University of Notre Dame
Aerospace and Mechanical Engineering

AME 469: Introduction to Robotics
Homework 2 Solutions

B. Goodwine
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1. (Craig, 2.3)

   The answer for this can be found in the Appendix. In the notation of the book, ${}^A_B R_{Z'X'Y'}(\alpha, \beta, \gamma)$ stands for the ZXY Euler angle matrix. We can look this up in the Appendix, and see that if $\alpha = \theta$, $\beta = \phi$ and $\gamma = 0$,, then

   $$ {}^A_B R_{Z'X'Y'}(\theta, \phi, 0) = \begin{bmatrix} \cos\theta & -\sin\theta\cos\phi & \sin\theta\sin\phi \\ \sin\theta & \cos\theta\cos\phi & -\cos\theta\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} $$

   You could of also used the formula for $R_{Z'X'Z'}$ to get the same answer since the third rotation was not specified.

2. (Craig, 2.5)

   The definition of an eigenvector is any vector ϕ that satisfies the equation

   $$ {}^A_B R\phi = \lambda\phi $$

   for some λ. If $\lambda = 1$, then the equation is

   $$ {}^A_B R\phi = \phi $$

   which indicates that ϕ is not rotated by ${}^A_B R$. The only vector that is not changed by a rotation is the **axis of rotation**.

3. (Craig, 2.14)

   We are looking for ${}^A_B T$, where, starting from initial coincidence, frame B is rotated by θ about K, where $\hat{K}$ (a unit vector, *not* a matrix) passes through the point ${}^A P$.

   To do this, we will define two other frames, A' and B', where the origins of A' and B' are located at the point ${}^A P$, A and A' have the same orientation, B' is rotated relative to A' by an amount θ about $\hat{K}$, and B' and B have the same orientation. See Figure 2.20 in the book. This idea is exactly the same idea as Example 2.9 in the book.

   Now, we know that

   $$ {}^A_B T = {}^A_{A'} T {}^{A'}_{B'} T {}^{B'}_B T. $$

Since A and A' only differ by a translation,

$$ {}^{A}_{A'}T = \left[\begin{array}{cc} I & {}^{A}P \\ 0 & 1 \end{array} \right] $$

where I is the 3×3 identity matrix.

Now, A' and B' only differ by a rotation about $\hat{K}$. Therefore,

$$ {}^{A'}_{B'}T = \left[\begin{array}{cc} R_K(\theta) & 0 \\ 0 & 1 \end{array} \right], $$

where $R_K(\theta)$ is given by Rodrigues' formula (like I like) or Equation 2.80 in the book.

Finally, B' and B differ only by translation along *negative* A^P, *i.e.*,

$$ {}^{B'}_{B}T = \left[\begin{array}{cc} I & -{}^{A}P \\ 0 & 1 \end{array} \right]. $$

Multiplying everything together gives:

$$ \boxed{ {}^{A}_{B}T = \left[\begin{array}{cc} I & {}^{A}P \\ 0 & 1 \end{array} \right] \left[\begin{array}{cc} R_K(\theta) & 0 \\ 0 & 1 \end{array} \right] \left[\begin{array}{cc} I & -{}^{A}P \\ 0 & 1 \end{array} \right] = \left[\begin{array}{cc} R_K(\theta) & -R_K(\theta){}^{A}P +{}^{A} P \\ 0 & 1 \end{array} \right]. } $$

4. (Craig, 2.20)

Let

$$ Q = \left[\begin{array}{c} q_x \\ q_y \\ q_z \end{array} \right] \qquad \text{and} \qquad K = \left[\begin{array}{c} k_x \\ k_y \\ k_z \end{array} \right]. $$

Then,

$$ Q' = R_K(\theta)Q = \left[\begin{array}{c} k_x k_x q_x v\theta + q_x c\theta + k_x k_y q_y v\theta - k_z q_y s\theta + k_x k_z q_z v\theta + k_y q_z s\theta \\ k_x k_y q_x v\theta + k_z q_x s\theta + k_y k_y q_y v\theta + q_y c\theta + k_y k_z q_z v\theta + k_z q_z s\theta \\ k_x k_z q_x v\theta - k_y q_x s\theta + k_y k_z q_y v\theta + k_x q_y s\theta + k_z k_z q_z v\theta + q_z c\theta \end{array} \right]. \tag{1} $$

Note that each row has a term of the form $q_? c\theta$. Separating these terms will give the $Q\cos\theta$ term.

Note also that

$$ (1-\cos\theta)(K \cdot Q)K = \left[\begin{array}{c} v(k_x q_x + k_y q_y + k_z q_z)k_x \\ v(k_x q_x + k_y q_y + k_z q_z)k_y \\ v(k_x q_x + k_y q_y + k_z q_z)k_z \end{array} \right], $$

where, as usual, $v = 1 - \cos\theta$.

