UNIVERSITY OF NOTRE DAME
Aerospace and Mechanical Engineering

ME 469: Introduction to Robotics
Homework 5 Solutions

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1. Hopefully, every group accomplished at least some work this week on project 2.

2. From the previous problem we know that the mechanism is singular whenever $\theta_2 = k\pi$. Let's pick $\theta_2 = 0$, and arbitrarily pick $\theta_1 = 0$ and $\theta_3 = \frac{\pi}{2}$.

   Substituting these values into the Jacobian gives:

$$J = \begin{bmatrix} -l_3 & -l_3 & -l_3 \\ l_1 + l_2 & l_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

   Let

$$F_e = \begin{bmatrix} -1000000 \\ 0 \\ l_3 1000000 \end{bmatrix}.$$

   (This came from drawing a free–body diagram for the last link, and determining what applied torque would be required so that there would be *no* toque at joint 3).

   Then, the joint torques are

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} -l_3 & -l_3 & -l_3 \\ l_1 + l_2 & l_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} -1000000 \\ 0 \\ -l_3 1000000 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   i.e., no joint torques are required to maintain this applied force and moment — F_e lies in the null space of J^T.

   Perhaps a more intuitive case would be to take $\theta_1 = \theta_2 = \theta_3 = 0$, and let F_e be a force in the x–direction. The same result would occur, *i.e.*, no joint torques required to resist this applied force.

3. (a) Figure 2 shows the manipulator with the link frame assignments determined in Homework 2, with a tool frame added at the end effector. The relationship between the tool frame and frame 3 is a pure displacement in the x direction, *i.e.*,

$${}^{T}_{3}T = \begin{bmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

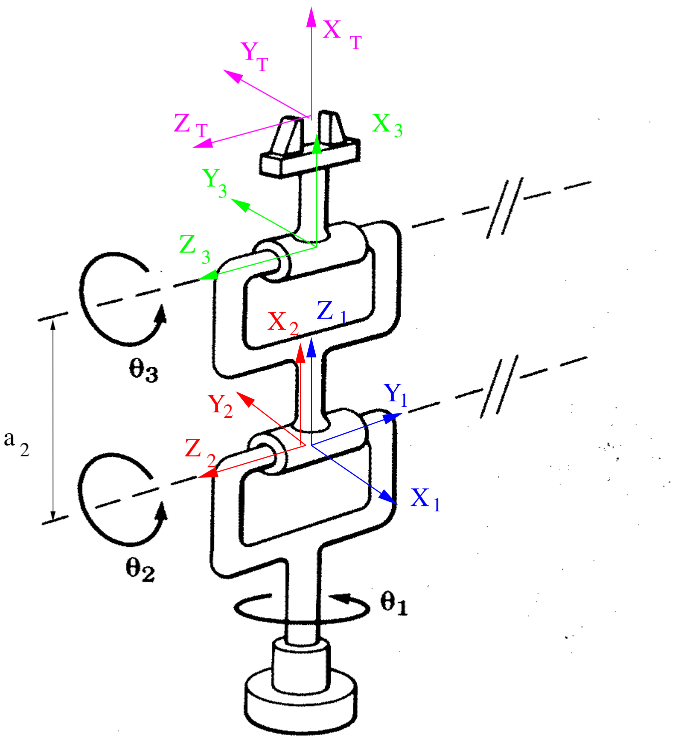

Figure 1. Frames for Problem 4.

Multiplying ${}^3_0T$ from Homework 2 and this gives

$$
{}^T_0T = {}^3_0 T {}^T_3 T = \begin{bmatrix} \cos(\theta_1)\,\cos(\theta_2+\theta_3) & -(\cos(\theta_1)\,\sin(\theta_2+\theta_3)) & \sin(\theta_1) & \cos(\theta_1)\,(a_2\,\cos(\theta_2)+a_3\cos(\theta_2+\theta_3)) \\ \cos(\theta_2+\theta_3)\,\sin(\theta_1) & -(\sin(\theta_1)\,\sin(\theta_2+\theta_3)) & -\cos(\theta_1) & (a_2\,\cos(\theta_2)+a_3\cos(\theta_2+\theta_3))\,\sin(\theta_1) \\ \sin(\theta_2+\theta_3) & \cos(\theta_2+\theta_3) & 0 & a_2\sin(\theta_2)+a_3\,\sin(\theta_2+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(Recall that the direction for the x axis for frame 3 is arbitrary. Therefore, you could have correctly put the x_3 axis in a different orientation. In such a case, the above reasoning would be the same, but the pure displacement would not necessarily be in the x direction.) Since we are only concerned with the (x, y, z) location of the end effector, the Jacobian can be determined by differentiating the displacement term of ${}^T_0T$, (the top three terms of the last column). Let's denote this vector by

