UNIVERSITY OF NOTRE DAME
Aerospace and Mechanical Engineering

AME 437: Control Systems Engineering
Homework 5 Solutions

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1. (5.8.1:1) Plot the root locus for
$$GH(s) = \frac{K(s+2)}{s(s+1)}.$$
There are poles at $s = 0$ and $s = -1$. There is a zero at $s = -2$.

   (a) Rule 5.3.1: the locus lies on the real axis between the pole at $s = 0$ and the pole at $s = -1$, and to the left of the zero at $s = -2$.

   (b) Rule 5.3.4: there are two poles and one zero, so there is only one asymptote at $\theta = 180°$. We do not need to use the asymptote angle formula for this one since, for $\theta = 180°$, it is automatically satisfied by Rule 5.3.1.

   (c) Rule 5.3.2 and 5.3.3: we can conclude that the loci *must* break out between the two poles and break into the locus to the left of the zero. There is no other way for it to start at the poles and end at either the zero or ∞.

   (d) Rule 5.3.6: Since
$$K = \frac{s(s+1)}{s+2},$$
then
$$\frac{dK}{ds} = \frac{(2s+1)(s+2) - s(s+1)}{(s+2)^2}$$
so $\frac{dK}{ds} = 0 \Longrightarrow s = -0.58, -3.41$ are the break out and break in points, respectively.

   (e) Rule 5.3.8: we do not have to compute the departure or arrival angles. Rule 5.3.1 takes care of this for poles and zeros on the real axis.

   The root locus is illustrated in Figure 1, which can be checked with `rlocus([1 2],[1 1 0])` in Matlab.

2. (5.8.1:5) Plot the root locus for
$$GH(s) = \frac{10K(s+10)}{s(s+4)(s+20)}.$$
There are poles at $s = 0, -4, -20$ and one zero at $s = -10$.

   (a) Rule 5.3.1: on the real axis, the locus lies between the poles at $s = 0$ and $s = -4$ and between the zero at $s = -10$ and the pole at $s = -20$.

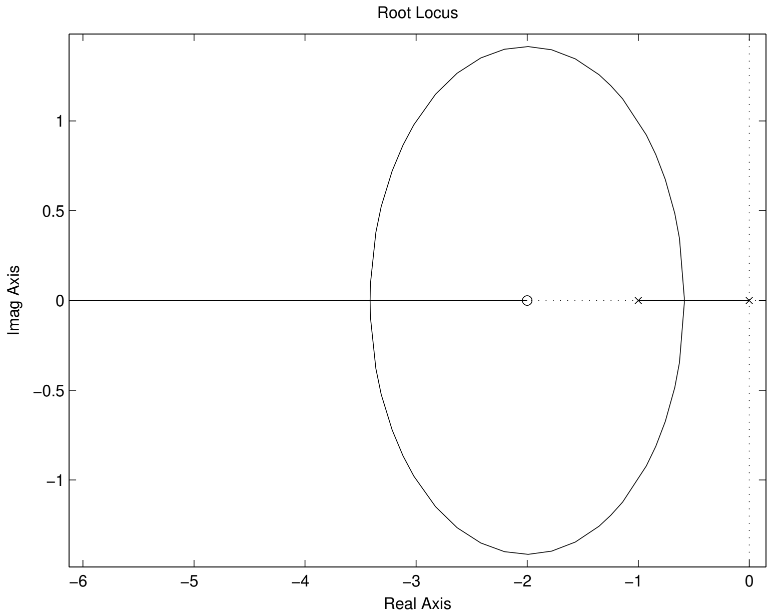

Figure 1. Root locus for 5.8.1:1.

