University of Notre Dame
Aerospace and Mechanical Engineering

AME 437: Control Systems Engineering
Homework 6 Solutions

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1. Final Value Theorem:
   (a)
   $$y_{ss} = \lim_{s\to 0} s\frac{1}{s}\frac{1}{s+2} = \frac{1}{2}.$$
   (b)
   $$y_{ss} = \lim_{s\to 0} s\frac{1}{s}\frac{1}{s-2} = -\frac{1}{2}.$$
   (c) `>> step(1,[1 -2])`
   produces the plot in Figure 1. This is different from the final value theorem because the final value theorem cannot be used on systems with unstable poles.

2. (6.8.1:1) Find K so that if
   $$G(s) = \frac{K}{(s+1)(s+5)},$$
   the closed loop dominate second order poles are such that
   $$\zeta = \frac{1}{\sqrt{2}}.$$
   (a) One could simply use
   `rlocus(1,[1 6 5])`
   in Matlab for this one and click on the locus until the right damping ratio is found.
   (b) Analytically, it is easy too. Since
   $$\frac{C(s)}{R(s)} = \frac{K}{(s+1)(s+5)+K} = \frac{K}{s^2+6s+(K+5)},$$
   solving
   $$\begin{aligned} \omega_n^2 &= K+5 \\ 2\zeta\omega_n &= 6 \end{aligned}$$
   for K gives $K = 13$.

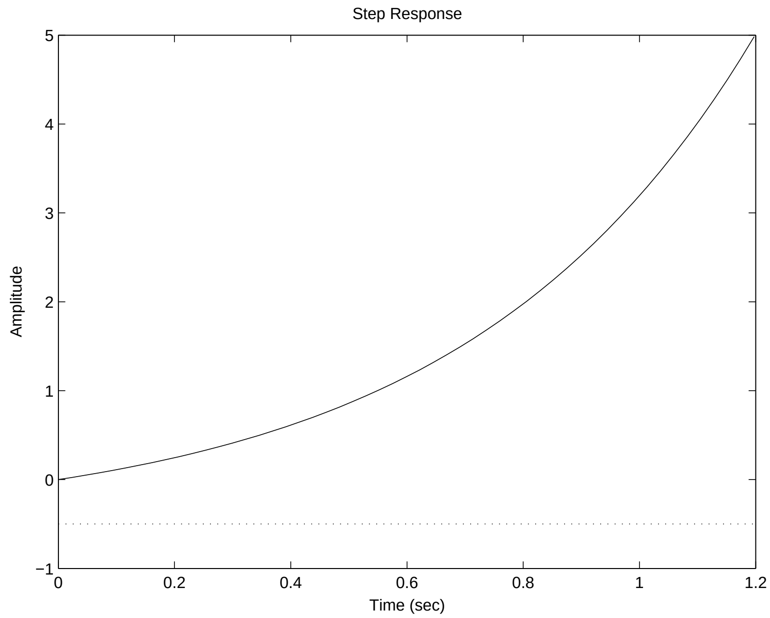

Figure 1. Unit step response for problem 1c.

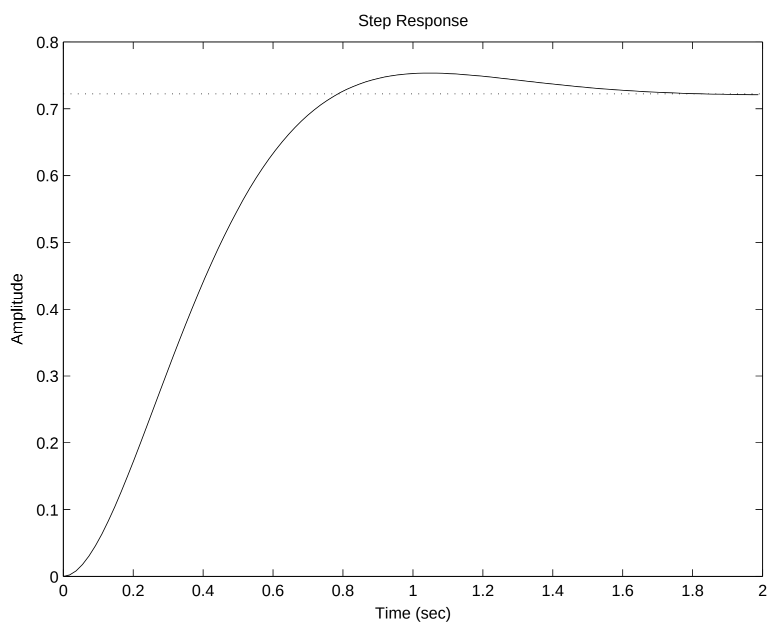

Figure 2. Step response for Problem 2.

