which implies

$$\|q(T') - q(t_2)\| < \frac{\alpha}{3}.$$

This is a contradiction.

Step Three Now we assume that

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

and that there exists a $T' < T$ such that $|B_{ap}(q(T'))v_i^a v_i^p - B_{ap}(q(t_2))v_i^a v_i^p| = \frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$ while conditions (ii), (iii) and $(v) - (vii)$ hold. We know from the mean value theorem and Schwartz inequality that

$$\|q(T') - q(t_2)\| \leq \sup_{t \in [t_2, T']} \|\dot{q}(t)\| |T' - t_2|$$

where

$$\sup_{t \in [t_2, T']} \|\dot{q}(t)\| \;\; = \;\; \sup_{t \in [t_2, T']} \|w^a(t) X_a(q(t)) + s(t) X_n(q(t))\|.$$

It follows from our assumptions that

$$\sup_{t \in [t_2, T']} \|\dot{q}(t)\| \;\; < \;\; C_0^3.$$

which implies

$$\|q(T') - q(t_2)\| < C_0^3 |T' - t_2|$$

where

$$\begin{aligned} C_2 &= \max_{f \in F} \sup(\|f'(q)\|), \\ C_3^3 &= \min(\frac{\alpha}{4}, \frac{|\lambda_i|}{8C_2}) \end{aligned}$$

and

$$C_2 = \max_{f \in F} \sup(\|f'(q)\|).$$

Since we assume that $T' < t_2 + \eta_3 \delta$ then $|T' - t_2| < \eta_3 \delta$ and

$$C_0^3 |T' - t_2| < C_0^3 \eta_3 \delta.$$

Furthermore, we assume that $\eta_3 < \frac{C_3^3}{C_0^3 \delta}$ which implies

$$\|q(T') - q(t_2)\| < C_3^3.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|B_{ap}(q(T'))v_i^a v_i^p - B_{ap}(q(t_2))v_i^a v_i^p| \leq \sup_{B_\alpha(q(t_0))} \|\nabla B_{ap}(q(t)) v_i^a v_i^p\| \|q(T') - q(t_2)\|$$

where

$$\sup_{B_\alpha(q(t_0))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \quad = \quad \sup_{B_\alpha(q(t_0))} \|\left(\frac{\partial B_{ap}(q(t))v_i^a v_i^p}{\partial q^1}, \ldots, \frac{\partial B_{ap}(q(t))v_i^a v_i^p}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_\alpha(q(t_0))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \quad \leq \quad C_2.$$

which implies

$$\begin{aligned} |B_{ap}(q(T'))v_i^a v_i^p - B_{ap}(q(t_2))v_i^a v_i^p| &\leq C_2\|q(T') - q(t_2)\|, \\ &< C_2 C_3^3. \end{aligned}$$

This gives us

$$|B_{ap}(q(T'))v_i^a v_i^p - B_{ap}(q(t_2))v_i^a v_i^p| \quad < \quad \frac{|\lambda_i|}{8}$$

for$a, p = 1, \ldots, m$. This is a contradiction.

Step Four Now we assume that

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

and that there exists a $T' < T$ such that $|S_a(q(T'))v_i^a - S_a(q(t_2))v_i^a| = \frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$ while conditions $(ii) - (iv)$, (vi) and (vii) hold. We assume that $\eta_3 < \frac{C_3^3}{C_0^3\delta}$ which implies

$$\|q(T') - q(t_2)\| < C_3^3.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|S_a(q(T'))v_i^a - S_a(q(t_2))v_i^a| \leq \sup_{B_\alpha(q(t_0))} \|\nabla S_a(q(t))v_i^a\| \|q(T') - q(t_2)\|$$

where

$$\sup_{B_\alpha(q(t_0))} \|\nabla S_a(q(t))v_i^a\| \;=\; \sup_{B_\alpha(q(t_0))} \|\left(\frac{\partial S_a(q(t))v_i^a}{\partial q^1}, \ldots, \frac{\partial S_a(q(t))v_i^a}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_\alpha(q(t_0))} \|\nabla S_a(q(t))v_i^a\| \;\leq\; C_2.$$

which implies

$$\begin{aligned} |S_a(q(T'))v_i^a - S_a(q(t_2))v_i^a| &\leq C_2\|q(T') - q(t_2)\|, \\ &< C_2 C_3^3. \end{aligned}$$

This gives us

$$|S_a(q(T'))v_i^a - S_a(q(t_2))v_i^a| \;<\; \frac{|\lambda_i|}{8}$$

for $a = 1, \ldots, m$. This is a contradiction.

Step Five Now we assume that

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

and that there exists a $T' < T$ such that $|T(q(T')) - T(q(t_2))| = \frac{|\lambda_i|}{4}$ while conditions $(ii) - (v)$ (vii) hold. We assume that $\eta_3 < \frac{C_3^3}{C_0^3\delta}$ which implies

$$\|q(T') - q(t_2)\| < C_3^3.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|T(q(T')) - T(q(t_2))| \leq \sup_{B_\alpha(q(t_0))} \|\nabla T(q(t))\| \|q(T') - q(t_2)\|$$

where

$$\sup_{B_\alpha(q(t_0))} \|\nabla T(q(t))\| \;\; = \;\; \sup_{B_\alpha(q(t_0))} \|\left(\frac{\partial T(q(t))}{\partial q^1}, \ldots, \frac{\partial T(q(t))}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_\alpha(q(t_0))} \|\nabla T(q(t))\| \;\; \leq \;\; C_2.$$

which implies

$$\begin{aligned} |T(q(T')) - T(q(t_2))| \;\; &\leq \;\; C_2\|q(T') - q(t_2)\|, \\ &< \;\; C_2 C_3^3. \end{aligned}$$

This gives us

$$|T(q(T')) - T(q(t_2))| \;\; < \;\; \frac{|\lambda_i|}{8}$$

for $a = 1, \ldots, m$. This is a contradiction.

Step Six Finally, we assume that

$$\eta_3 < \min(1, \frac{\alpha}{3C_0^3\delta}, \frac{\epsilon}{C_1^3\delta}, \frac{C_3^3}{C_0^3\delta})$$

and that there exists a $T' < T$ such that $\|X_j(q(T')) - X_j(q(t_2))\| = \frac{|\lambda_i|}{4}$ for some $j = 1, \ldots, K$ while conditions $(ii) - (vi)$ hold. We assume that $\eta_3 < \frac{C_3^3}{C_0^3\delta}$ which implies

$$\|q(T') - q(t_2)\| < C_3^3.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$\|X_j(q(T')) - X_j(q(t_2))\| \leq \sup_{B_\alpha(q(t_0))} \|X_j(q(t))'\| \|q(T') - q(t_2)\|$$

where $X_j(q(t))'$ is the Jacobian matrix with the i, k entry $\frac{\partial X_j^i(q(t))}{\partial q^k}$ and $\|\cdot\|$ is the appropriate matrix norm. We have

$$\sup_{B_\alpha(q(t_0))} \|\nabla X_j(q(t))'\| \leq C_2.$$

which implies

$$\begin{aligned} \|X_j(q(T')) - X_j(q(t_2))\| &\leq C_2\|q(T') - q(t_2)\|, \\ &< C_2 C_3^3. \end{aligned}$$

This gives us

$$\|X_j(q(T')) - X_j(q(t_2))\| < \frac{|\lambda_i|}{8}$$

for all $j = 1, \ldots, K$. This is a contradiction.

$\square$

6.5 Proof of Secondary Technical Lemmas

Lemma 6.5.1. *Given the piecewise control law* (6.6)*,* A*,* v_i*,* δ*,* $w(t_0)$*,* $s(t_0)$*,* s_T*,* w_T*, if*

$$\eta_1 < \min(1, \frac{\alpha}{3C_0^1\delta}, \frac{\epsilon}{C_1^1\delta}, \frac{C_3^1}{C_0^1\delta})$$

then

(i) $w(t_1) = w_T$*,*

(ii) $|s(t) - s(t_0)| < \epsilon$ *,*

(iii) $\|q(t) - q(t_0)\| < \frac{\alpha}{3}$*,*

(iv) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_0))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ *for* $a, p = 1, \ldots, m$*,*

(v) $|S_a(q(t))v_i^a - S_a(q(t_0))v_i^a| < \frac{|\lambda_i|}{4}$ *for* $a = 1, \ldots, m$*,*

(vi) $|T(q(t)) - T(q(t_0))| < \frac{|\lambda_i|}{4}$*,*

(vii) $\|X_j(q(t)) - X_j(q(t_0))\| < \frac{|\lambda_i|}{4}$ *for each* $j = 1, \ldots, K$*,*

for all $t \in [t_0, t_1]$*.*

Proof of Lemma 6.5.1. The control law (6.6) along with straightforward integration of Equation (6.1) yields condition (i) for any choice of t_1 such that $t_1 > t_0$ and $t_1 - t_0 < \delta$. The remainder of the proof can be found in the proof of Lemma 6.2.1. $\square$

Lemma 6.5.2. *If* $\delta < \min(L_0, L_1)$ *then* $N_0 < 1$ *and* $N_0 < N_1$*.*

Proof of Lemma 6.5.2. The condition $N_0 < 1$ is equivalent to

$$
\begin{aligned}
& \frac{|s_T - s(t_0)| + \epsilon}{\frac{M_0}{\delta} - M_1 - M_2\delta} < 1 \\
\implies \quad & |s_T - s(t_0)| + \epsilon < \frac{M_0}{\delta} - M_1 - M_2\delta \\
\implies \quad & (|s_T - s(t_0)| + \epsilon)\delta < M_0 - M_1\delta - M_2\delta^2 \\
\implies \quad & 0 < M_0 - (M_1 + (|s_T - s(t_0)| + \epsilon))\delta - M_2\delta^2 \\
\implies \quad & 0 < M_0 - M_1'\delta - M_2\delta^2.
\end{aligned}
$$

It suffices to show that if $\delta < \min(L_0, L_1)$ then $0 < M_0 - M_1'\delta - M_2\delta^2$ and $N_0 < N_1$.