(b) Rule 5.3.4: there are three poles and one zero, so there are two asymptotes

$$\begin{aligned}\theta_1 &= 90^\circ \\ \theta_2 &= 270^\circ\end{aligned}$$

which intersect the real axis at

$$\sigma = \frac{0 - 4 - 20 - (-10)}{2} = -7.$$

(c) There must be a break out point somewhere between $s = 0$ and $s = -4$. Since

$$K = \frac{s(s+4)(s+20)}{10(s+10)},$$

then

$$\frac{dK}{ds} = 0 \quad \Longrightarrow \quad s^3 + 27s^2 + 240s + 400 = 0 \quad \Longrightarrow \quad s = -2.14, -12.4 \pm 5.68i,$$

so the break out point is at $s = -2.14$.

(d) There is probably no need to use Routh's array to see if the locus crosses the imaginary axis since it is only a third order transfer function, it will most likely be attracted to the asymptotes after it breaks out.

The root locus is illustrated in Figure 2. Using `rlocus(10*[1 10],conv([1 4 0],[1 20]))` in Matlab verifies the result.

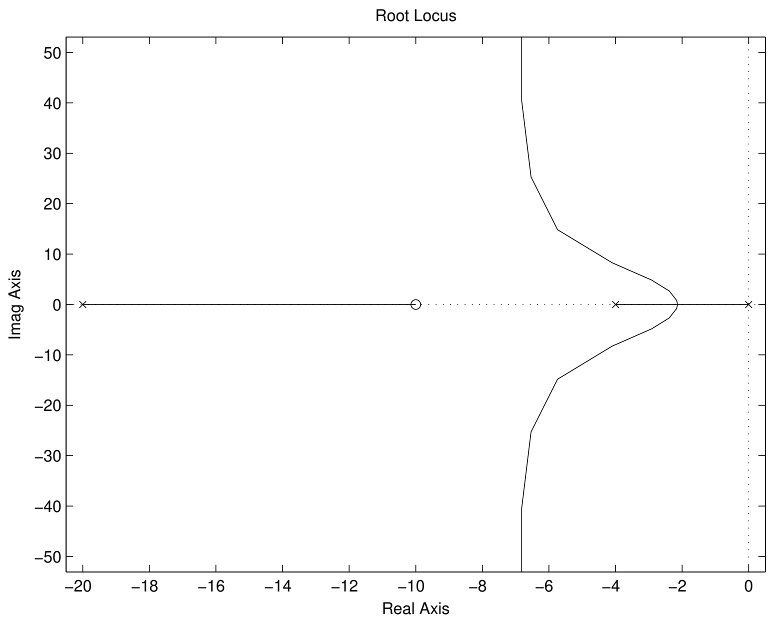

Figure 2. Root locus for 5.8.1:5.

3. (5.8.1:9) Plot the root locus for

$$GH(s) = \frac{K}{(s+2)(s+4)(s+10)}.$$

There are poles at $s = -2, -4, -10$ and no zeros.

(a) Rule 5.3.1: on the real axis, the locus lies between the poles at $s = -2$ and $s = -4$ and to the left of the pole at $s = -10$.

(b) Rule 5.3.4: there are three asymptotes. Their angles are 60°, 180° and 300°, and they intersect the real axis at

$$\sigma = \frac{-2-4-10}{3} = -5.33.$$

(c) Rule 5.3.6:

$$\frac{dK}{ds} = \frac{d}{ds}(s+2)(s+4)(s+10) = 0 \quad \Longrightarrow \quad s = -2.92, -7.74.$$

The first is the break out point between the poles at $s = -2$ and $s = -4$. The second is not on the locus.

(d) You could more accurately sketch the locus by using Routh's criterion to find the gain for marginal stability, which can be used to exactly compute where the locus crosses the imaginary axis.

The root locus is illustrated in Figure 3. Using `rlocus(1,conv([1 6 8],[1 10]))` in Matlab verifies the result.

