University of Notre Dame
Aerospace and Mechanical Engineering

ME 469: Introduction to Robotics
Homework 1 Solutions

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1. (Craig, 2.3)

   The answer for this can be found in the Appendix. In the notation of the book, ${}^A_BR_{Z'X'Y'}(\alpha,\beta,\gamma)$ stands for the ZXY Euler angle matrix. We can look this up in the Appendix, and see that if $\alpha=\theta$, $\beta=\phi$ and $\gamma=0$,, then

   $$ {}^A_BR_{Z'X'Y'}(\theta,\phi,0) = \begin{bmatrix} \cos\theta & -\sin\theta\cos\phi & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} $$

   You could of also used the formula for $R_{Z'X'Z'}$ to get the same answer since the third rotation was not specified.

2. (Craig, 2.5)

   The definition of an eigenvector is any vector ϕ that satisfies the equation

   $$ {}^A_BR\phi = \lambda\phi $$

   for some λ. If $\lambda=1$, then the equation is

   $$ {}^A_BR\phi = \phi $$

   which indicates that ϕ is not rotated by ${}^A_BR$. The only vector that is not changed by a rotation is the **axis of rotation**.

3. (Craig, 2.6)

   ???????? Hopefully, someone will get this problem right.

4. (Craig, 2.14)

   We are looking for ${}^A_BT$, where, starting from initial coincidence, frame B is rotated by θ about K, where $\hat{K}$ (a unit vector, *not* a matrix) passes through the point ${}^AP$.

   To do this, we will define two other frames, A' and B', where the origins of A' and B' are located at the point ${}^AP$, A and A' have the same orientation, B' is rotated relative to A' by an amount θ about $\hat{K}$, and B' and B have the same orientation. See Figure 2.20 in the book. This idea is exactly the same idea as Example 2.9 in the book.

Now, we know that

$$ {}^A_BT = {}^A_{A'}T {}^{A'}_{B'}T {}^{B'}_{B}T. $$

Since A and A' only differ by a translation,

$$ {}^A_{A'}T = \begin{bmatrix} I & {}^AP \\ 0 & 1 \end{bmatrix} $$

where I is the 3×3 identity matrix.

Now, A' and B' only differ by a rotation about $\hat{K}$. Therefore,

$$ {}^{A'}_{B'}T = \begin{bmatrix} R_K(\theta) & 0 \\ 0 & 1 \end{bmatrix}, $$

where $R_K(\theta)$ is given by Rodrigues' formula (like I like) or Equation 2.80 in the book.

Finally, B' and B differ only by translation along *negative* A^P, *i.e.*,

$$ {}^{B'}_{B}T = \begin{bmatrix} I & -{}^AP \\ 0 & 1 \end{bmatrix}. $$

Multiplying everything together gives:

$$ \boxed{ {}^A_BT = \begin{bmatrix} I & {}^AP \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_K(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -{}^AP \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_K(\theta) & -R_K(\theta){}^AP + {}^AP \\ 0 & 1 \end{bmatrix}. } $$

5. (Craig, 2.16)

   Covered in class on Wednesday, September 16, 1998.

6. (Craig, 2.20)

   Let

$$ Q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \qquad \text{and} \qquad K = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}. $$

Then,

$$ Q' = R_K(\theta)Q = \begin{bmatrix} k_xk_xq_xv\theta + q_xc\theta + k_xk_yq_yv\theta - k_zq_ys\theta + k_xk_zq_zv\theta + k_yq_zs\theta \\ k_xk_yq_xv\theta + k_zq_xs\theta + k_yk_yq_yv\theta + q_yc\theta + k_yk_zq_zv\theta + k_zq_zs\theta \\ k_xk_zq_xv\theta - k_yq_xs\theta + k_yk_zq_yv\theta + k_xq_ys\theta + k_zk_zq_zv\theta + q_zc\theta \end{bmatrix}. \tag{1} $$

Note that each row has a term of the form $q_?c\theta$. Separating these terms will give the $Q\cos\theta$ term.

Note also that

$$ (1-\cos\theta)(K \cdot Q)K = \begin{bmatrix} v(k_xq_x + k_yq_y + k_zq_z)k_x \\ v(k_xq_x + k_yq_y + k_zq_z)k_y \\ v(k_xq_x + k_yq_y + k_zq_z)k_z \end{bmatrix}, $$

where, as usual, $v = 1 - \cos\theta$.

Finally, note that

$$\sin\theta(K \times Q) = \begin{bmatrix} \sin\theta(-k_z q_y + k_y q_z) \\ \sin\theta(k_z q_x - k_x q_z) \\ \sin\theta(k_y q_x - k_x q_y) \end{bmatrix}.$$

Inspecting Equation 1 verifies that

$$Q' = Q\cos\theta + \sin\theta(K \times Q) + (1 - \cos\theta)(K \cdot Q)K,$$

as required.

7. (Craig, 2.38)

For two unit vectors, we know that the cosine of the angle between them is given by their dot product.

Since,

$$v_1 \cdot v_2 = v_{1_x} v_{2_x} + v_{1_y} v_{2_y} + v_{1_z} v_{2_z},$$

where

$$v_1 = \begin{bmatrix} v_{1_x} \\ v_{1_y} \\ v_{1_z} \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} v_{2_x} \\ v_{2_y} \\ v_{2_z} \end{bmatrix},$$

these dot products can be written in vector form

$$v_1 \cdot v_2 = v_1^T v_2 = \begin{bmatrix} v_{1_x} & v_{1_y} & v_{1_z} \end{bmatrix} \begin{bmatrix} v_{2_x} \\ v_{2_y} \\ v_{2_z} \end{bmatrix}.$$

Let R be a rotation matrix. If the angle between vectors is preserved by a rigid body rotation, the cosine of the angle is preserved. Therefore,

$$\begin{aligned} v_1^T v_2 &= (Rv_1)^T (Rv_2) \\ &= v_1^T R^T R v_2. \end{aligned}$$

Since this is true for any two vectors v_1 and v_2, it must be the case that

$$R^T R = I,$$

which shows that the transpose of a rotation matrix is its inverse.