Now suppose

$$\delta < L_0,$$

$$
\begin{aligned}
\implies \quad & \delta < \frac{1}{2}\sqrt{\frac{(M_1')^2 + 4M_0M_2}{M_2^2}} - \frac{M_1'}{2M_2} \\
\implies \quad & \delta + \frac{M_1'}{2M_2} < \frac{1}{2}\sqrt{\frac{(M_1')^2 + 4M_0M_2}{M_2^2}} \\
\implies \quad & \left(\delta + \frac{M_1'}{2M_2}\right)^2 < \frac{(M_1')^2 + 4M_0M_2}{4M_2^2} \\
\implies \quad & \delta^2 + 2\delta\frac{M_1'}{2M_2} + \frac{(M_1')^2}{4M_2^2} < \frac{(M_1')^2 + 4M_0M_2}{4M_2^2} \\
\implies \quad & \delta^2 + 2\delta\frac{M_1'}{2M_2} < \frac{4M_0M_2}{4M_2^2} \\
\implies \quad & \delta^2 + \delta\frac{M_1'}{M_2} < \frac{M_0}{M_2} \\
\implies \quad & 0 < M_0 - \delta M_1' - \delta^2 M_2.
\end{aligned}
$$

The condition $N_0 < N_1$ is equivalent to

$$
\begin{aligned}
& \frac{|s_T - s(t_0)| + \epsilon}{\frac{M_0}{\delta} - M_1 - M_2\delta} < \frac{C_3^2}{P_0 + P_1\delta} \\
\implies \quad & \frac{(|s_T - s(t_0)| + \epsilon)\delta}{M_0 - M_1\delta - M_2\delta^2} < \frac{C_3^2}{P_0 + P_1\delta} \\
\implies \quad & (|s_T - s(t_0)| + \epsilon)\delta(P_0 + P_1\delta) < C_3^2(M_0 - M_1\delta - M_2\delta^2) \\
\implies \quad & 0 < M_0 - M_1''\delta - M_2''\delta^2.
\end{aligned}
$$

It suffices to show that if $\delta < \min(L_0, L_1)$ then $0 < M_0 - M_1'\delta - M_2\delta^2$ and $0 < M_0 - M_1''\delta - M_2''\delta^2$. Now suppose

$$\delta < L_1,$$

$$
\begin{aligned}
\implies \quad & \delta < \frac{1}{2}\sqrt{\frac{(M_1'')^2 + 4M_0M_2''}{(M_2'')^2}} - \frac{M_1''}{2M_2''} \\
\implies \quad & 0 < M_0 - \delta M_1'' - \delta^2 M_2''.
\end{aligned}
$$

Since we assume $\delta < \min(L_0, L_1)$ our claim holds. □

Lemma 6.5.3. *Given the piecewise control law* (6.6)*, A, v_i and δ, if conditions (i)–(vii) of Lemma 6.2.1 hold for all $t \in [t_0, t_1]$ and $\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)|+\epsilon)}{C_1^2\delta})$ then*

(i) $|s(t) - s(t_1)| < 2\max(|s_T|, |s(t_0)| + \epsilon)$,

(ii) $\|q(t) - q(t_1)\| < \frac{\alpha}{3}$,

(iii) $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| < \frac{|\lambda_i|}{4}$ *for* $a, p = 1, \ldots, m$,

(iv) $|S_a(q(t))v_i^a - S_a(q(t_1))v_i^a| < \frac{|\lambda_i|}{4}$ *for* $a = 1, \ldots, m$,

(v) $|T(q(t)) - T(q(t_1))| < \frac{|\lambda_i|}{4}$,

(vi) $\|X_j(q(t)) - X_j(q(t_1))\| < \frac{|\lambda_i|}{4}$ *for each* $j = 1, \ldots, K$,

for all $t \in [t_1, t_2']$.

Proof of Lemma 6.5.3. We use proof by contradiction for conditions $(i) - (vi)$. This requires six similar steps. In each step, we assume that there exists a time $t_2'' < t_2'$ such that one of the conditions $(i) - (vi)$ is violated while the remaining conditions hold. We show that each of the six cases lead to a contradiction.

Step One We begin by assuming

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

and that there exists a $t_2'' < t_2'$ such that $|s(t) - s(t_1)| = 2\max(|s_T|, |s(t_0)| + \epsilon)$ while conditions $(ii) - (vi)$ hold. We know from the mean value theorem and Schwartz inequality that

$$|s(t_2'') - s(t_1)| \leq \sup_{t \in [t_1, t_2'']} |\dot{s}(t)||t_2'' - t_1|$$

where

$$\begin{aligned} \sup_{t \in [t_1, t_2'']} |\dot{s}(t)| &= \sup_{t \in [t_1, t_2'']} |B_{ap}(q(t)) w^a(t) w^p(t) \\ &\quad + 2S_a(q(t)) w^a(t) s(t) + T(q(t)) s(t) s(t)|. \end{aligned}$$

It follows from our assumptions that

$$\sup_{t \in [t_1, t_2'']} |\dot{s}(t)| < C_1^2$$

where

$$C_1^2 = A^2 3\frac{|\lambda_i|}{2} + 4A(|S_a(q(t_0))v_i^a| + \frac{|\lambda_i|}{2})\max(|s_T|, |s(t_0)| + \epsilon)$$
$$+(|T(q(t_0))| + \frac{|\lambda_i|}{2})(2\max(|s_T|, |s(t_0)| + \epsilon))^2.$$

This implies

$$|s(t_2'') - s(t_1)| < C_1^2|t_2'' - t_1|.$$

Since we assume that $t_2'' < t_1 + \eta_2\delta$ then $|t_2'' - t_1| < \eta_2\delta$ and

$$C_1^2|t_2'' - t_1| < C_1^2\eta_2\delta.$$

Furthermore, we assume that $\eta_2 < \frac{2\max(|s_T|,|s(t_0)|+\epsilon)}{C_1^2\delta}$ which implies

$$|s(t_2'') - s(t_1)| < 2\max(|s_T|, |s(t_0)| + \epsilon).$$

This is a contradiction.

Step Two Now we assume that

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2\delta})$$

and that there exists a $t_2'' < t_2'$ such that $\|q(t_2'') - q(t_1)\| = \frac{\alpha}{3}$ while conditions (i) and $(iii)-(vi)$ hold. We know from the mean value theorem and Schwartz inequality that

$$\|q(t_2'') - q(t_1)\| \leq \sup_{t\in[t_1,t_2'']} \|\dot{q}(t)\| |t_2'' - t_1|$$

where

$$\sup_{t\in[t_1,t_2'']} \|\dot{q}(t)\| \quad = \quad \sup_{t\in[t_1,t_2'']} \|w^a(t)X_a(q(t)) + s(t)X_K(q(t))\|.$$

It follows from our assumptions that

$$\sup_{t\in[t_1,t_2'']} \|\dot{q}(t)\| \quad < \quad C_0^2.$$

where

$$\begin{aligned} C_0^2 \quad &= \quad A(\|X_1(q(t_0))\| + \frac{|\lambda_i|}{2}) + \cdots + A(\|X_m(q(t_0))\| + \frac{|\lambda_i|}{2}) \\ &\quad +(\|X_n(q(t_0))\| + \frac{|\lambda_i|}{2})2\max(|s_T|, |s(t_0)| + \epsilon). \end{aligned}$$

This implies

$$\|q(t_2'') - q(t_1)\| < C_0^2|t_2'' - t_1|.$$

Since we assume that $t_2'' < t_1 + \eta_2\delta$ then $|t_2'' - t_1| < \eta_2\delta$ and

$$C_0^2|t_2'' - t_1| \quad < \quad C_0^2\eta_2\delta.$$

Furthermore, we assume that $\eta_2 < N_1$ which implies

$$\|q(t_2'') - q(t_1)\| < \frac{\alpha}{4}.$$

This is a contradiction.

Step Three Now we assume that

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

and that there exists a $t_2'' < t_2'$ such that $|B_{ap}(q(t_2''))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| = \frac{|\lambda_i|}{4}$ for $a, p = 1, \ldots, m$ while conditions (i), (ii) and $(iv) - (vi)$ hold. We know from the mean value theorem and Schwartz inequality that

$$\|q(t_2'') - q(t_1)\| \leq \sup_{t \in [t_1, t_2'']} \|\dot{q}(t)\| |t_2'' - t_1|$$

where

$$\sup_{t \in [t_1, t_2'']} \|\dot{q}(t)\| \quad = \quad \sup_{t \in [t_1, t_2'']} \|w^a(t) X_a(q(t)) + s(t) X_K(q(t))\|.$$

It follows from our assumptions that

$$\sup_{t \in [t_1, t_2'']} \|\dot{q}(t)\| \quad < \quad C_0^2.$$

which implies

$$\|q(t_2'') - q(t_1)\| < C_0^2 |t_2'' - t_1|.$$

Since we assume that $t_2'' < t_1 + \eta_2\delta$ then $|t_2'' - t_1| < \eta_2\delta$ and

$$C_0^2|t_2'' - t_1| \quad < \quad C_0^2\eta_2\delta.$$

Furthermore, we assume that $\eta_2 < N_1$ which implies

$$\|q(t_2'') - q(t_1)\| < C_3^2$$

where

$$C_3^2 \quad = \quad \min(\frac{\alpha}{4}, \frac{|\lambda_i|}{8C_2})$$

and

$$C_2 \quad = \quad \max_{f \in F} \sup(\|f'(q)\|).$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|B_{ap}(q(t_2''))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \|q(t_2'') - q(t_1)\|$$

where

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \quad = \quad \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\left(\frac{\partial B_{ap}(q(t))v_i^a v_i^p}{\partial q^1}, \ldots, \frac{\partial B_{ap}(q(t))v_i^a v_i^p}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla B_{ap}(q(t))v_i^a v_i^p\| \quad \leq \quad C_2.$$

which implies

$$\begin{aligned}|B_{ap}(q(t_2''))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| &\leq C_2\|q(t_2'') - q(t_1)\|,\\ &< C_2 C_3^2.\end{aligned}$$

This gives us

$$|B_{ap}(q(t_2''))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p| < \frac{|\lambda_i|}{8}$$

for $a, p = 1, \ldots, m$. This is a contradiction.

Step Four Now we assume that

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$$

and that there exists a $t_2'' < t_2'$ such that $|S_a(q(t_2''))v_i^a - S_a(q(t_1))v_i^a| = \frac{|\lambda_i|}{4}$ for $a = 1, \ldots, m$ while conditions $(i) - (iii)$, (v) and (vi) hold. We assume that $\eta_2 < N_1$ which implies

$$\|q(t_2'') - q(t_1)\| < C_3^2.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|S_a(q(t_2''))v_i^a - S_a(q(t_1))v_i^a| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla S_a(q(t))v_i^a\| \|q(t_2'') - q(t_1)\|$$

where

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla S_a(q(t))v_i^a\| \;=\; \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\left(\frac{\partial S_a(q(t))v_i^a}{\partial q^1}, \ldots, \frac{\partial S_a(q(t))v_i^a}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla S_a(q(t))v_i^a\| \;\leq\; C_2.$$

which implies

$$\begin{aligned} |S_a(q(t_2''))v_i^a - S_a(q(t_1))v_i^a| &\leq C_2\|q(t_2'') - q(t_1)\|, \\ &< C_2 C_3^2. \end{aligned}$$

This gives us

$$|S_a(q(t_2''))v_i^a - S_a(q(t_1))v_i^a| \;<\; \frac{|\lambda_i|}{8}$$

for $a = 1, \ldots, m$. This is a contradiction.

Step Five Now we assume that

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2\delta})$$

and that there exists a $t_2'' < t_2'$ such that $|T(q(t)) - T(q(t_0))| = \frac{|\lambda_i|}{4}$ while conditions $(i) - (iv)$ and (vi) hold. We assume that $\eta_2 < N_1$ which implies

$$\|q(t_2'') - q(t_1)\| < C_3^2.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$|T(q(t_2''))-T(q(t_1))| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla T(q(t))\| \|q(t_2'')-q(t_1)\|$$

where

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla T(q(t))\| = \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\left(\frac{\partial T(q(t))}{\partial q^1}, \ldots, \frac{\partial T(q(t))}{\partial q^n}\right)\|.$$

We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla T(q(t))\| \leq C_2.$$

which implies

$$\begin{aligned} |T(q(t_2''))-T(q(t_1))| &\leq C_2\|q(t_2'')-q(t_1)\|, \\ &< C_2 C_3^2. \end{aligned}$$

This gives us

$$|T(q(t_2''))-T(q(t_1))| < \frac{|\lambda_i|}{8}$$

for $a = 1, \ldots, m$. This is a contradiction.

