1. (Craig, 2.3)
The answer for this can be found in the Appendix. In the notation of the book, $^A_B R_{Z'X'Y'}(\alpha, \beta, \gamma)$ stands for the $ZXY$ Euler angle matrix. We can look this up in the Appendix, and see that if $\alpha = \theta$, $\beta = \phi$ and $\gamma = 0$, then
\[
^A_B R_{Z'X'Y'}(\theta, \phi, 0) = \begin{bmatrix}
\cos \theta & -\sin \theta \cos \phi & \sin \theta \sin \phi \\
\sin \theta & \cos \theta \cos \phi & -\cos \theta \sin \phi \\
0 & \sin \phi & \cos \phi
\end{bmatrix}
\]

You could of also used the formula for $R_{Z'X'Y'}$ to get the same answer since the third rotation was not specified.

2. (Craig, 2.5)
The definition of an eigenvector is any vector $\phi$ that satisfies the equation
\[
^A_B R \phi = \lambda \phi
\]
for some $\lambda$. If $\lambda = 1$, then the equation is
\[
^A_B R \phi = \phi
\]
which indicates that $\phi$ is not rotated by $^A_B R$. The only vector that is not changed by a rotation is the **axis of rotation**.

3. (Craig, 2.14)
We are looking for $^A_B T$, where, starting from initial coincidence, frame $B$ is rotated by $\theta$ about $K$, where $\hat{K}$ (a unit vector, not a matrix) passes through the point $^A P$.
To do this, we will define two other frames, $A'$ and $B'$, where the origins of $A'$ and $B'$ are located at the point $^A P$, $A$ and $A'$ have the same orientation, $B'$ is rotated relative to $A'$ by an amount $\theta$ about $\hat{K}$, and $B'$ and $B$ have the same orientation. See Figure 2.20 in the book. This idea is exactly the same idea as Example 2.9 in the book.
Now, we know that
\[
^A_B T = ^A_{A'} T'_{B'} T_B.
\]
Since $A$ and $A'$ only differ by a translation,

\[ A_A T = \begin{bmatrix} I & A^P \\ 0 & 1 \end{bmatrix} \]

where $I$ is the $3 \times 3$ identity matrix.

Now, $A'$ and $B'$ only differ by a rotation about $\hat{K}$. Therefore,

\[ A_{B'} T = \begin{bmatrix} R_K(\theta) & 0 \\ 0 & 1 \end{bmatrix}, \]

where $R_K(\theta)$ is given by Rodrigues’ formula (like I like) or Equation 2.80 in the book.

Finally, $B'$ and $B$ differ only by translation along negative $A^P$, i.e.,

\[ B_B T = \begin{bmatrix} I & -A^P \\ 0 & 1 \end{bmatrix}. \]

Multiplying everything together gives:

\[ A_{B} T = \begin{bmatrix} I & A^P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_K(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & -A^P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_K(\theta) & -R_K(\theta)A^P + A^P \\ 0 & 1 \end{bmatrix}. \]

4. (Craig, 2.20)

Let

\[ Q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}. \]

Then,

\[ Q' = R_K(\theta)Q = \begin{bmatrix} k_xk_yq_xv\theta + q_xc\theta + k_xk_yq_yv\theta - k_xq_yq_zc\theta + k_xk_yq_zs\theta + k_xk_yq_zs\theta \\ k_xk_yq_xv\theta + k_xq_yq_zc\theta + k_xk_yq_yv\theta + k_xk_yq_zs\theta + k_xq_yq_zs\theta + k_xq_yq_zs\theta \\ k_xk_yq_xv\theta - k_yq_zc\theta + k_yk_xq_yv\theta + k_yq_zs\theta + k_yq_zs\theta + k_yq_zs\theta \end{bmatrix}. \]

Note that each row has a term of the form $q_zc\theta$. Separating these terms will give the $Q\cos\theta$ term.

Note also that

\[(1 - \cos\theta)(K \cdot Q)K = \begin{bmatrix} v(k_xq_x + k_yq_y + k_zq_z)k_x \\ v(k_xq_x + k_yq_y + k_zq_z)k_y \\ v(k_xq_x + k_yq_y + k_zq_z)k_z \end{bmatrix}, \]

where, as usual, $v = 1 - \cos\theta$.