$$
p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}.
$$

Then the Jacobian is

$$
\begin{aligned}
J &= \begin{bmatrix} \frac{\partial p_x}{\theta_1} & \frac{\partial p_x}{\theta_2} & \frac{\partial p_x}{\theta_3} \\ \frac{\partial p_y}{\theta_1} & \frac{\partial p_y}{\theta_2} & \frac{\partial p_y}{\theta_2} \\ \frac{\partial p_z}{\theta_1} & \frac{\partial p_z}{\theta_2} & \frac{\partial p_z}{\theta_3} \end{bmatrix} \\
&= \begin{bmatrix} -((a_2\,\cos(\theta_2)+a_3\cos(\theta_2+\theta_3))\,\sin(\theta_1)) & -(\cos(\theta_1)\,(a_2\,\sin(\theta_2)+a_3\sin(\theta_2+\theta_3))) & -(a_3\cos(\theta_1)\,\sin(\theta_2+\theta_3)) \\ \cos(\theta_1)\,(a_2\,\cos(\theta_2)+a_3\cos(\theta_2+\theta_3)) & -(\sin(\theta_1)\,(a_2\,\sin(\theta_2)+a_3\sin(\theta_2+\theta_3))) & -(a_3\sin(\theta_1)\,\sin(\theta_2+\theta_3)) \\ 0 & a_2\,\cos(\theta_2)+a_3\cos(\theta_2+\theta_3) & a_3\cos(\theta_2+\theta_3) \end{bmatrix}.
\end{aligned}
$$

(b) A quick mental calculation shows that

$$
\det(J) = -\left(a_2 a_3\left(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\right)\,\sin(\theta_3)\right).
$$

Therefore, the mechanism is singular if $\theta_3 = 0$.

4. (a) Figure 3 shows the manipulator with the link frame assignments determined in Homework 2, with a tool frame added at the end effector. For simplicity, assume that the final joint is "straight," , *i.e.*, it is aligned with the frames so that the relationship between the tool frame and frame 3 is a pure displacement in the x direction, *i.e.*,

$$
{}^T_3T = \begin{bmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Multiplying ${}^3_0T$ from Homework 2 and this gives

$$
{}^T_0T = {}^3_0 T {}^T_3 T = \begin{bmatrix} \cos(\theta_2+\theta_3) & -\sin(\theta_2+\theta_3) & 0 & a_1+\cos(\theta_2)\,a_2+\cos(\theta_2+\theta_3)\,a_3 \\ \sin(\theta_2+\theta_3) & \cos(\theta_2+\theta_3) & 0 & \sin(\theta_2)\,a_2+\sin(\theta_2+\theta_3)\,a_3 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