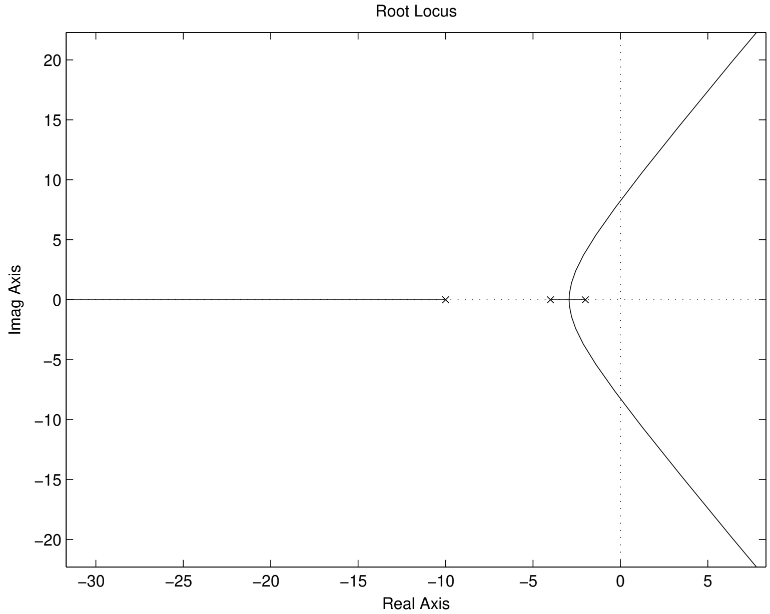

Figure 3. Root locus for 5.8.1:9.

4. (5.8.2:1) Sketch the root locus for

$$G(s) = \frac{K}{s+1} \qquad \text{and} \qquad H(s) = \frac{s+2}{s-4}$$

and determine the range of values for K for which the closed loop transfer function is stable. For the root locus, notice that $GH(s)$ has poles at $s = 4$ and $s = -1$ and a zero at $s = -2$.

(a) Rule 5.3.1: on the real axis, the locus lies between the two poles and then to the left of the zero.

(b) Rule 5.3.3 and 5.4.4: the locus must break out from between the two poles and then break in to the left of the zero.

(c) Rule 5.3.6:

$$K = \frac{(s+1)(s-4)}{s+2} \quad \Longrightarrow \quad s^2 + 4s - 2 = 0 \quad \Longrightarrow \quad s = .45, -4.45.$$

The first value of s is the break out point, and the second is the break in point.

The root locus is illustrated in Figure 3. Using `rlocus(1,conv([1 6 8],[1 10]))` in Matlab verifies the result.

To determine the range of K values for stability, we use Routh's criterion on the characteristic polynomial for the overall transfer function.

$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s+1}}{1 + \frac{K}{s+1}\frac{s+2}{s-4}},$$

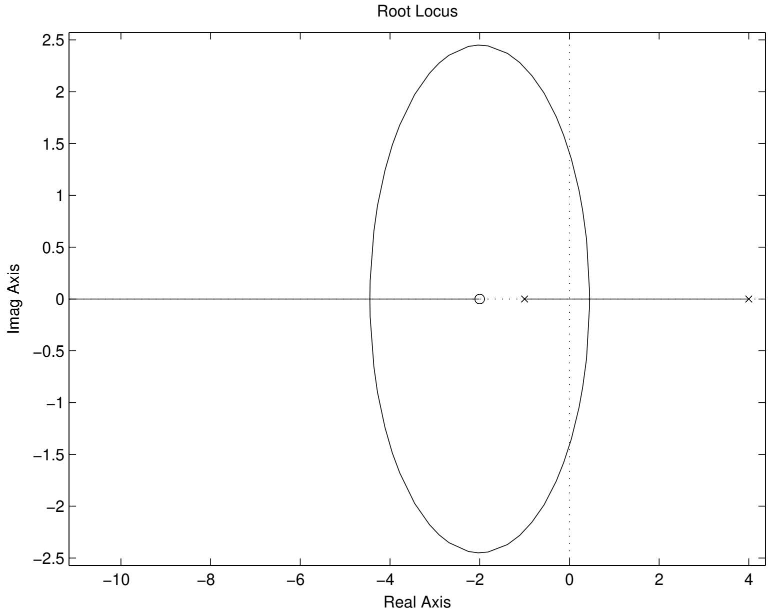

Figure 4. Root locus for 5.8.2:1.

so the characteristic polynomial is $s^2 + (K-3)s + (2K-4)$.

$$\begin{array}{c|cc} s^2 & 1 & 2K-4 \\ s^1 & K-3 & \\ s^0 & 2K-4 & \end{array}.$$

From the second row, $K > 3$.