Step Six Finally, we assume that

$$\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)|+\epsilon)}{C_1^2\delta})$$

and that there exists a $t_2'' < t_2'$ such that $\|X_j(q(t_1')) - X_j(q(t_0))\| = \frac{|\lambda_i|}{4}$ for some $j = 1, \ldots, K$ while conditions $(i) - (v)$ hold. We assume that $\eta_2 < N_1$ which implies

$$\|q(t_2'') - q(t_1)\| < C_3^2.$$

Again, it follows from the mean value theorem and Schwartz inequality that

$$\|X_j(q(t_2'')) - X_j(q(t_1))\| \leq \sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|X_j(q(t))'\| \|q(t_2'') - q(t_1)\|$$

where $X_j(q(t))'$ is the Jacobian matrix with the i, k entry $frac\partial X_j^i(q(t))\partial q^k$ and $\|\cdot\|$ is the appropriate matrix norm. We have

$$\sup_{B_{\frac{\alpha}{3}}(q(t_1))} \|\nabla X_j(q(t))'\| \leq C_2.$$

which implies

$$\begin{aligned} \|X_j(q(t_2'')) - X_j(q(t_1))\| &\leq C_2 \|q(t_2'') - q(t_1)\|, \\ &< C_2 C_3^2. \end{aligned}$$

This gives us

$$\|X_j(q(t_2'')) - X_j(q(t_1))\| < \frac{|\lambda_i|}{8}$$

for all $j = 1, \ldots, K$. This is a contradiction.

$\square$

Lemma 6.5.4. *Given the piecewise control law* (6.6)*, A, v_i and δ, if conditions*

$(i){-}(vi)$ of Lemma 6.5.3 hold for all $t \in [t_1, t_2']$ and $N_0 < \eta_2 < \min(1, N_1, \frac{2\max(|s_T|,|s(t_0)|+\epsilon)}{C_1^2\delta})$ then

(i) $|B_{ap}(q(t))Av_i^a Av_i^p| > |2S_a(q(t))Av_i^a s(t) + T(q(t))s(t)s(t)|$ *for all* $t \in [t_1, t_2']$,

(ii) $|s(t_2') - s(t_1)| > |s_T - s(t_1)|$,

(iii) $\operatorname{sgn}(B_{ap}(q(t))Av_i^a Av_i^p) = \operatorname{sgn}(B_{ap}(q(t_1))Av_i^a Av_i^p)$ *for all* $t \in [t_1, t_2']$.

Proof of Lemma 6.5.4. By Lemma 6.5.3, if $\eta_2 < \min(1, N_1, \frac{2\max(|s_T|,|s(t_0)|+\epsilon)}{C_1^2\delta})$ then $|s(t) - s(t_1)| < 2\max(|s_T|, |s(t_0)| + \epsilon)$ and each of the following $|B_{ap}(q(t))v_i^a v_i^p - B_{ap}(q(t_1))v_i^a v_i^p|$, $|S_a(q(t))v_i^a - S_a(q(t_1))v_i^a|$, $|T(q(t)) - T(q(t_1))|$ and $\|X_j(q(t)) - X_j(q(t_1))\|$ are less than $\frac{|\lambda_i|}{4}$ for all $t \in [t_1, t_2']$. This implies that

$$\begin{aligned} \sup_{\tau\in[t_1,t_2']} |\dot{q}(\tau)|\eta_2\delta &< (AP_0 + P_1)\eta_2\delta, \\ A^2|B_{ap}(q(t))v_i^a v_i^p| &> A^2 M_0, \quad \forall\ t \in [t_1, t_2'], \\ |S_a(q(t))Av_i^a s(t) + T(q(t))| &< AM_1 + M_2, \quad \forall\ t \in [t_1, t_2']. \end{aligned}$$

It suffices to show that if conditions $(i){-}(vi)$ of Lemma 6.5.3 hold for all $t \in [t_1, t_2']$ and $N_0 < \eta_2 < \min(1, N_1, \frac{2\max(|s_T|,|s(t_0)|+\epsilon)}{C_1^2\delta})$ then

(i) $M_0 - \delta M_1 - \delta^2 M_2 > 0$,

(ii) $|s(t_2') - s(t_1)| > |s_T - s(t_1)|$.

(iii) $\operatorname{sgn}(B_{ap}(q(t))Av_i^a Av_i^p) = \operatorname{sgn}(B_{ap}(q(t_1))Av_i^a Av_i^p)$ for all $t \in [t_1, t_2']$.

We begin with condition (i). We have already shown that $\delta < L_0$

$$\implies 0 < M_0 - \delta M_1' - \delta^2 M_2.$$

Furthermore, $0 < M_0 - \delta M_1' - \delta^2 M_2$

$$\begin{aligned}
&\Longrightarrow \quad \delta(|s_T - s(t_0)| + \epsilon) < M_0 - \delta M_1 - \delta^2 M_2 \\
&\Longrightarrow \quad 0 < M_0 - \delta M_1 - \delta^2 M_2.
\end{aligned}$$

Now we consider condition (ii). We know from the mean value theorem and Schwartz inequality that

$$|s(t_2') - s(t_1)| \quad \geq \quad \inf_{t \in [t_1, t_2']} |\dot{s}(t)| \eta_2 \delta.$$

It follows from Equation (6.5) that

$$\inf_{t \in [t_1, t_2']} |\dot{s}(t)| \quad = \quad \inf_{t \in [t_1, t_2']} |B_{ap}(q(t)) w^a(t) w^p(t) + 2 S_a(q(t)) w^a(t) s(t) + T(q(t))|.$$

We know that if $\eta_2 < \min(1, N_1, \frac{2\max(|s_T|, |s(t_0)| + \epsilon)}{C_1^2 \delta})$ and condition (ii) holds then

$$\inf_{t \in [t_1, t_2']} |\dot{s}(t)| \quad > \quad \frac{1}{\delta^2} M_0 - \frac{1}{\delta} M_1 - M_2.$$

This implies

$$|s(t_2') - s(t_1)| \quad > \quad \left(\frac{M_0}{\delta} - M_1 - \delta M_2 \right) \eta_2.$$

By assumption,

$$\eta_2 > \frac{|s_T - s(t_0)| + \epsilon}{\frac{M_0}{\delta} - M_1 - M_2 \delta}$$

$$
\begin{aligned}
&\Longrightarrow \quad \left(\frac{M_0}{\delta} - M_1 - M_2\delta\right)\eta_2 > |s_T - s(t_0)| + \epsilon \\
&\Longrightarrow \quad |s(t_2') - s(t_1)| > |s_T - s(t_0)| + \epsilon.
\end{aligned}
$$

Since $|s_T - s(t_0)| + \epsilon > |s_T - s(t_1)|$, we have $|s(t_2') - s(t_1)| > |s_T - s(t_1)|$.

Finally, we consider condition (iii). By assumption, v_i is an eigenvector with unit length of the symmetric bilinear form $B(q(t_0)) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ corresponding to the eigenvalue λ_i where $\operatorname{sgn}(\lambda_i) = \operatorname{sgn}(s_T - s(t_0))$. This implies that $B_{ap}(q(t_0))Av_i^a Av_i^p = A^2\lambda_i$. Therefore, the $\operatorname{sgn}(B_{ap}(q(t_0))Av_i^a Av_i^p) = \operatorname{sgn}(\lambda_i)$. By definition, $\operatorname{sgn}(\lambda_i) = \operatorname{sgn}(s_T - s(t_0))$. It follows from Lemma 6.5.3 that

$$
B_{ap}(q(t_1))v_i^a v_i^p - \frac{|\lambda_i|}{4} < B_{ap}(q(t))v_i^a v_i^p < B_{ap}(q(t_1))v_i^a v_i^p + \frac{|\lambda_i|}{4}
$$

for all $t \in [t_1, t_2']$. Futhermore, we have

$$
B_{ap}(q(t_0))v_i^a v_i^p - \frac{|\lambda_i|}{2} < B_{ap}(q(t))v_i^a v_i^p < B_{ap}(q(t_0))v_i^a v_i^p + \frac{|\lambda_i|}{2}
$$

for all $t \in [t_1, t_2']$ which is equivalent to

$$
\lambda_i - \frac{|\lambda_i|}{2} < B_{ap}(q(t))v_i^a v_i^p < \lambda_i + \frac{|\lambda_i|}{2}.
$$

This implies that

$$
\operatorname{sgn}(B_{ap}(q(t))Av_i^a Av_i^p) = \operatorname{sgn}(\lambda_i) \tag{6.9}
$$

for all $t \in [t_1, t_2']$.

$\square$

Lemma 6.5.5. *If conditions $(i)-(iii)$ of Lemma 6.5.4 hold for all $t \in [t_1, t_2']$ then* *there exists $t_2 < t_2'$ such that $s(t_2) = s_T$.*

Proof of Lemma 6.5.5. Conditions (i) and (iii) of Lemma 6.5.4 ensure that the $\mathrm{sgn}(\dot{s}(t)) = \mathrm{sgn}(s_T - s(t_1))$ for all $t \in [t_1, t_2']$. We need to consider two possible cases. In case one, we have $s_T > s(t_1)$. This implies that the $\mathrm{sgn}(\dot{s}(t))$ is positive for all $t \in [t_1, t_2']$. Therefore, $s(t)$ is monotonically increasing over the interval $[t_1, t_2']$. It follows from the continuity of $s(t)$ on $[t_1, t_2']$ and condition (ii) of Lemma 6.5.4 that $s(t)$ will travel far enough such that it passes through s_T for some $t_2 < t_2'$. In case two, we have $s(t_1) > s_T$. This implies that the $\mathrm{sgn}(\dot{s}(t))$ is negative for all $t \in [t_1, t_2']$. Therefore, $s(t)$ is monotonically decreasing over the interval $[t_1, t_2']$. It follows from the continuity of $s(t)$ on $[t_1, t_2']$ and condition (ii) of Lemma 6.5.4 that $s(t)$ will travel far enough such that it passes through s_T for some $t_2 < t_2'$. $\square$

6.6 Velocity to Velocity Algorithm

In this section we apply our theoretical results to our motivating examples. In addition, we provide several numerical simulations of the velocity to velocity control law that follows from our constructive proofs of Theorem 6.1.2, Theorem 6.1.3 and Theorem 6.1.4. Our piecewise control takes the form

$$u^a(t) = \begin{cases} \frac{Av_i^a - w^a(t_0)}{t_1 - t_0}, & \text{if } t \in [t_0, t_1) \\ 0, & \text{if } t \in [t_1, t_2) \\ \frac{w_T^a - Av_i^a}{T - t_2}, & \text{if } t \in [t_2, T]. \end{cases}$$

The proof of Theorem 6.1.2 provides details on the choice of parameters A and v_i when $B(q(t_0))$ is indefinite while Theorem 6.1.3 and Theorem 6.1.4 indicate the choice of v_i for the positive and negative definite cases. In addition, Lemma 6.2.1, Lemma 6.3.1 and Lemma 6.5.1 guide the choice of t_1, t_2 and T. The simulations serve to illustrate the constructive nature of our results. Note that though our results allow us to prescribe a bound on configuration $q(t)$, the plots of our numerical simulations are restricted to the time evolution of the actuated and unactuated velocity states. In each example, we prescribe $\epsilon = \alpha = \Delta = 0.1$.