Finally, note that

$$ \sin\theta(K \times Q) = \left[\begin{array}{c} \sin\theta(-k_z q_y + k_y q_z) \\ \sin\theta(k_z q_x - k_x q_z) \\ \sin\theta(k_y q_x - k_x q_y) \end{array} \right]. $$

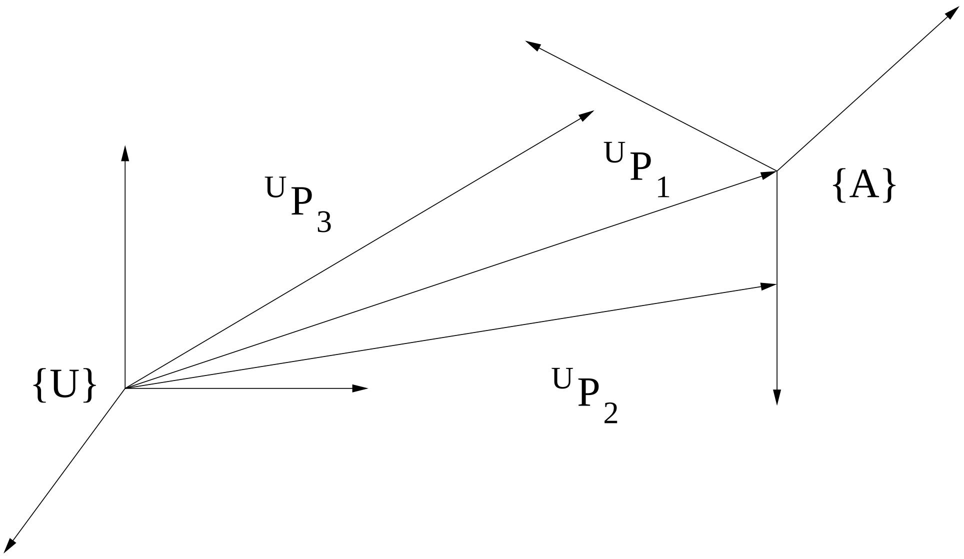

Figure 1. Frames for problem 2.23.

Inspecting Equation 1 verifies that

$$Q' = Q\cos\theta + \sin\theta(K \times Q) + (1 - \cos\theta)(K \cdot Q)K,$$

as required.

5. (Craig, 2.23)

We are need to determine

$$^U_AT = \begin{bmatrix} ^U_AR & ^UP_{AORG} \\ 000 & 1 \end{bmatrix},$$

so we need to determine $^UP_{AORG}$ and ^U_AR where the first column of ^U_AR is $\hat{X}_A$ written in the U frame, the second column in the $\hat{Y}_A$ written in the U frame, *etc.* The relevant data is illustrated in Figure 1.

Here is the algorithm:

(a) $^UP_{AORG} =^U P_1$ (this is given).

(b) Since $^UP_2$ is on the $\hat{X}_A$ axis, then it follows that $^UP_2 -^U P_1$ points along the $\hat{X}_A$ axis, but it may not be a unit vector. Therefore,

$$\hat{X}_A = \frac{^UP_2 -^U P_1}{\|^UP_2 -^U P_1\|}.$$

(c) By the same reasoning as for the $\hat{X}_A$ component of ${}^U_A R$, we know that ${}^U P_3 - {}^U P_1$ points from the origin of A to a point near the $\hat{Y}_A$ axis and is in the XY-plane of frame A. Therefore, we can find the component of ${}^U P_3 - {}^U P_1$ that is in the direction of $\hat{X}_A$ and subtract from ${}^U P_3 - {}^U P_1$, then what remains will point along the $\hat{Y}_A$ axis, *i.e*,

$$\hat{Y}_A = \frac{({}^U P_3 - {}^U P_1) - \left(({}^U P_3 - {}^U P_1) \cdot \hat{X}_A\right) \hat{X}_A}{\|({}^U P_3 - {}^U P_1) - \left(({}^U P_3 - {}^U P_1) \cdot \hat{X}_A\right) \hat{X}_A\|}.$$

(d) $\hat{Z}_A$ is easy:

$$\hat{Z}_A = \hat{X}_A \times \hat{Y}_A.$$

Now, all the information contained in ${}^U_A T$ has been determined.

6. (Craig, 2.35)

   This problem required you to remember two facts from linear algebra:

   (a) $\det(AB) = \det(A)\det(B)$ and

   (b) $\det(A^T) = \det(A)$.

   So,

$$\begin{aligned} & R^T R = I \\ \Longrightarrow\quad & \det(R^T R) = \det(I) = 1 \\ \Longrightarrow\quad & \det(R^T)\det(R) = 1 \\ \Longrightarrow\quad & \det(R)^2 = 1 \\ \Longrightarrow\quad & \det(R) = \pm 1. \end{aligned}$$

   To prove the answer is $+1$ and not -1 requires another fact (I gave full credit if you got this far). Let

$$R = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

   and note that

$$\det(R) = r_1 \cdot (r_2 \times r_3).$$

   Since

$$(r_2 \times r_3) = r_1$$

   then

$$\det(R) = +1.$$

7. (Craig, 2.38)

   For two unit vectors, we know that the cosine of the angle between them is given by their dot product.

Since,

$$v_1 \cdot v_2 = v_{1_x} v_{2_x} + v_{1_y} v_{2_y} + v_{1_z} v_{2_z},$$

where

$$v_1 = \begin{bmatrix} v_{1_x} \\ v_{1_y} \\ v_{1_z} \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} v_{2_x} \\ v_{2_y} \\ v_{2_z} \end{bmatrix},$$

these dot products can be written in vector form

$$v_1 \cdot v_2 = v_1^T v_2 = \begin{bmatrix} v_{1_x} & v_{1_y} & v_{1_z} \end{bmatrix} \begin{bmatrix} v_{2_x} \\ v_{2_y} \\ v_{2_z} \end{bmatrix}.$$

Let R be a rotation matrix. If the angle between vectors is preserved by a rigid body rotation, the cosine of the angle is preserved. Therefore,

$$\begin{aligned} v_1^T v_2 &= (Rv_1)^T (Rv_2) \\ &= v_1^T R^T R v_2. \end{aligned}$$

Since this is true for any two vectors v_1 and v_2, it must be the case that

$$R^T R = I,$$

which shows that the transpose of a rotation matrix is its inverse.