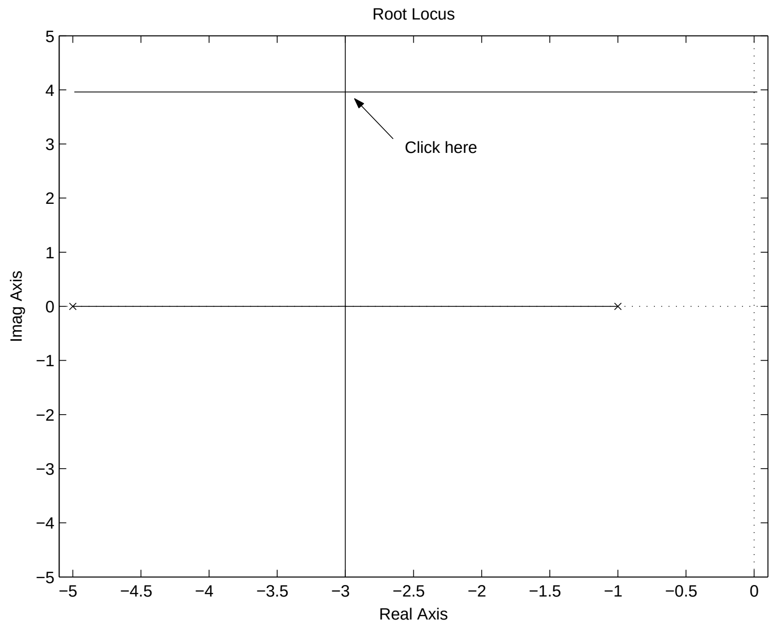

Figure 3. Root locus for problem 3.

(c) The step response is illustrated in Figure 2. The Matlab command is

```
>> step(13,[1 6 18])
```

.

3. (6.8.1:2) If

$$G(s) = \frac{K}{(s+1)(s+5)},$$

find K so that for the closed loop system, $\omega_d = 4$.

(a) Again, in Matlab,

```
>>rlocus(1,[1 6 5])
```

and changing the axis limits by right-clicking on the figure produces the root locus in Figure 3a. Clicking at the point where $\omega_d = 4$ gives the gain $K = 20$.

(b) Analytically, solving

$$\begin{aligned} \omega_n^2 &= K + 5 \\ 2\zeta\omega_n &= 6 \\ \omega_n\sqrt{1-\zeta^2} &= 4 \end{aligned}$$

also gives $K = 20$.

(c) `>> step(20,[1 6 25])`
is plotted in Figure 4.

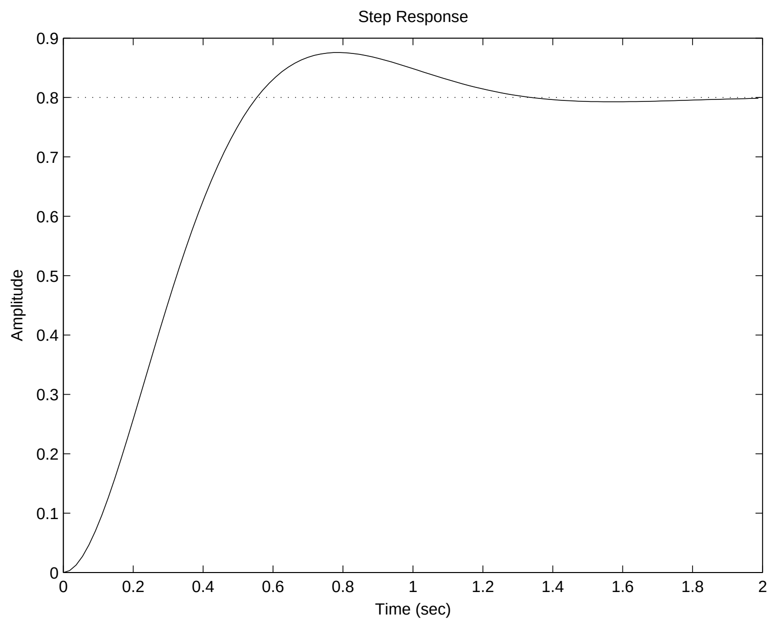

Figure 4. Step response for problem 3.