6.6.1 Planar Rigid Body

Let us consider the planar rigid body with the control set $\{F^1, F^2\}$. Recall that the symmetric bilinear form is

$$\begin{pmatrix} 0 & -\frac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2m+J}\right)^{3/2}}{2\left(\frac{1}{m}\right)^{3/2}} \\ -\frac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2m+J}\right)^{3/2}}{2\left(\frac{1}{m}\right)^{3/2}} & 0 \end{pmatrix}.$$

The symmetric bilinear form is independent of the configuration and indefinite for all parameter values. It satisfies the sufficient conditions given in Theorem 6.1.2. Figure 6.1 and Figure 6.2 are simulations of the velocity to velocity algorithm for the planar rigid body given the parameter values $m = 1$, $h = 1$ and $J = 1$. We prescribe our constants to be $\epsilon = \alpha = \Delta = 0.1$.

Let us consider the planar rigid body with the control set $\{F^1, F^3\}$. Recall that the symmetric bilinear form is

$$\begin{pmatrix} 0 & -\frac{\sqrt{\frac{1}{J}}}{2} \\ -\frac{\sqrt{\frac{1}{J}}}{2} & 0 \end{pmatrix}.$$

The symmetric bilinear form is independent of the configuration and indefinite for all parameter values. It satisfies the sufficient conditions given in Theorem 6.1.2.

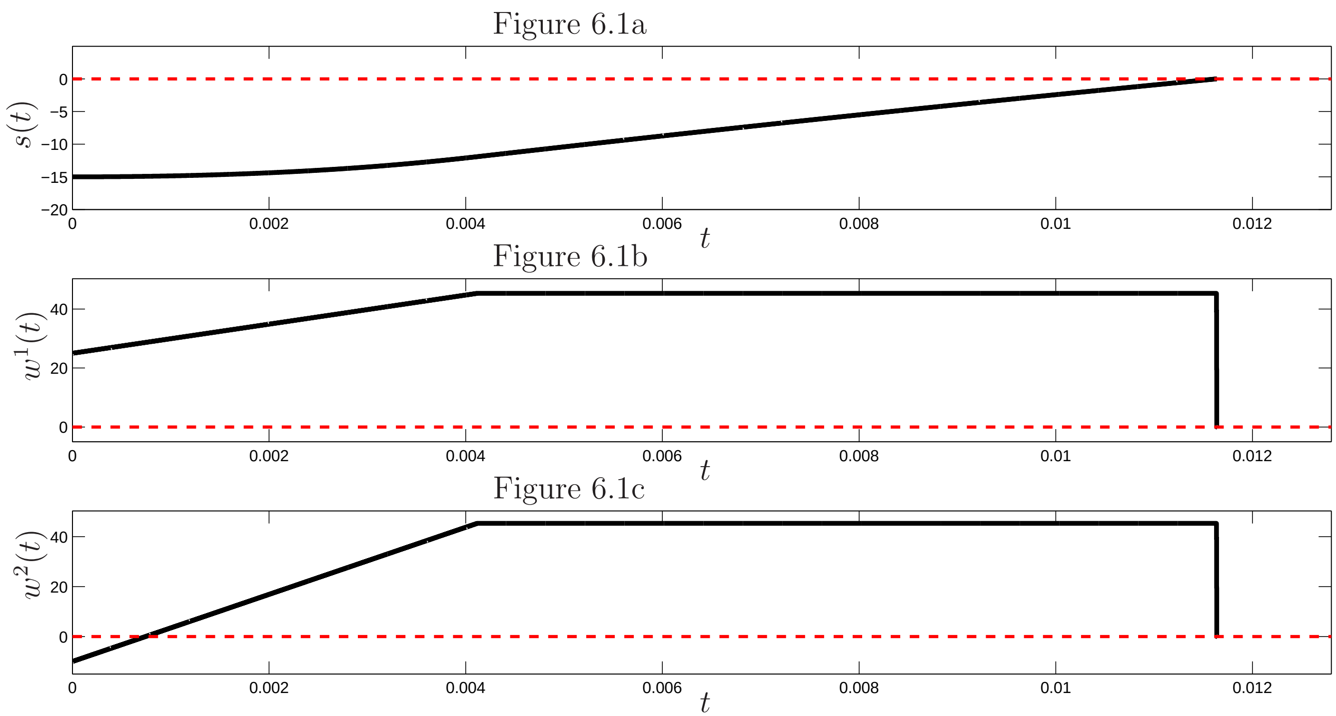

Figure 6.1: A simulation of the velocity to velocity algorithm for the planar rigid body. In each subplot, the trajectory of the velocity component is a solid line and the target velocity is a dashed line. Plot A displays the unactuated velocity component being driven from $s(t_0) = -15$ to $s(T) = 0$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 25$ to $w^1(T) = 0$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = 0$. Note that the instantaneous change in slope found in Plot B and C corresponds to switching between stages in the control law.

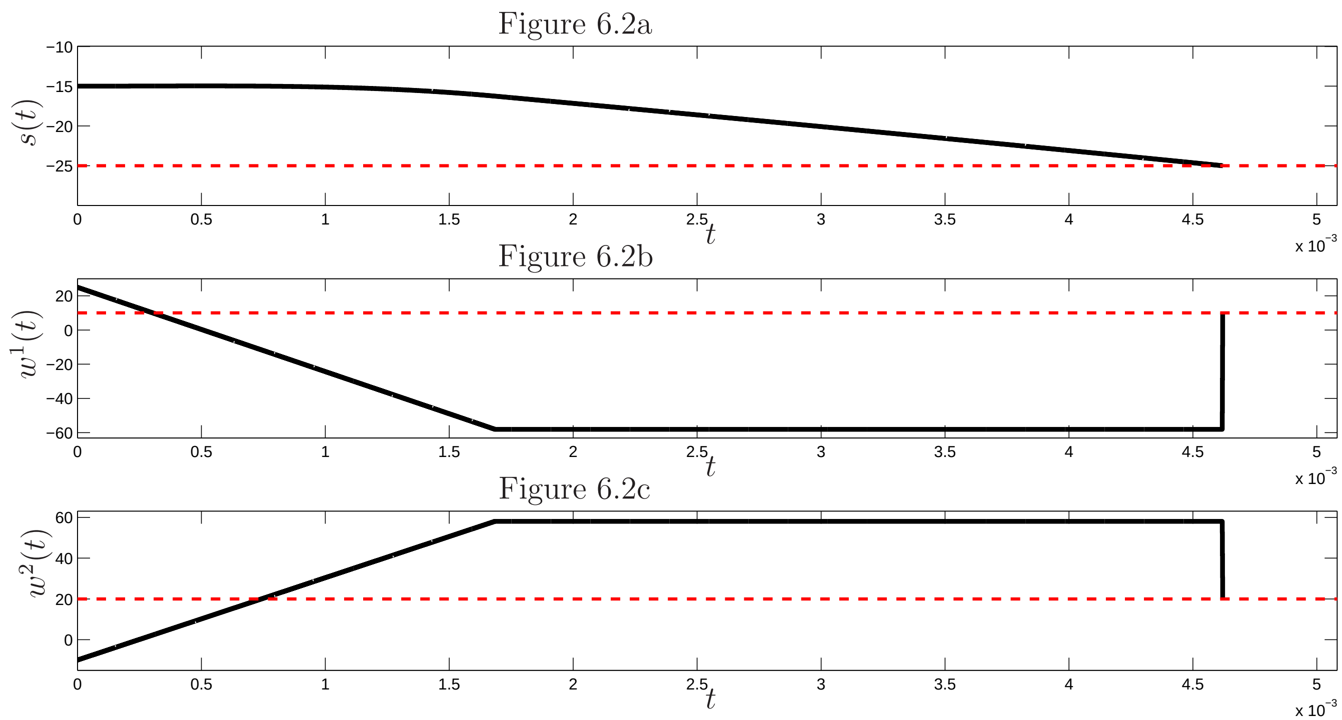

Figure 6.2: A simulation of the velocity to velocity algorithm for the planar rigid body. Plot A displays the unactuated velocity component being driven from $s(t_0) = -15$ to $s(T) = -25$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 25$ to $w^1(T) = 10$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = 20$.

Figure 6.3 and Figure 6.4 are simulations of the velocity to velocity algorithm for the planar rigid body given the parameter values $m = 1$, $h = 1$ and $J = 1$. We prescribe our constants to be $\epsilon = \Delta = 0.1$.

6.6.2 Roller Racer

Let us consider the roller racer with the control set $\{F^1\}$. Recall that the symmetric bilinear form is

$$B(\psi) = \frac{2m(L_1 + L_2\cos(\psi))(I_1L_2 - I_2L_1\cos(\psi))}{(L_1\cos(\psi) + L_2)\,(I_2\cos(2\psi)\,(L_1^2m - I_1) + I_1\,(I_2 + 2L_2^2m) + I_2L_1^2m)\,K(\psi)}.$$

The symmetric bilinear form depends on the configuration. We take the initial angle to be $\psi(t_0) = 0$. The symmetric bilinear form is negative definite at this configuration. We specify a target velocity whose unactuated velocity component is below the initial state. This satisfies the sufficient conditions given in Theorem 6.1.4. Figure 6.5 is a simulation of the velocity to velocity algorithm for the roller racer given the parameter values $m = 10$, $L1 = 1$, $L2 = 2$, $I1 = 10$, and $I2 = 1$. We prescribe our constants to be $\epsilon = \alpha = \Delta = 0.1$.