5. (5.8.3:1) Plot the root locus for

$$GH(s) = \frac{K(s+2)}{[(s+1)^2+1](s+20)(s+30)}.$$

There are four poles at $s = -1 \pm i, -20, -30$ and a zero at $s = -2$.

(a) Rule 5.3.1: on the real axis, the locus lies between the zero at $s = -2$ and the pole at $s = -20$, and to the left of the pole at $s = -30$.

(b) Since there are four poles and one zero, there are three asymptotes. The angles are $60°, 180°$ and $300°$, and they intersect the real axis at

$$\sigma = \frac{-1-1-20-30-(-2)}{3} = -16.66.$$

(c) Rule 5.3.8: The locus departs the complex conjugate pole with positive imaginary part at an angle determined by the $180°$ rule:

$$\Sigma_{i=1}^{m}\psi_i - \Sigma_{i=1}^{n}\phi_i = 180,$$

where the ψ_i are the angles from the zeros to the s value and the ϕ_i are the angles from the poles. Note, $\psi_1 = 45°$, $\phi_1 = 90°$, $\phi_2 = tan^{-1}\frac{1}{19} = 3.01°$ and $\phi_3 = tan^{-1}\frac{1}{29} = 1.97°$. Thus

$$45 - 90 - \phi_{dep} - 3.01 - 1.97 = 180,$$

so

$$\phi_{dep} = 130°.$$

(d) Rule 5.3.6: if there are no break in or break out points, then the branches leaving the complex conjugate pair of poles simply tend toward the asymptotes and go to ∞. If there are break in and break out points, then the one to the right will be a break in point from the complex conjugate poles and the one to the left will be a break out point for the asymptotes.

Since

$$K = \frac{(s^2+2s+2)(s+20)(S+30)}{s+2},$$

boring math gives

$$\frac{dK}{ds} = 0 \quad \Longrightarrow \quad 3s^4 + 112s^3 + 1014s^2 + 2808s + 1400 = 0.$$

Thus,

$$s = -25.4826, -7.2057, -4.0115, -0.6335.$$

The middle two lie on the locus on the real axis. $s = -4.01$ must be a break in point, and $s = -7.21$ must be a break out point.

The root locus is illustrated in Figure 5, and Figure 6 shows a close-up of the locus near the complex conjugate pairs of poles of $GH(s)$.

6. (5.8.6:1) All the root locus plotting rules, including rule 1, are based upon $\angle GH(s) = 180°$. For a given s-value, $\angle GH(s)$ is the sum of the angles from all the zeros of $GH(s)$ minus the sum of the angles from all the poles of $GH(s)$. On the real axis, however, the contribution to $\angle GH(s)$ from each member of a complex conjugate pair or poles or zeros will exactly cancel each other since they must occur in complex conjugate pairs.

7. The following code numerically solves the Lorenz equations using the fourth order Runge-Kutta method. A plot of the solution is in Figure 7.

```
#include<stdio.h>
#include<math.h>

main() {

  double *x,dt,tfinal,t;
  double k1,k2,k3,k4,l1,l2,l3,l4,m1,m2,m3,m4;
  double sig, beta, rho;
  FILE *fp;

  x = (double *)malloc(3*sizeof(double));
  fp = fopen("rk.d","w");
```

