1. From Newton’s law, we know that $I\ddot{\theta}$ is equal to the sum of the moments. Therefore,

$$I\ddot{\theta} = \tau - mgl \cos \theta.$$  

For small $\theta$, $\cos \theta \approx 1$. Thus

$$I\ddot{\theta} = \tau - mgl.$$  

Letting $I = 1$ and $mgl = 1$, the equation of motion is

$$\ddot{\theta} = \tau - 1.$$  

Converting to two first order equations where $x_1 = \theta$ and $x_2 = \dot{\theta}$ gives

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x_2.$$  

2. Implementing a PID controller:

(a) For proportional control,

$$\tau = k_p(\theta_d - \theta).$$  

Using `ode45()` in Matlab the following is saved as `pid.m`:

```matlab
function out = odefile(t,y,options,kp,kd,ki,td);
out = [y(2); -kd*y(2) - kp*y(1) + ki*y(3) + kp*td - 1; td - y(1)];
```

i. In Matlab, letting

```matlab
kp = 5;
kd = 0;
ki = 0;
td = 1;
options = odeset('OutputFcn', 'odeplot', 'OutputSel', [1]);
[t y] = ode45('pid', [0 20], [0 0 0], options, kp, kd, ki, td);
```
produced the result in Figure 1. Clearly the response is oscillatory and the average value is approximately $x_1 = 0.8$.

ii. Increasing $k_p = 10$ produced the result in Figure 2. In this case the solution is still oscillatory, but the average value has increased to $x_1 = 0.9$ and the frequency has increased. Since the mean steady state value has increased from 0.8 to 0.9, the steady state error for $\theta_d = 1$ has decreased.

(b) Adding derivative control introduces damping into the response.

i. Now we let $k_p = 5$ and $k_d = 0.5$. The response is illustrated in Figure 3.

ii. Increasing $k_d = 1$, produces the response in Figure 4, which has a reduced settling time compared to the case where $k_d = 0.5$.

iii. Increasing $k_d = 10$, produces the response in Figure 5, which displays no oscillatory response.

iv. Increasing $k_p = 10$ and keeping $k_d = 10$ produces the response in Figure 6, which has decreased steady state error.

v. Figure 6 also illustrates that the rise time decreases if $k_p$ is increased.

(c) Adding integral control eliminates the steady state error, so $\theta_d \rightarrow 1$ as $t \rightarrow \infty$.

i. For $k_p = 1$, $k_d = 1$ and $k_i = 0.5$, the response of the robot arm is illustrated in Figure 7. There is no steady state error.

ii. Increasing $k_i = 0.75$, produces the response illustrated in Figure 8, which displays increased overshoot and settling time.

iii. Letting $k_p = 10$, $k_d = 1$ and $k_i = 0.5$, compared to 7 for problem 2(c)i, Figure 9 illustrates that overshoot is increased from approximately 25% to approximately 50%.
Figure 2: Response for proportional control where $k_p = 10$.

Figure 3: Response for proportional plus derivative control where $k_p = 5$ and $k_d = 0.5$. 
Figure 4: Response for proportional plus derivative control where $k_p = 5$ and $k_d = 1$.

Figure 5: Response for proportional plus derivative control where $k_p = 5$ and $k_d = 10$. 
Figure 6: Response for proportional plus derivative control where $k_p = 10$ and $k_d = 1$.

Figure 7: Response for proportional plus derivative plus integral control where $k_p = 1$, $k_d = 1$ and $k_i = 0.5$. 

Figure 8: Response for proportional plus derivative plus integral control where $k_p = 1$, $k_d = 1$ and $k_i = 0.75$.

Figure 9: Response for proportional plus derivative plus integral control where $k_p = 10$, $k_d = 1$ and $k_i = 0.5$. 