6.6.3 Snakeboard

Let us consider the snakeboard with the control set $\{F^1, F^2\}$. Recall that the symmetric bilinear form is

$$\begin{pmatrix} 0 & -\frac{J_r\sqrt{\frac{1}{J_w}}\cos(\phi)\sqrt{\frac{l^2m}{2J_r^2\cos(2\phi)-2J_r^2+4J_rl^2m}}}{\sqrt{l^2m}} \\ -\frac{J_r\sqrt{\frac{1}{J_w}}\cos(\phi)\sqrt{\frac{l^2m}{2J_r^2\cos(2\phi)-2J_r^2+4J_rl^2m}}}{\sqrt{l^2m}} & 0 \end{pmatrix}.$$

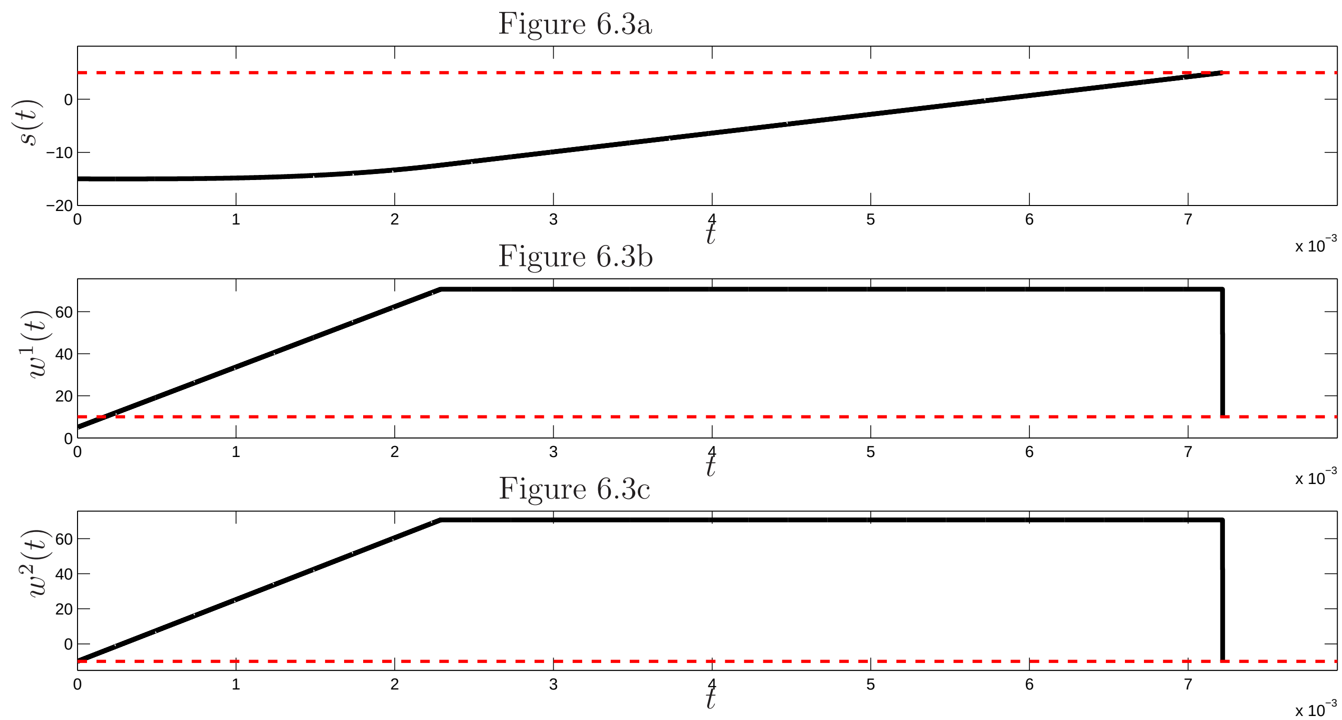

Figure 6.3: A simulation of the velocity to velocity algorithm for the planar rigid body. Plot A displays the unactuated velocity component being driven from $s(t_0) = -15$ to $s(T) = 5$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 5$ to $w^1(T) = 10$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = -10$.

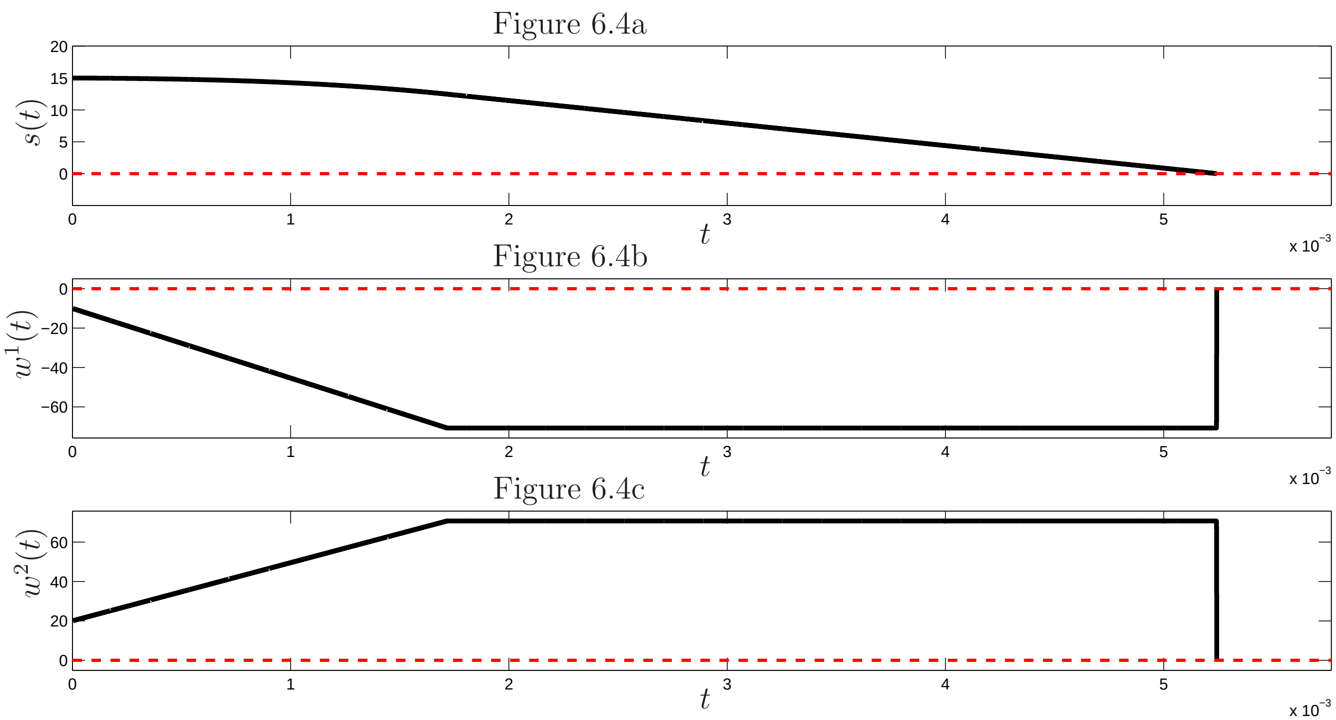

Figure 6.4: A simulation of the velocity to velocity algorithm for the planar rigid body. Plot A displays the unactuated velocity component being driven from $s(t_0) = 15$ to $s(T) = 0$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = -10$ to $w^1(T) = 0$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = 20$ to $w^2(T) = 0$.

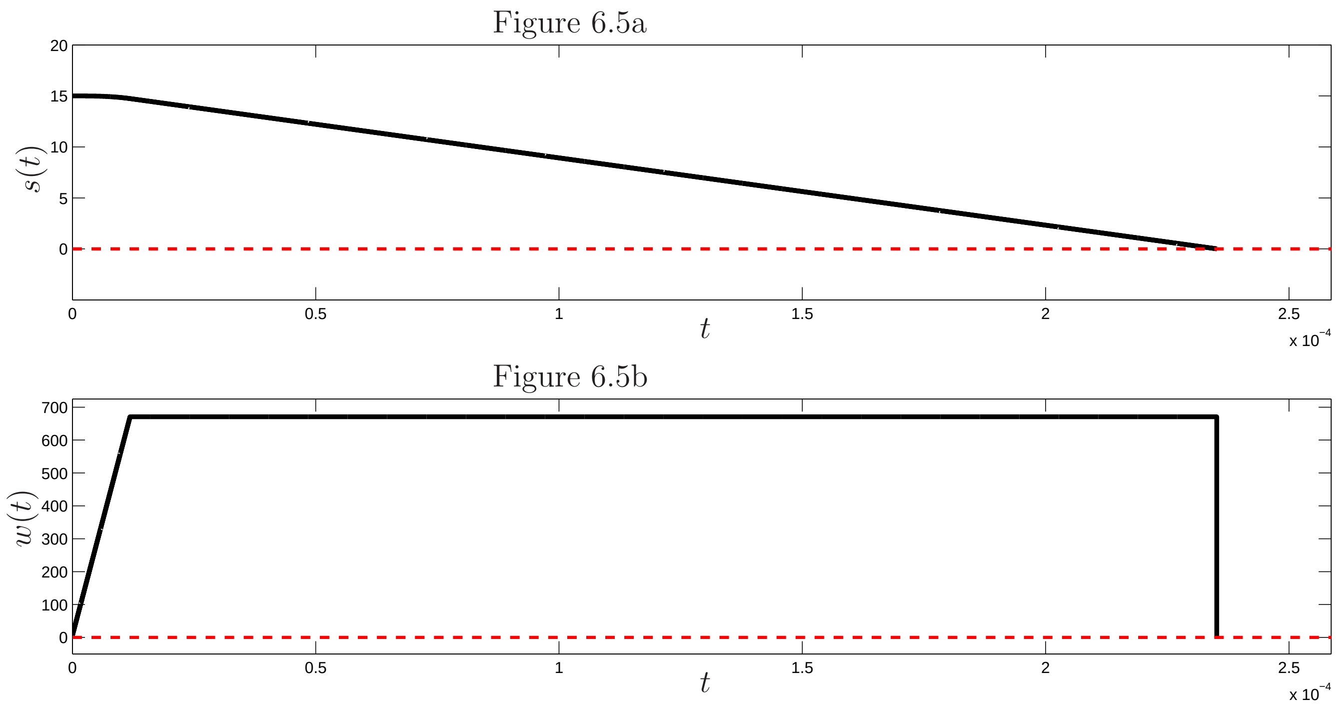

Figure 6.5: A simulation of the velocity to velocity algorithm for the roller racer. Plot A displays the unactuated velocity component being driven from $s(t_0) = 15$ to $s(T) = 0$. Plot B displays the actuated velocity component being driven from $w^1(t_0) = 5$ to $w^1(T) = 0$.

The symmetric bilinear form depends ont the configuration ϕ. The symmetric bilinear form is indefinite for all parameter values and values of ϕ away from $\{\frac{\Pi}{2}, -\frac{\Pi}{2}, \frac{3\Pi}{2}, -\frac{3\Pi}{2}\}$. It satisfies the sufficient conditions given in Theorem 6.1.2. Figure 6.6 and Figure 6.7 are simulations of the velocity to velocity algorithm for the snakeboard given the parameter values $m = 1$, $l = 1$, $J_r = 1$ and $J_w = 1$. We prescribe our constants to be $\epsilon = \alpha = \Delta = 0.1$.

6.6.4 Three Link Manipulator

Let us consider the three link manipulator with the control set $\{F^1, F^2\}$. Recall that the symmetric bilinear form is

$$\begin{pmatrix} B_{11}(\theta) & B_{12}(\theta) \\ B_{21}(\theta) & B_{22}(\theta) \end{pmatrix}$$

where

$$B_{11}(\theta) = -B_{22}(\theta) = \frac{L^2 m \sin(2\theta)\sqrt{\frac{1}{I_c + L^2 m}}}{2I_c - L^2 m \cos(2\theta) + L^2 m}$$

and

$$B_{12}(\theta) = B_{21}(\theta) = \frac{L^2\left(\frac{1}{I_c+L^2m}\right)^{3/2}\sqrt{\frac{I_c+L^2m}{m\left(2I_c-L^2m\cos(2\theta)+L^2m\right)}}\left(L^2m(\cos(4\theta)+3)\left(2I_c+L^2m\right)-4\cos(2\theta)\left(2I_c^2+2I_cL^2m+L^4m^2\right)\right)}{4I_c^2\left(\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}\right)^{3/2}}.$$

The symmetric bilinear form depends on the configuration θ. However, it is indefinite for all parameter values and all θ. It satisfies the sufficient conditions given in Theorem 6.1.2. Figure 6.8 and Figure 6.9 are simulations of the velocity to velocity algorithm for the three link manipulator given the parameter values $m = 1$, $L = 1$ and $I_c = 1$. We prescribe our constants to be $\epsilon = \alpha = \Delta = 0.1$.