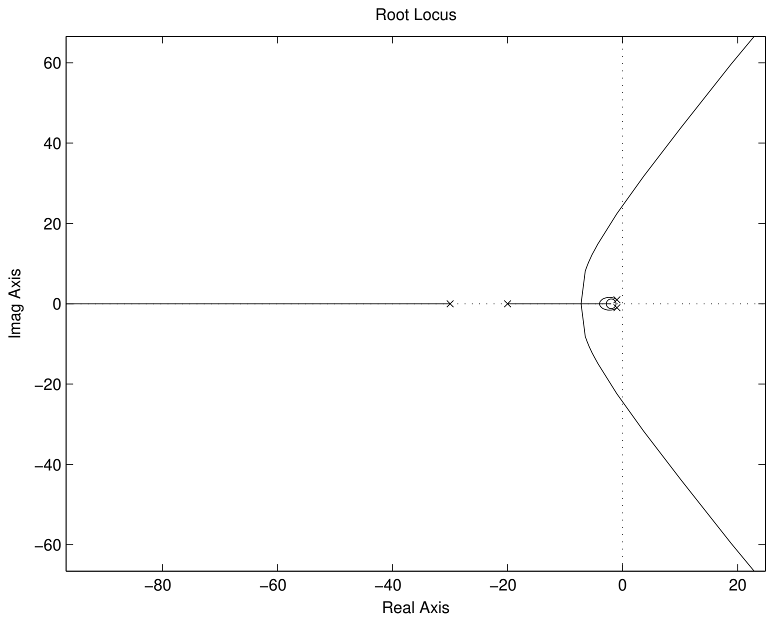

Figure 5. Root locus for 5.8.3:1.