4. (6.8.2:2) For the closed loop transfer function,

$$T_c(s) = \frac{K(s+\delta)}{(s+\sigma-i\omega_d)(s+\sigma+i\omega_d)},$$

find K, δ, σ and ω_d such that $\omega_n = 5$, $\zeta = 0.8$, $e_{ss} = 0$ for a step and $e_{ss} = 10\%$ for a ramp. Since $(s+\sigma-i\omega_d)(s+\sigma+i\omega_d) = s^2 + 2\sigma s + (\sigma^2 + \omega_d^2)$, solve

$$\begin{aligned} \sigma^2 + \omega_d^2 &= 25 \\ \sigma &= 5(0.8) \end{aligned}$$

gives $\sigma = 4$ and $\omega_d = 3$.
Now

$$T_c(s) = \frac{K(s+\delta)}{s^2+8s+25}.$$

Using the final value theorem, for a step input

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{1}{s}\left(1 - \frac{K(s+\delta)}{s^2+8s+25}\right) \\ &= \lim_{s\to 0}\left(\frac{s^2+8s+25-K(s+\delta)}{s^2+8s+25}\right) \\ &= \frac{25-K\delta}{25}. \end{aligned}$$

Thus, $K\delta = 25$.

For a ramp input,

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\frac{1}{s^2}\left(1 - \frac{K(s+\delta)}{s^2+8s+25}\right) \\ &= \lim_{s\to 0} \frac{1}{s}\left(\frac{s^2+8s+25-K(s+\delta)}{s^2+8s+25}\right) \\ &= \lim_{s\to 0} \frac{1}{s}\left(\frac{s^2+(8-K)s}{s^2+8s+25}\right) \\ &= \frac{8-K}{25}. \end{aligned}$$

Thus, $K = 5.5$ and $\delta = 4.54$.

To check our answers, first use

```
>> roots([1 8 25])

ans =

  -4.0000 + 3.0000i
  -4.0000 - 3.0000i
```

which verifies the $\omega_n = 5$ and $\delta = 0.8$ specifications.

```
>> step(5.5*[1 4.54],[1 8 25])
```

is plotted in Figure 5, which verifies the step input specification.

```
>> step([1 2.5 0],[1 8 25 0])
```

is plotted in Figure 6, which verifies the ramp input specification.

5. (6.8.3:1) Find K such that the closed loop system with open loop transfer function

$$G(s) = \frac{K}{s(s+2)(s+20)}$$

has dominant second order poles such that $\omega_n = 3$.

The root locus plot, determined from

```
>> rlocus(1,[1 22 40 0])
```

is illustrated in Figure 7. Zooming in and clicking on the locus shows that $\omega_n = 3$ when $K = 185$. Note that for this K value, the dominant second order poles are located at $s = -.74 \pm 2.9i$.

For

$$G(s) = \frac{185}{s(s+2)(s+20)},$$

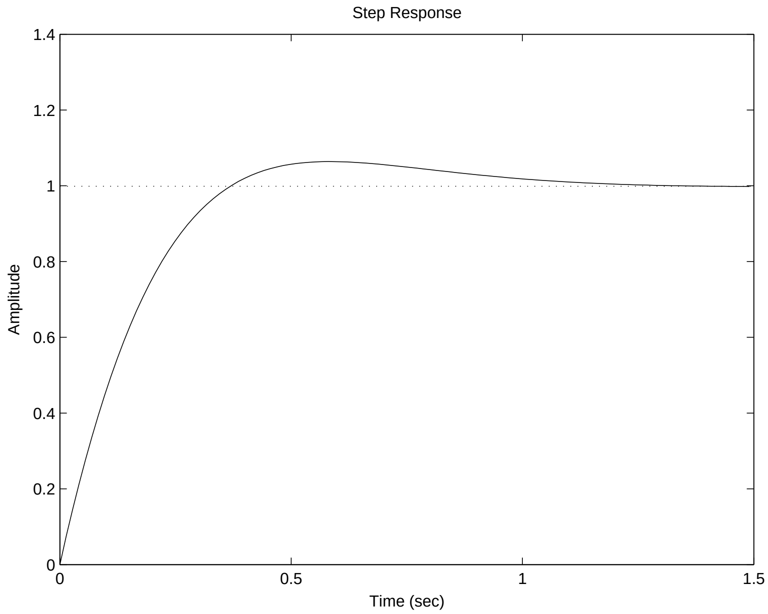

Figure 5. Step response for problem 4.