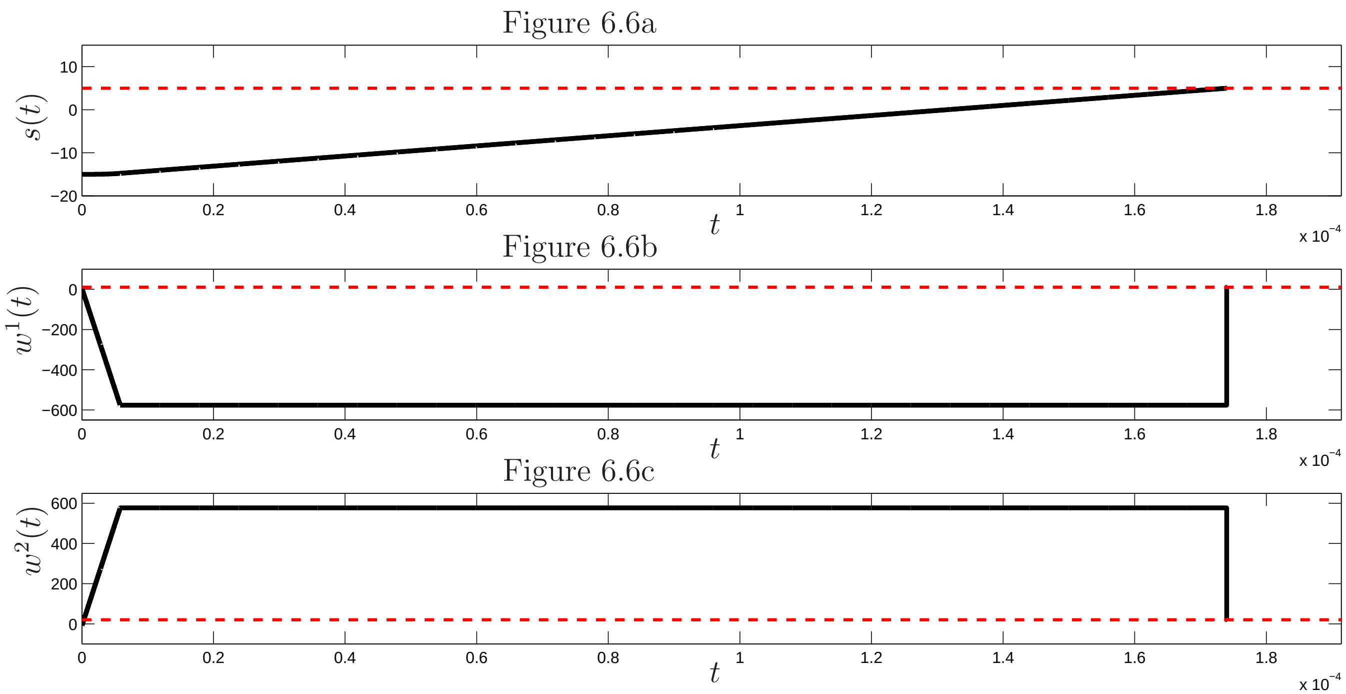

Figure 6.6: A simulation of the velocity to velocity algorithm for the snakeboard. Plot A displays the unactuated velocity component being driven from $s(t_0) = -15$ to $s(T) = 5$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 5$ to $w^1(T) = 10$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = 20$.

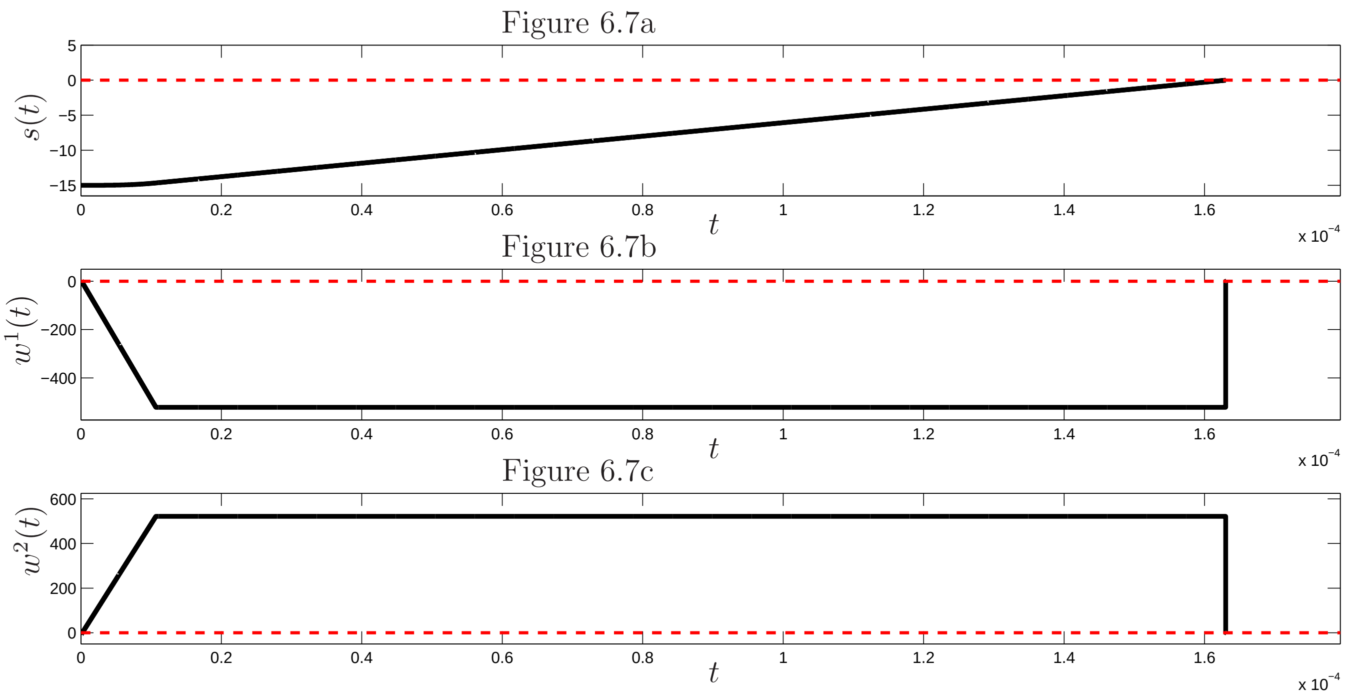

Figure 6.7: A simulation of the velocity to velocity algorithm for the snakeboard. Plot A displays the unactuated velocity component being driven from $s(t_0) = -15$ to $s(T) = 0$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 5$ to $w^1(T) = 0$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = 0$.

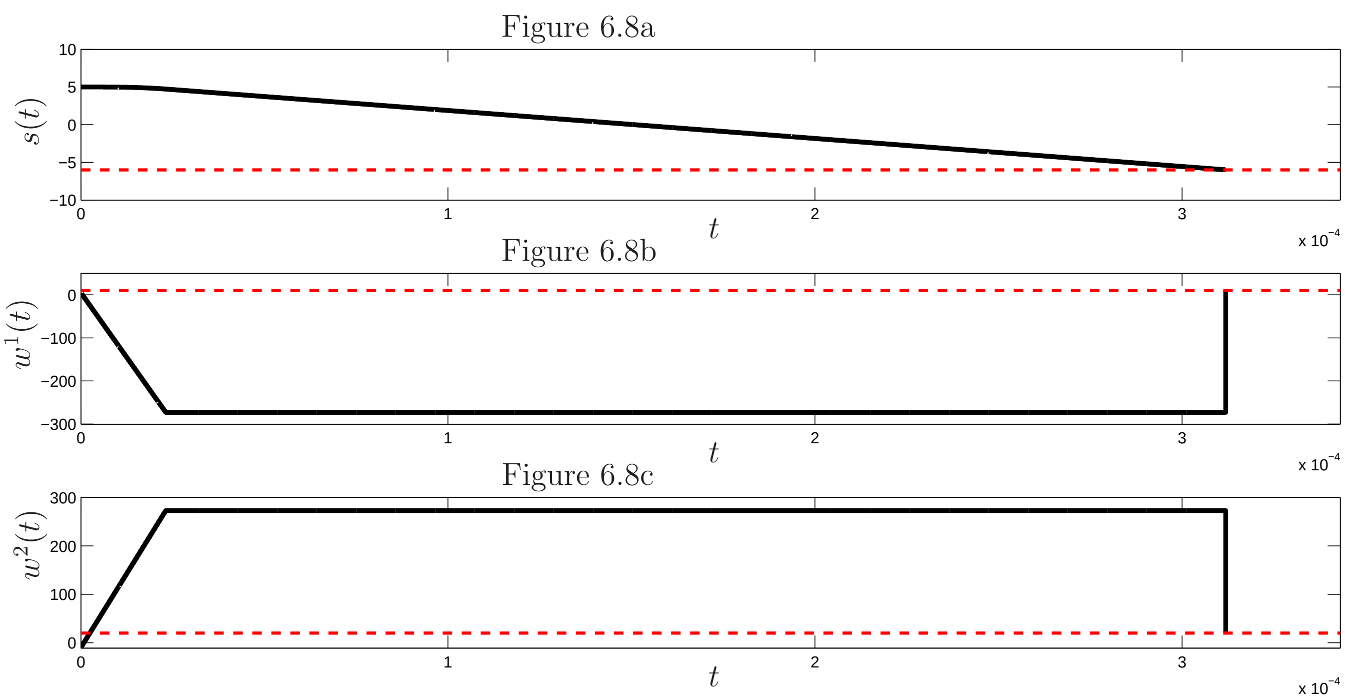

Figure 6.8: A simulation of the velocity to velocity algorithm for the three link manipulator. Plot A displays the unactuated velocity component being driven from $s(t_0) = 5$ to $s(T) = -6$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 5$ to $w^1(T) = 10$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = 20$.