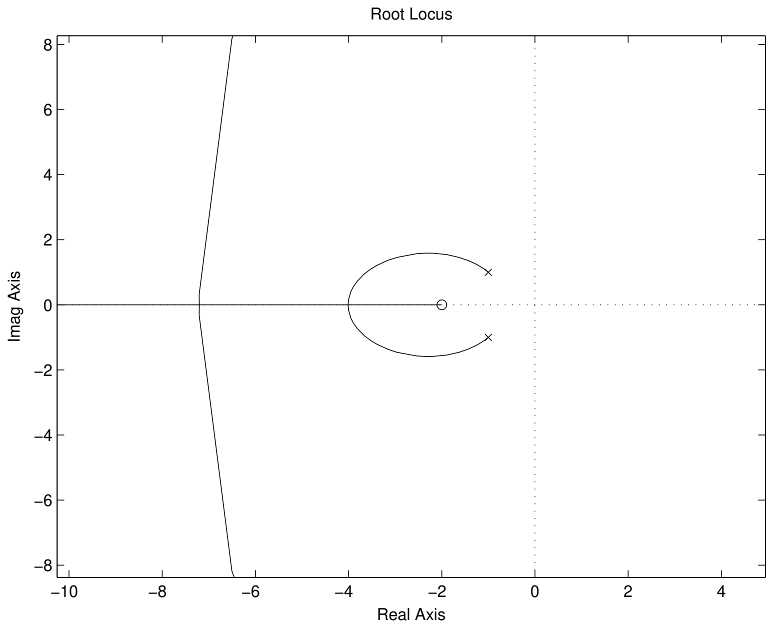

Figure 6. Root locus for 5.8.3:1.

```
  tfinal = 50.0;
  dt = 0.001;

  sig = 10.0;
  beta = 8.0/3.0;
  rho = 28.0;

  x[0] = 0.0;
  x[1] = 0.05;
  x[2] = 0.05;

  for(t=0;t<=tfinal;t+=dt) {

    k1 = dt*(sig*(x[1] - x[0]));
    l1 = dt*(rho*x[0] - x[1] - x[0]*x[2]);
    m1 = dt*(-beta*x[2] + x[0]*x[1]);

    k2 = dt*(sig*((x[1]+l1/2) - (x[0]+k1/2)));
    l2 = dt*(rho*(x[0]+k1/2) - (x[1]+l1/2) - (x[0]+k1/2)*(x[2]+m1/2));
    m2 = dt*(-beta*(x[2]+m1/2) + (x[0]+k1/2)*(x[1]+l1/2));

    k3 = dt*(sig*((x[1]+l2/2) - (x[0]+k2/2)));
    l3 = dt*(rho*(x[0]+k2/2) - (x[1]+l2/2) - (x[0]+k2/2)*(x[2]+m2/2));
    m3 = dt*(-beta*(x[2]+m2/2) + (x[0]+k2/2)*(x[1]+l2/2));

    k4 = dt*(sig*((x[1]+l3) - (x[0]+k3)));
    l4 = dt*(rho*(x[0]+k3) - (x[1]+l3) - (x[0]+k3)*(x[2]+m3));
    m4 = dt*(-beta*(x[2]+m3) + (x[0]+k3)*(x[1]+l3));

    x[0] += (k1+k2*2+k3*2+k4)/6.0;
    x[1] += (l1+l2*2+l3*2+l4)/6.0;
    x[2] += (m1+m2*2+m3*2+m4)/6.0;

    fprintf(fp,"%f\t%f\t%f\t%f\n",t,x[0],x[1],x[2]);
  }

  fclose(fp);
}
```

8. From http://userwww.sfsu.edu/ rsauzier/LorenzE.html:

   Lorenz, Edward N(orton) (1917-)

   Meteorologist, born in West Hartford, CT. Working at the Massachusetts Institute of Technology from 1946, he was the first to describe what is known as "deterministic chaos" as a shaper of weather, and was the originator of the term "the butterfly effect"- the flapping wings of a butterfly in China could alter the weather over America a few days later. Among other major meteorology awards, he received the 1991 Kyoto Prize.

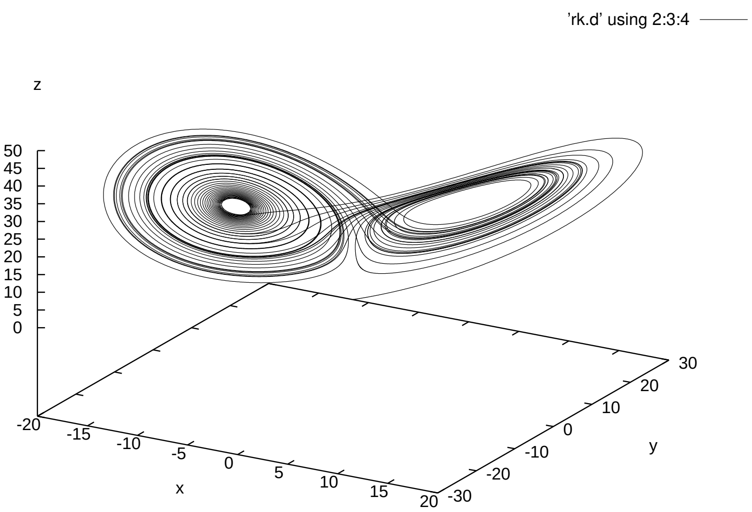

Figure 7. Solution to the Lorenz equations.

From `http://mathforum.org/library/drmath/view/61791.html`:

> The Lorenz equations are actually a system of nonlinear differential equations. The significance of these equations, which were discovered by Edward Lorenz back in the 60s, is that relatively simple systems such as these could exhibit rather complex (specifically, chaotic) behavior. The chaotic aspect of this system demonstrates that, despite being given the initial conditions to any arbitrary degree of accuracy, one cannot predict a sufficiently advanced state of the system. In other words, the system has built into itself the property of amplifying small perturbations until they become so significant they affect the accuracy of the results.
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> Lorenz, a meteorologist, made his discovery by observing weather phenomena - in particular, convection of fluids (and to a weatherman, the "fluid" he's most interested in is air). He took various mathematical models of fluid convection and simplified them into a system of differential equations that basically led him to the now-famous Lorenz equations.