Finally, note that

\[ \sin\theta(K \times Q) = \begin{bmatrix} \sin\theta(-k_zq_y + k_yq_z) \\ \sin\theta(k_zq_x - k_xq_z) \\ \sin\theta(k_yq_x - k_xq_y) \end{bmatrix}. \]
Inspecting Equation 1 verifies that

\[ Q' = Q \cos \theta + \sin \theta (K \times Q) + (1 - \cos \theta)(K \cdot Q)K, \]

as required.

5. (Craig, 2.23)

We are need to determine

\[ U_A T = \begin{bmatrix} \overset{\text{I}}{U}_A R & U_{PA_{0RG}} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

so we need to determine \( U_{PA_{0RG}} \) and \( \overset{\text{I}}{U}_A R \) where the first column of \( \overset{\text{I}}{U}_A R \) is \( \hat{X}_A \) written in the U frame, the second column in the \( \hat{Y}_A \) written in the U frame, etc. The relevant data is illustrated in Figure 1.

Here is the algorithm:

(a) \( U_{PA_{0RG}} = U P_1 \) (this is given).

(b) Since \( U P_2 \) is on the \( \hat{X}_A \) axis, then it follows that \( U P_2 - U P_1 \) points along the \( \hat{X}_A \) axis, but it may not be a unit vector. Therefore,

\[ \hat{X}_A = \frac{U P_2 - U P_1}{||U P_2 - U P_1||}. \]
(c) By the same reasoning as for the $\hat{X}_A$ component of $U_A R$, we know that $U P_3 - U P_1$ points from the origin of $A$ to a point near the $\hat{Y}_A$ axis and is in the $XY$-plane of frame $A$. Therefore, we can find the component of $U P_3 - U P_1$ that is in the direction of $\hat{X}_A$ and subtract from $U P_3 - U P_1$, then what remains will point along the $\hat{Y}_A$ axis, i.e.,

$$
\hat{Y}_A = \frac{(U P_3 - U P_1) - \left( (U P_3 - U P_1) \cdot \hat{X}_A \right) \hat{X}_A}{\| (U P_3 - U P_1) - \left( (U P_3 - U P_1) \cdot \hat{X}_A \right) \hat{X}_A \|}.
$$

(d) $\hat{Z}_A$ is easy:

$$
\hat{Z}_A = \hat{X}_A \times \hat{Y}_A.
$$

Now, all the information contained in $U_A T$ has been determined.

6. (Craig, 2.35)

This problem required you to remember two facts from linear algebra:

(a) $\det(AB) = \det(A)\det(B)$ and

(b) $\det(A^T) = \det(A)$.

So,

$$
R^T R = I
$$

$$
\implies \det(R^T R) = \det(I) = 1
$$

$$
\implies \det(R^T)\det(R) = 1
$$

$$
\implies \det(R)^2 = 1
$$

$$
\implies \det(R) = \pm 1.
$$

To prove the answer is $+1$ and not $-1$ requires another fact (I gave full credit if you got this far). Let

$$
R = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}
$$

and note that

$$
\det(R) = r_1 \cdot (r_2 \times r_3).
$$

Since

$$
(r_2 \times r_3) = r_1
$$

then

$$
\det(R) = +1.
$$

7. (Craig, 2.38)

For two unit vectors, we know that the cosine of the angle between them is given by their dot product.
Since,

\[ v_1 \cdot v_2 = v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z}, \]

where

\[ v_1 = \begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{bmatrix}, \]

these dot products can be written in vector form

\[ v_1 \cdot v_2 = v_1^T v_2 = \begin{bmatrix} v_{1x} & v_{1y} & v_{1z} \end{bmatrix} \begin{bmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{bmatrix}. \]

Let \( R \) be a rotation matrix. If the angle between vectors is preserved by a rigid body rotation, the cosine of the angle is preserved. Therefore,

\[ v_1^T v_2 = (Rv_1)^T (Rv_2) \]

\[ = v_1^T R^T R v_2. \]

Since this is true for any two vectors \( v_1 \) and \( v_2 \), it must be the case that

\[ R^T R = I, \]

which shows that the transpose of a rotation matrix is its inverse.