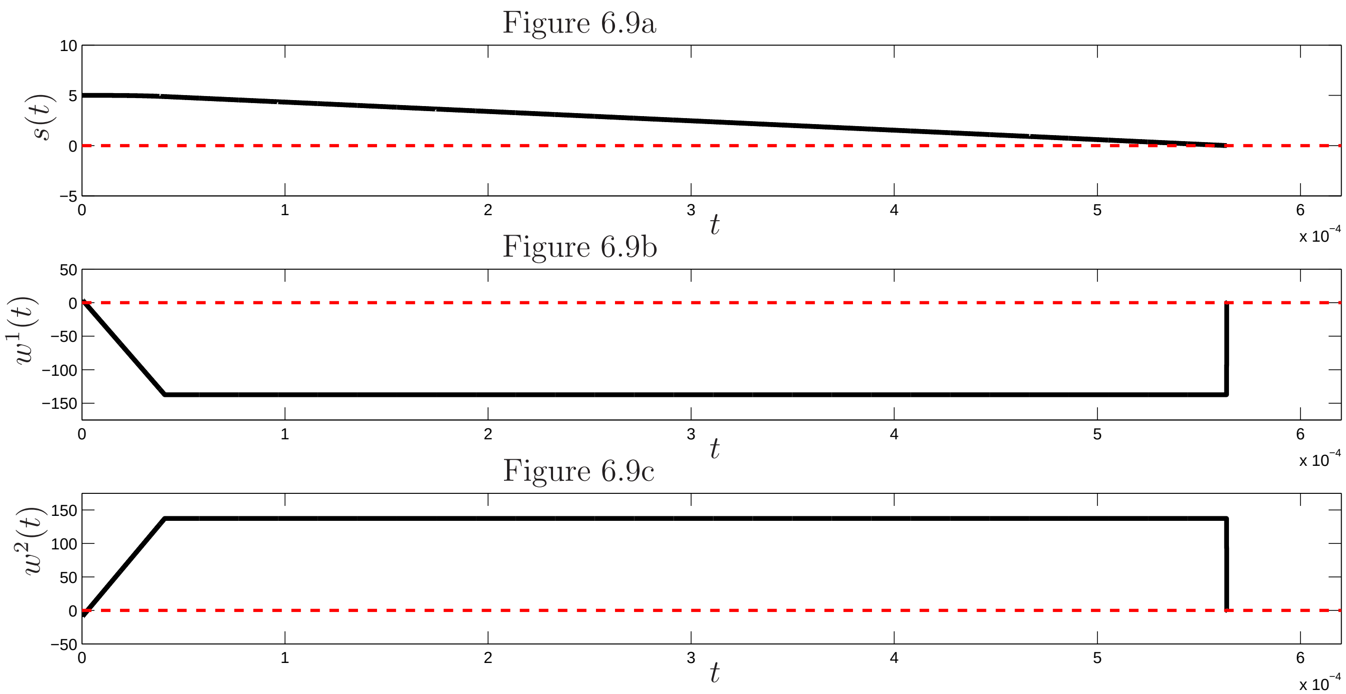

Figure 6.9: A simulation of the velocity to velocity algorithm for the three link manipulator. Plot A displays the unactuated velocity component being driven from $s(t_0) = 5$ to $s(T) = 0$. Plot B displays the first actuated velocity component being driven from $w^1(t_0) = 5$ to $w^1(T) = 0$. Plot C displays the second actuated velocity component being driven from $w^2(t_0) = -10$ to $w^2(T) = 0$.

CHAPTER 7

CONCLUSIONS

In this thesis we have presented contributions to modeling, analysis and control of underactuated mechanical systems. Specifically, we introduce two alternative refinements of the basic geometric framework for mechanical control systems. Our geometric models account for the additional structure resulting from the underactuated nature of this class of mechanical control systems. The key feature of these models is a general partitioning of the actuated and unactuated dynamics. We introduce a general feedback linearization control law for the actuated dynamics that gives rise to a linear subsystem. We characterize the coupling between the linearized subsystem and the unactuated dynamics using a symmetric bilinear form. Our main analytic result is a theorem on velocity reachability for mechanical systems underactuated by one control. The sufficient conditions for this theorem depends on the definiteness of the symmetric bilinear form. A significant advantage of this result is that the formulation is still valid for the extended class of underactuated mechanical systems with linear velocity constraints. A natural consequence of the constructive proof of our main result is a velocity to velocity algorithm. The general algorithm can be applied to a large class of systems such as the forced planar rigid body, roller racer, snakeboard, and three link manipulator.

7.1 Future Work

This section provides potential future directions of research in continuation of this work.

7.1.1 Discrete Underactuated Mechanical Control Systems

We propose the construction of a systematic framework for modeling, analysis, control and simulation of discrete mechanical control systems which combines numerical and differential geometric techniques. We have begun to develop a numerical approximation of a continuous mechanical system and extend our stopping algorithm to this general class of underactuated systems. The planar ice skater is a nontrivial example of an underactuated mechanical system. This problem is not unique to our method, rather, the vast majority of existing tests used to motivate and implement motion planning algorithms for mechanical systems require symbolic computations that do not scale well with the increase in dimensionality of the system or the existence of multiple constraints. The coordinate invariant results mask the necessary computations. These computations often generate results which cannot be easily interpreted or in the worse case cannot be fully computed. This severely limits the applicability of the aforementioned methods to relatively low dimensional systems. An alternative approximate technique motivated by and consistent with the underlying geometric framework would aid in the practical implementation of the existing analytic tools.

7.1.2 Hybrid Mechanical Control Systems

The governing physics of a hybrid mechanical control system impose a stratified structure on the tangent bundle of the configuration manifold. A stratification naturally occurs when changes in the dynamics arise from switches in the constraints (holonomic or nonholonomic) describing the interaction between the mechanical system and the environment [26]. Typical representatives of this general class of mechanical systems include legged locomotion, grasping devices and skidding wheels. Mechanical systems that switch between constraints at specified boundaries of the tangent bundle cannot be analyzed using methods derived for smooth mechanical systems. Additionally, current results that do account for the geometric structure are limited to systems that switch between constraints at arbitrary configurations and zero velocity [14]. These results are not applicable to legged locomotion because switching occurs at nonzero velocity when the leg hits the ground.

7.1.3 Mechanical Systems Underactuated by More Than One Control

We have been able to show that real-valued symmetric (quadratic) forms play a critical role in the velocity reachability analysis in the nonzero velocity setting for mechanical systems underactuated by one control [50], [51]. The definiteness of the form can be used as necessary and sufficient conditions for velocity reachability results. However, the computational tests for definiteness of a vector-valued symmetric form, which can be associated with a mechanical system underactuated by more than one control, are known to be complex [15]. It has been observed that computational complexity is an unresolved problem in general nonlinear control systems [5]. Several efforts have been made to obtain conditions in the zero velocity setting from properties of a certain intrinsic vector-valued quadratic form which does not depend upon the choice of basis for the input distribution [9], [31]. Recently, it has been observed that vector-valued quadratic forms come up in a variety of areas in control theory which have motivated a new initiative to understand the geometry of these forms [15].

APPENDIX A

PLANAR RIGID BODY

The coefficients in the actuated and unactuated dynamic equations for the planar rigid body with the control set $\{Y_1, Y_2\}$ are

$$\begin{array}{ll}
\mathbb{G}(\nabla_{X_1} X_2, X_2) & -\frac{h}{\sqrt{\frac{1}{m}}(h^2 m+J)} \\
\mathbb{G}(\nabla_{X_1} X_2, X_3) & -\frac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{\left(\frac{1}{m}\right)^{3/2}} \\
\mathbb{G}(\nabla_{X_1} X_3, X_2) & \frac{J\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{\sqrt{\frac{1}{m}}} \\
\mathbb{G}(\nabla_{X_1} X_3, X_3) & \frac{h}{\sqrt{\frac{1}{m}}(h^2 m+J)} \\
\mathbb{G}(\nabla_{X_2} X_2, X_1) & \frac{h}{\sqrt{\frac{1}{m}}(h^2 m+J)} \\
\mathbb{G}(\nabla_{X_2} X_3, X_1) & -\frac{J\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{\sqrt{\frac{1}{m}}} \\
\mathbb{G}(\nabla_{X_3} X_2, X_1) & \frac{h^2\sqrt{\frac{h^2}{J}+\frac{1}{m}}\left(\frac{1}{h^2 m+J}\right)^{3/2}}{\left(\frac{1}{m}\right)^{3/2}} \\
\mathbb{G}(\nabla_{X_3} X_3, X_1) & -\frac{h}{\sqrt{\frac{1}{m}}(h^2 m+J)}
\end{array}$$

The coefficients in the actuated and unactuated dynamic equations for the planar rigid body with the control set $\{Y_1, Y_3\}$ are

$$\begin{array}{ll}
\mathbb{G}(\nabla_{X_1} X_2, X_3) & \sqrt{\frac{1}{J}} \\
\mathbb{G}(\nabla_{X_3} X_2, X_1) & -\sqrt{\frac{1}{J}}
\end{array}$$

APPENDIX B

ROLLER RACER

The first orthonormal basis vector field is

$$H_{1o} = \begin{pmatrix} \dfrac{\cos(\theta)}{\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ \dfrac{\sin(\theta)}{\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ \dfrac{\sin(\psi)}{(L_1\cos(\psi)+L_2)\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ 0 \end{pmatrix}.$$

The second orthonormal basis vector field is

$$H_{2o} = \begin{pmatrix} \dfrac{2\cos(\theta)\sin(\psi)(I_2L_1\cos(\psi)-I_1L_2)\sqrt{-\frac{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m}{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}}}{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m} \\ \dfrac{2\sin(\theta)\sin(\psi)(I_2L_1\cos(\psi)-I_1L_2)\sqrt{-\frac{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m}{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}}}{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m} \\ \dfrac{\left(-I_2\cos(2\psi)+I_2+2L_1L_2m\cos(\psi)+2L_2^2m\right)\sqrt{-\frac{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m}{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}}}{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m} \\ \dfrac{1}{\sqrt{-\frac{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m}{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}}} \end{pmatrix}.$$

The control vector field projected onto the constraint distribution $\mathcal{H}$

$$\overset{\mathcal{H}}{Y}_1 = \left(\begin{array}{c} 0 \\ \dfrac{1}{\sqrt{-\frac{I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m}{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}}}. \end{array} \right) \tag{B.1}$$

The nonzero generalized Christoffel symbols associated with the constrained connection are

$$\begin{array}{ll}
\widehat{\Gamma}^1_{12} & \dfrac{(I_1+I_2)\sin(\psi)(L_1+L_2\cos(\psi))}{(L_1\cos(\psi)+L_2)^3} \\
\widehat{\Gamma}^2_{12} & \dfrac{2m(L_1+L_2\cos(\psi))(I_2L_1\cos(\psi)-I_1L_2)}{(L_1\cos(\psi)+L_2)\left(-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m\right)} \\
\widehat{\Gamma}^1_{22} & \dfrac{2m(L_1+L_2\cos(\psi))(I_1L_2-I_2L_1\cos(\psi))}{(L_1\cos(\psi)+L_2)\left(-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m\right)} \\
\widehat{\Gamma}^2_{22} & \dfrac{4m\sin(\psi)(I_1L_2-I_2L_1\cos(\psi))\left(L_2\cos(\psi)\left(L_1^2m-I_1\right)+L_1\left(I_2+L_2^2m\right)\right)}{\left(-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m\right)^2}.
\end{array}$$