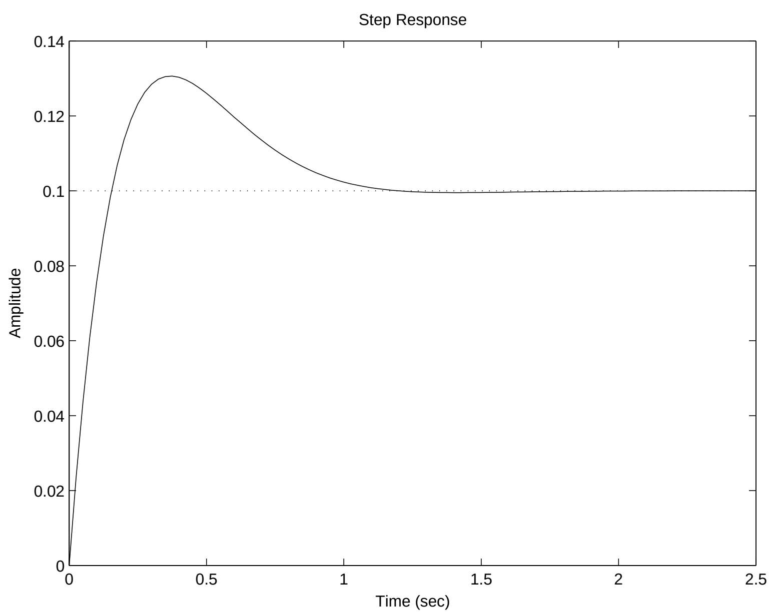

Figure 6. Ramp error response for problem 4.

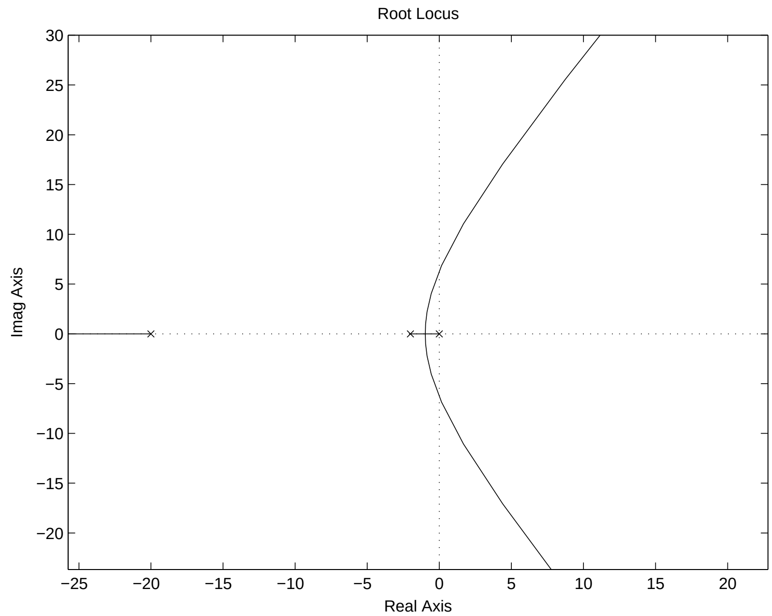

Figure 7. Root locus for problem 5.

the closed loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{185}{s^3 + 22s^2 + 40s + 185}.$$

The step response for the original system is given by

```
>> step(185,[1 22 40 185])
```

and the result is illustrated in Figure 8.

Considering only the dominant poles,

$$G(s) = \frac{K}{(s + .74 + 2.9i)(s + .74 - 2.9i)} = \frac{K}{s^2 + 1.48s + 9}.$$

To have zero steady state error to be able to compare it with the previous response, let $K = 9$. Using Matlab,

```
>> step(9,[1 1.48 9])
```

is illustrated in Figure 9 along with the previous step response.