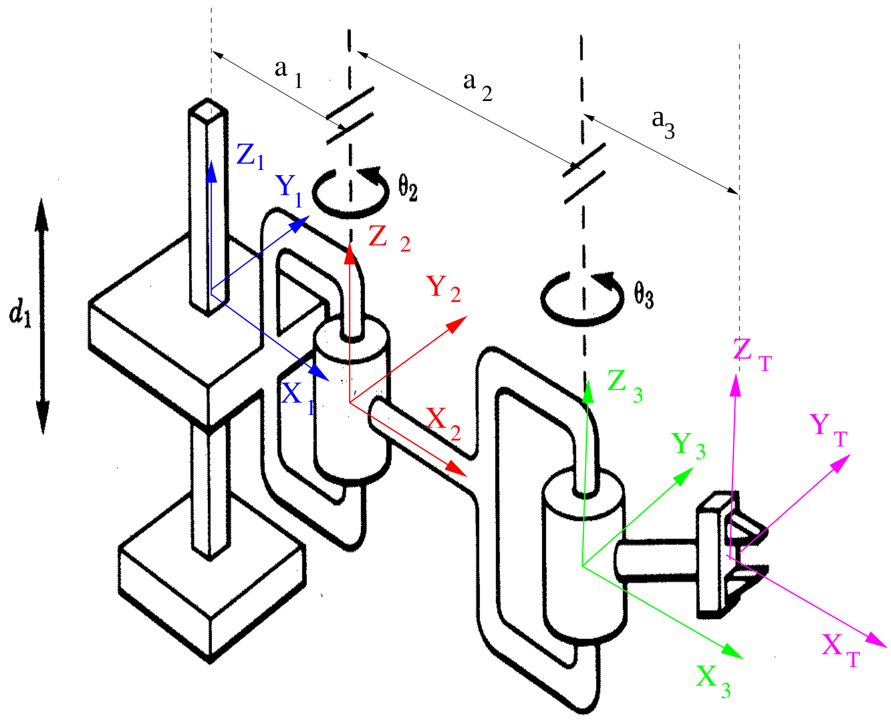

Figure 2. Frames for Problem 5.

(Recall that the direction for the x axis for frame 3 is arbitrary. Therefore, you could have correctly put the x_3 axis in a different orientation. In such a case, the above reasoning would be the same, but the pure displacement would not necessarily be in the x direction.) Since we are only concerned with the (x, y, z) location of the end effector, the Jacobian can be determined by differentiating the displacement term of ${}^T_0T$, (the top three terms of the last column). Let's denote this vector by

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}.$$

Then the Jacobian is

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial p_x}{d_1} & \frac{\partial p_x}{\theta_2} & \frac{\partial p_x}{\theta_3} \\ \frac{\partial p_y}{d_1} & \frac{\partial p_y}{\theta_2} & \frac{\partial p_y}{\theta_2} \\ \frac{\partial p_z}{d_1} & \frac{\partial p_z}{\theta_2} & \frac{\partial p_z}{\theta_3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -(a_2 \sin(\theta_2)) - a_3 \sin(\theta_2 + \theta_3) & -(a_3 \sin(\theta_2 + \theta_3)) \\ 0 & a_2 \cos(\theta_2) + a_3 \cos(\theta_2 + \theta_3) & a_3 \cos(\theta_2 + \theta_3) \\ 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

(b) A quick mental calculation shows that

$$\det(J) = a_2 a_2 \sin\theta_3.$$

Therefore, the mechanism is singular if $\theta_3 = 0$.

5. (a) Figure 4 shows the manipulator with the link frame assignments determined in Homework 3, with a tool frame added at the end effector. The relationship between the tool frame and frame 3 is a pure displacement in the $-y$ direction, *i.e.*,

$${}^T_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiplying ${}^3_0T$ from Homework 3 and this gives

$${}^T_0T =^3_0 T^T_3T = \begin{bmatrix} 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & d_2 \\ 1 & 0 & 0 & a + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Recall that the direction for the x axis for frame 3 is arbitrary. Therefore, you could have correctly put the x_3 axis in a different orientation. In such a case, the above reasoning would be the same, but the pure displacement would not necessarily be in the x direction.) Since we are only concerned with the (x, y, z) location of the end effector, the Jacobian can be determined by differentiating the displacement term of ${}^T_0T$, (the top three terms