The constrained $\mathbb{G}$-orthonormal frame is $\{\overset{\mathcal{H}}{X}_1, \overset{\mathcal{H}}{X}_2\}$ where

$$\overset{\mathcal{H}}{X}_1 = \left(\begin{array}{c}
\dfrac{2\cos(\theta)\sin(\psi)(I_2L_1\cos(\psi)-I_1L_2)}{\left(I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m\right)\sqrt{\frac{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}{I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m}}} \\
\dfrac{2\sin(\theta)\sin(\psi)(I_2L_1\cos(\psi)-I_1L_2)}{\left(I_2\cos(2\psi)\left(I_1-L_1^2m\right)-I_1\left(I_2+2L_2^2m\right)-I_2L_1^2m\right)\sqrt{\frac{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}{I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m}}} \\
\dfrac{I_2\cos(2\psi)-I_2-2L_1L_2m\cos(\psi)-2L_2^2m}{\sqrt{\frac{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}{I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m}}\left(I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m\right)} \\
\sqrt{\dfrac{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}{I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m}}
\end{array} \right),$$

and

$$\overset{\mathcal{H}}{X}_2 = \begin{pmatrix} \frac{\cos(\theta)}{\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ \frac{\sin(\theta)}{\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ \frac{\sin(\psi)}{(L_1\cos(\psi)+L_2)\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ 0 \end{pmatrix}.$$

It will be convenient to introduce the following term

$$C(\psi) \;=\; \sqrt{\frac{-\cos(2\psi)\left(I_1+I_2+L_1^2(-m)\right)+I_1+I_2+L_1^2m+4L_1L_2m\cos(\psi)+2L_2^2m}{I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m}}$$

The coefficients in the constrained actuated and unactuated dynamic equations for the roller race with the control set $\{\overset{\mathcal{H}}{Y}_1\}$ are

$$\begin{array}{ll} \mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_1}\overset{\mathcal{H}}{X}_1,\overset{\mathcal{H}}{X}_2) & \frac{2m(L_1+L_2\cos(\psi))(I_1L_2-I_2L_1\cos(\psi))}{(L_1\cos(\psi)+L_2)\left(I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m\right)\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \\ \mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_2}\overset{\mathcal{H}}{X}_1,\overset{\mathcal{H}}{X}_1) & \frac{2m(L_1+L_2\cos(\psi))(I_2L_1\cos(\psi)-I_1L_2)}{(L_1\cos(\psi)+L_2)\left(I_2\cos(2\psi)\left(L_1^2m-I_1\right)+I_1\left(I_2+2L_2^2m\right)+I_2L_1^2m\right)\sqrt{\frac{(I_1+I_2)\sin^2(\psi)}{(L_1\cos(\psi)+L_2)^2}+m}} \end{array}$$

APPENDIX C

SNAKEBOARD

The first orthonormal basis vector field is

$$H_{1o} = \begin{pmatrix} \frac{l\cos(\theta)\cos(\phi)}{\sqrt{l^2 m}} \\ \frac{l\sin(\theta)\cos(\phi)}{\sqrt{l^2 m}} \\ -\frac{\sin(\phi)}{\sqrt{l^2 m}} \\ 0 \\ 0 \end{pmatrix}.$$

The second orthonormal basis vector field is

$$H_{2o} = \begin{pmatrix} \frac{\sqrt{2}J_r\cos(\theta)\sin(\phi)\cos(\phi)}{lm\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ \frac{\sqrt{2}J_r\sin(\theta)\sin(\phi)\cos(\phi)}{lm\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ -\frac{\sqrt{2}J_r\sin^2(\phi)}{l^2m\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ \frac{\sqrt{2}}{\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ 0 \end{pmatrix}.$$

The third orthonormal basis vector field is

$$H_{2o} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{J_w}} \end{pmatrix}.$$

The first control vector field projected onto the constraint distribution $\mathcal{H}$ is

$$\overset{\mathcal{H}}{Y}_1 = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ 0 \end{pmatrix}. \tag{C.1}$$

The second control vector field projected onto the constraint distribution $\mathcal{H}$ is

$$\overset{\mathcal{H}}{Y}_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{J_w}} \end{pmatrix}. \tag{C.2}$$

The nonzero generalized Christoffel symbols associated with the constrained connection are

$$\begin{array}{ll} \widehat{\Gamma}^2_{13} & -\frac{\sqrt{2}J_r\cos(\phi)}{\sqrt{J_w}\sqrt{l^2m}\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\ \widehat{\Gamma}^1_{23} & \frac{\sqrt{2}J_r\cos(\phi)}{\sqrt{J_w}\sqrt{l^2m}\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \end{array}.$$

The nonzero coefficients in the actuated and unactuated dynamic equations

are

$$
\begin{array}{ll}
\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_1}\overset{\mathcal{H}}{X}_2,\overset{\mathcal{H}}{X}_3) & \dfrac{\sqrt{2}J_r\cos(\phi)}{\sqrt{J_w}\sqrt{l^2m}\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}} \\
\mathbb{G}(\nabla_{\overset{\mathcal{H}}{X}_3}\overset{\mathcal{H}}{X}_2,\overset{\mathcal{H}}{X}_1) & -\dfrac{\sqrt{2}J_r\cos(\phi)}{\sqrt{J_w}\sqrt{l^2m}\sqrt{\frac{J_r\left(J_r\cos(2\phi)-J_r+2l^2m\right)}{l^2m}}}
\end{array}
$$

APPENDIX D

THREE LINK MANIPULATOR

The first control vector field is

$$Y_1 = \begin{pmatrix} \frac{I_c m - L^2 m^2 \cos^2(\theta) + L^2 m^2}{I_c m^2 - L^2 m^3 \sin^2(\theta) - L^2 m^3 \cos^2(\theta) + L^2 m^3} \\ -\frac{L^2 m^2 \sin(\theta)\cos(\theta)}{I_c m^2 - L^2 m^3 \sin^2(\theta) - L^2 m^3 \cos^2(\theta) + L^2 m^3} \\ \frac{L m^2 \sin(\theta)}{I_c m^2 - L^2 m^3 \sin^2(\theta) - L^2 m^3 \cos^2(\theta) + L^2 m^3} \end{pmatrix}.$$

The second control vector field is

$$Y_2 = \begin{pmatrix} -\frac{L^2 m^2 \sin(\theta)\cos(\theta)}{I_c m^2 - L^2 m^3 \sin^2(\theta) - L^2 m^3 \cos^2(\theta) + L^2 m^3} \\ \frac{I_c m - L^2 m^2 \sin^2(\theta) + L^2 m^2}{I_c m^2 - L^2 m^3 \sin^2(\theta) - L^2 m^3 \cos^2(\theta) + L^2 m^3} \\ -\frac{L m^2 \cos(\theta)}{I_c m^2 - L^2 m^3 \sin^2(\theta) - L^2 m^3 \cos^2(\theta) + L^2 m^3} \end{pmatrix}.$$

The nonzero Christoffel symbols associated with the Levi-Civita connection are

$$\begin{array}{ll} \Gamma^3_{13} & -L\cos(\theta) \\ \Gamma^3_{23} & -L\sin(\theta) \end{array}.$$

The nonzero coefficients of the actuated and unactuated dynamic equations

are

$$
\begin{array}{ll}
\mathbb{G}(\nabla_{X_1}X_1, X_2) & -\dfrac{2L^3m\sin(\theta)\cos^2(\theta)}{(2I_c-L^2m\cos(2\theta)+L^2m)^2\sqrt{\frac{I_c+L^2m}{4I_cm-2L^2m^2\cos(2\theta)+2L^2m^2}}} \\
\mathbb{G}(\nabla_{X_1}X_1, X_3) & \dfrac{L^2m\sin(2\theta)\sqrt{\frac{1}{I_c+L^2m}}}{2I_c-L^2m\cos(2\theta)+L^2m} \\
\mathbb{G}(\nabla_{X_1}X_2, X_2) & \dfrac{2\sqrt{2}L^3\cos^3(\theta)}{I_c(I_c+L^2m)\left(\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}\right)^{3/2}} \\
\mathbb{G}(\nabla_{X_1}X_2, X_3) & -\dfrac{2L^2m\cos^2(\theta)\left(\frac{1}{I_c+L^2m}\right)^{3/2}\sqrt{\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}}}{\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}}} \\
\mathbb{G}(\nabla_{X_1}X_3, X_2) & -\dfrac{2L^2m\cos^2(\theta)\left(\frac{1}{I_c+L^2m}\right)^{3/2}\sqrt{\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}}}{\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}}} \\
\mathbb{G}(\nabla_{X_1}X_3, X_3) & \dfrac{\sqrt{2}L\cos(\theta)}{(I_c+L^2m)\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}}} \\
\mathbb{G}(\nabla_{X_2}X_1, X_1) & \dfrac{2L^3m\sin(\theta)\cos^2(\theta)}{(2I_c-L^2m\cos(2\theta)+L^2m)^2\sqrt{\frac{I_c+L^2m}{4I_cm-2L^2m^2\cos(2\theta)+2L^2m^2}}} \\
\mathbb{G}(\nabla_{X_2}X_1, X_3) & 2L^2m^3\sin^2(\theta)\left(\frac{1}{I_c+L^2m}\right)^{3/2}\left(\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}\right)^{3/2}\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}} \\
\mathbb{G}(\nabla_{X_2}X_2, X_1) & -\dfrac{2\sqrt{2}L^3\cos^3(\theta)}{I_c(I_c+L^2m)\left(\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}\right)^{3/2}} \\
\mathbb{G}(\nabla_{X_2}X_2, X_3) & \dfrac{L^2m\sin(2\theta)\sqrt{\frac{1}{I_c+L^2m}}}{-2I_c+L^2m\cos(2\theta)+L^2(-m)} \\
\mathbb{G}(\nabla_{X_2}X_3, X_1) & \dfrac{2L^2m\cos^2(\theta)\left(\frac{1}{I_c+L^2m}\right)^{3/2}\sqrt{\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}}}{\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}}} \\
\mathbb{G}(\nabla_{X_2}X_3, X_3) & \dfrac{\sqrt{2}Lm\sin(\theta)\sqrt{\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}}}{I_c+L^2m} \\
\mathbb{G}(\nabla_{X_3}X_1, X_1) & \dfrac{L^2m\sin(2\theta)\sqrt{\frac{1}{I_c+L^2m}}}{-2I_c+L^2m\cos(2\theta)+L^2(-m)} \\
\mathbb{G}(\nabla_{X_3}X_1, X_2) & -2L^2m^3\sin^2(\theta)\left(\frac{1}{I_c+L^2m}\right)^{3/2}\left(\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}\right)^{3/2}\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}} \\
\mathbb{G}(\nabla_{X_3}X_2, X_1) & \dfrac{2L^2m\cos^2(\theta)\left(\frac{1}{I_c+L^2m}\right)^{3/2}\sqrt{\frac{I_c+L^2m}{m(2I_c-L^2m\cos(2\theta)+L^2m)}}}{\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}}} \\
\mathbb{G}(\nabla_{X_3}X_2, X_2) & \dfrac{L^2m\sin(2\theta)\sqrt{\frac{1}{I_c+L^2m}}}{2I_c-L^2m\cos(2\theta)+L^2m} \\
\mathbb{G}(\nabla_{X_3}X_3, X_1) & -\dfrac{\sqrt{2}L\cos(\theta)}{(I_c+L^2m)\sqrt{\frac{2I_c-L^2m\cos(2\theta)+L^2m}{I_cm}}} \\
\mathbb{G}(\nabla_{X_3}X_3, X_2) & -\dfrac{2Lm\sin(\theta)\sqrt{\frac{I_c+L^2m}{4I_cm-2L^2m^2\cos(2\theta)+2L^2m^2}}}{I_c+L^2m}
\end{array}
$$

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