6. (6.8.4:1) Consider

$$G(s) = \frac{0.1}{(s - 10i)(s + 10i)} = \frac{0.1}{s^2 + 100}$$

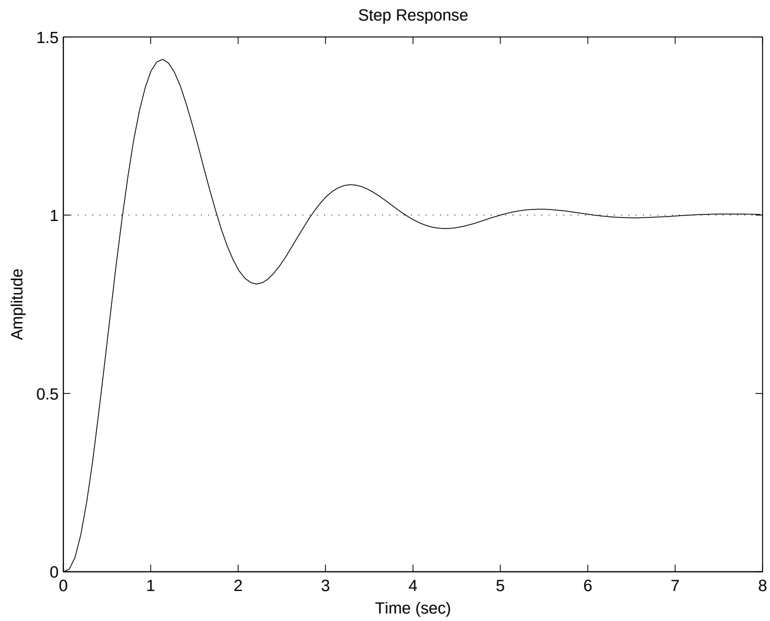

Figure 8. Step response for problem 5.

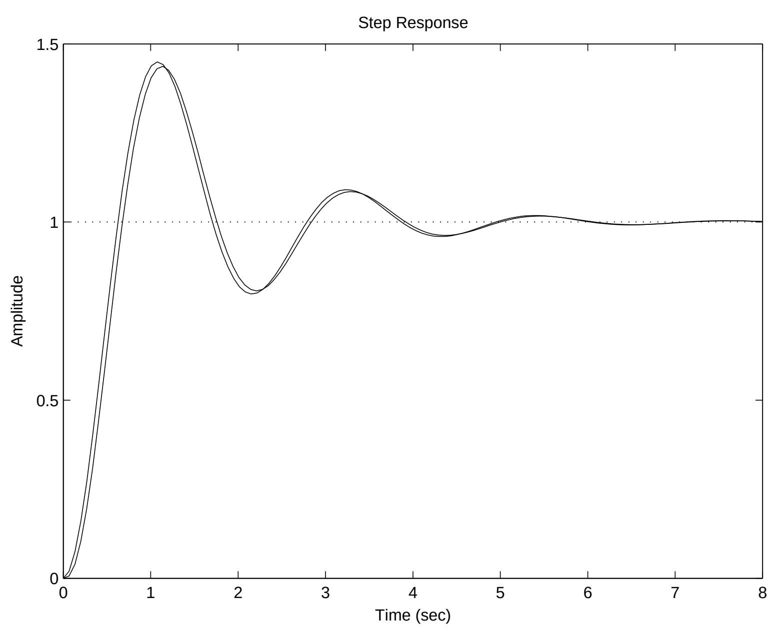

Figure 9. Both step responses for problem 5.

and

$$H(s) = Ks.$$

(a) The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{0.1}{s^2 + 0.1Ks + 100}.$$

Since the poles of the closed loop transfer function are at

$$s = \frac{-0.1K \pm \sqrt{0.01K^2 - 400}}{2},$$

the minimum gain that will give two real poles is $K = 200$.

(b) If the system is given a step input,

$$c_{ss} = \lim_{s \to 0} s \frac{1}{s} \frac{0.1}{s^2 + 20s + 100} = 0.001,$$

so $e_{ss} = .999$.

(c) Adding additional gain in the feedforward loop is better because that will increase the magnitude of the numerator in c_{ss} which will reduce e_{ss}.

7. Note:

(a) comparing 1 and 2 indicates that 2 will have a higher frequency, lower peak time and less damping than 1;

(b) comparing 3 and 4 indicates that each will have no oscillation and 4 will have a much faster rise time since the slower pole is nearly canceled by a zero;

(c) 5 is nonminimum phase; and

(d) 6 will have an increased overshoot compared with 1 due to the additional zero in the LHP.

Thus, the pairs are (1,b), (2,e), (3,a), (4,d), (5,f) and (6.c).