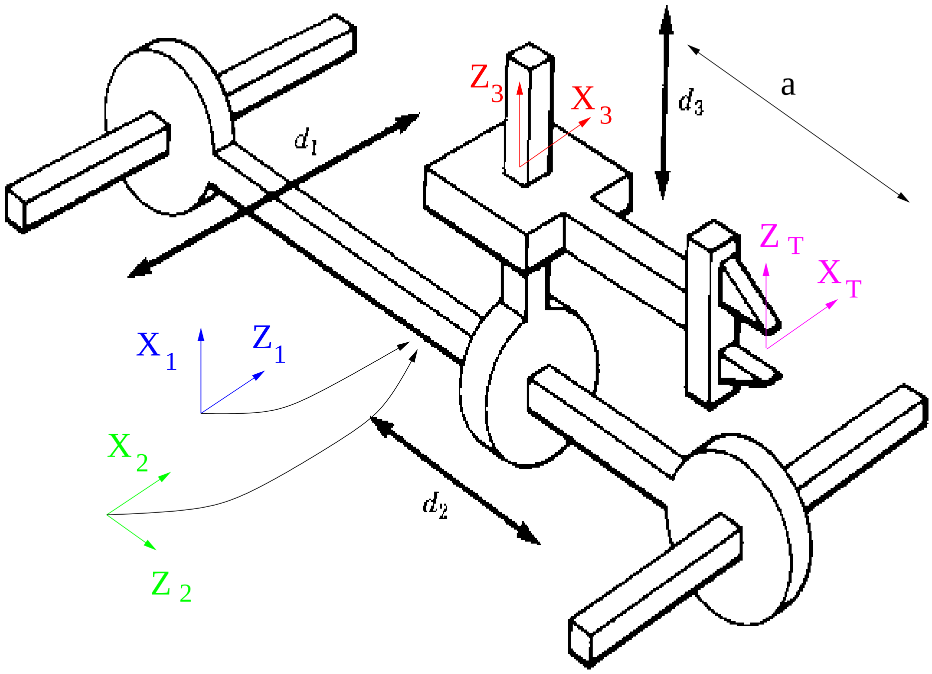

Figure 3. Frames for Problem 6.

of the last column). Let's denote this vector by

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}.$$

Then the Jacobian is

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial p_x}{d_1} & \frac{\partial p_x}{d_2} & \frac{\partial p_x}{d_3} \\ \frac{\partial p_y}{d_1} & \frac{\partial p_y}{d_2} & \frac{\partial p_y}{d_2} \\ \frac{\partial p_z}{d_1} & \frac{\partial p_z}{d_2} & \frac{\partial p_z}{d_3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & d_3 \\ 0 & -1 & 0 & d_2 \\ 1 & 0 & 0 & a+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

This matrix is *never* singular, and so the mechanism has *no singularities.*
This should be clear to you by inspcetion; however, if it is not clear to you, compute the determinant and you will see that it is never zero, regardless of the values of the d_i.

6. We computed the Jacobian for the SCARA robot in class. The Mathematica we generated is available on the course web page: `http://controls.ame.nd.edu/me469/scara.nb.ps`

From that, we have

$$\mathcal{V} = \begin{bmatrix} -\left(\dot\theta_1 \sin(\theta_1)\, l_1\right) - \left(\dot\theta_1 + \dot\theta_2\right) \sin(\theta_1+\theta_2)\, l_2 \\ \dot\theta_1 \cos(\theta_1)\, l_1 + \left(\dot\theta_1 + \dot\theta_2\right) \cos(\theta_1+\theta_2)\, l_2 \\ \dot d_4 \\ -\dot\theta_1 - \dot\theta_2 - \dot\theta_3 \end{bmatrix}.$$

"Factoring out" the joint velocity terms gives:

$$\mathcal{V} = \begin{bmatrix} -(\sin(\theta_1)\, l_1) - \sin(\theta_1+\theta_2)\, l_2 & -(\sin(\theta_1+\theta_2)\, l_2) & 0 & 0 \\ \cos(\theta_1)\, l_1 + \cos(\theta_1+\theta_2)\, l_2 & \cos(\theta_1+\theta_2)\, l_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \\ \dot d_4 \end{bmatrix},$$

so

$$J = \begin{bmatrix} -(\sin(\theta_1)\, l_1) - \sin(\theta_1+\theta_2)\, l_2 & -(\sin(\theta_1+\theta_2)\, l_2) & 0 & 0 \\ \cos(\theta_1)\, l_1 + \cos(\theta_1+\theta_2)\, l_2 & \cos(\theta_1+\theta_2)\, l_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Substituting $\theta_1 = \theta_2 = \theta_3 = 30°$ and $d_4 = 0$, and also $l_1 = l_2 = 1$ gives

$$J = \begin{bmatrix} -1.36603 & -0.866025 & 0 & 0 \\ 1.36603 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Mathematica's `Eigensystem[J . Transpose[J]]` gives

```
{{6.86611,1.,0.821624,0.0443155},
{{0.600767,-0.526938,0.,0.601178},
{0.,0.,1.,0.},
{0.375466,-0.477923,0.,-0.794113},
{0.705765,0.702799,0.,-0.0892733}}}
```

where the first element of the list contains the four Eigenvalues. The Eigenvector corresponding to the smallest Eigenvalue is the last one. Therefore, the direction of maximum mechanical advantage is in the direction of the last Eigenvector.

The direction of maximum velocity amplitude is in the direction of the Eigenvector corresponding to the largest Eigenvalue. Therefore, the direction of maximum velocity amplitude is in the direction of the first Eigenvector.