AME 469: Introduction to Robotics  
Homework 3 Solutions  

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1. This is simply Equation 3.6 implemented in Mathematica:  

\[ T[\text{alph}_-, a_-, d_-, \text{th}_-] := \]  
\[ \{\{\cos[\text{th}], -\sin[\text{th}], 0, a\}, \]  
\[ \{\sin[\text{th}]*\cos[\text{alph}], \cos[\text{th}]*\cos[\text{alph}], -\sin[\text{alph}], -\sin[\text{alph}]*d\}, \]  
\[ \{\sin[\text{th}]*\sin[\text{alph}], \cos[\text{th}]*\sin[\text{alph}], \cos[\text{alph}], \cos[\text{alph}]*d\}, \]  
\[ \{0, 0, 0, 1\}\} \]  

Can be used, for example, as  

\[ T[\pi/2, 0, 0, \text{theta}] \text{ //MatrixForm} \]  

which will print out transformation with those parameter values in a nice matrix form in the Mathematica notebook.  

2. (Craig, 3.9)  
We were given $^0T$, and want to find the location of the tip in fram 0.  
Clearly,  

\[ ^0P_{\text{tip}} = ^0T \cdot ^2P_{\text{tip}}, \]  

where  

\[ ^2P_{\text{tip}} = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}, \]  

or  

\[ ^0P_{\text{tip}} = \begin{bmatrix} l_2 \cos\theta_1 \cos\theta_2 + l_1 \cos\theta_1 \\ l_2 \sin\theta_1 \cos\theta_2 + l_1 \sin\theta_1 \\ l_2 \sin\theta_2 \end{bmatrix}. \]
Figure 1. Frames for problem 3.
3. (Craig, 3.17)

Following the usual rules for affixing frames to manipulators results in the frames illustrated in Figure 1.

Consulting the figure, and naming the distance \( a_2 \), the following link parameters are apparent:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( a_{i-1} )</th>
<th>( a_{i-1} )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \theta_1 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \theta_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( a_2 )</td>
<td>( d_3 )</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Determining \( ^0T_{\frac{1}{2}}T \) and \( ^3T \) is now a simple matter of substituting these values into Equation 3.6, or, even simpler, using the Mathematica function from Problem 1.

\[
<<\text{forward.m} \\
T[0,0,0,t1] . T[\Pi/2,0,0,t2] . T[\Pi/2,a2,d3,0] //\text{Simplify}
\]

produces the mess

\[
\{(\text{Cos}[t1], \text{Cos}[t2], \text{Sin}[t1], \text{Cos}[t1] \text{Sin}[t2]), \text{Cos}[t1] (a2, \text{Cos}[t2]+d3, \text{Sin}[t2])\},\{\text{Cos}[t2], \text{Sin}[t1], -\text{Cos}[t1], \text{Sin}[t1] (a2, \text{Cos}[t2]+d3, \text{Sin}[t2])\},\{\text{Sin}[t2], 0, -\text{Cos}[t2], -d3, \text{Cos}[t2]+a2, \text{Sin}[t2]\},\{0,0,0,1\}\}
\]

or

\[
^3T_{0} = \begin{bmatrix}
\cos(\theta_1) \cos(\theta_2) & \sin(\theta_1) & \cos(\theta_1) \sin(\theta_2) & \cos(\theta_1) (a_2 \cos(\theta_2) + d_3 \sin(\theta_2)) \\
\cos(\theta_2) \sin(\theta_1) & -\cos(\theta_1) & \sin(\theta_1) \sin(\theta_2) & \sin(\theta_1) (a_2 \cos(\theta_2) + d_3 \sin(\theta_2)) \\
\sin(\theta_2) & 0 & -\cos(\theta_2) & \sin(\theta_2) (a_2 \cos(\theta_2) + d_3 \sin(\theta_2)) \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

To check to make sure it makes sense, take all the joint angles, \( \theta_1, \theta_2 \) and \( d_3 \) to be zero:

\[
<<\text{forward.m} \\
T[0,0,0,0] . T[\Pi/2,0,0,0] . T[\Pi/2,a2,0,0] //\text{Simplify}
\]

which gives the expected answer

\[
^0T_{3} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & a_2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Figure 2. Frames for problem 4.
4. (Craig, 3.18) 
Following the usual rules for affixing frames to manipulators results in the frames illustrated in Figure 2.

Consulting the figure, and naming the distance $a_2$, the following link parameters are apparent:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$a_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
<td>$\theta_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$a_2$</td>
<td>$\theta_3$</td>
<td></td>
</tr>
</tbody>
</table>

Determining $^0T_{2}T$ and $^3T$ is now a simple matter of substituting these values into Equation 3.6, or, even simpler, using the Mathematica function from Problem 1.

<<forward.m
T[0,0,0,t1] . T[Pi/2,0,0,t2] . T[0,a2,0,t3] //Simplify

gives

$$^3T_0 = \begin{bmatrix} \cos(\theta_1) \cos(\theta_2 + \theta_3) & -\cos(\theta_1) \sin(\theta_2 + \theta_3) & \sin(\theta_1) & a_2 \cos(\theta_1) \cos(\theta_2) \\ \cos(\theta_2 + \theta_3) \sin(\theta_1) & -\sin(\theta_2 + \theta_3) \sin(\theta_1) & -\cos(\theta_1) & a_2 \cos(\theta_2) \sin(\theta_1) \\ \sin(\theta_2 + \theta_3) & \cos(\theta_2 + \theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To check to make sure it makes sense, take all the joint angles, $\theta_1, \theta_2$ and $\theta_3$ to be zero:

<<forward.m
T[0,0,0,0] . T[Pi/2,0,0,Pi/2] . T[0,a2,0,0] //Simplify

which gives the expected answer

$$^0T_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. (Craig, 3.19)

For this problem, take the zero position of the links as follows:

- for link 2, let $d_2 = 0$ when the link is all the way “down” and
- for link 3, let $d_3 = 0$ when the wrist is flush against the joint stop, i.e., all the way to the left.
Figure 3. Link frame assignments for problem 5.
The link frame assignments are illustrated in Figure 3.
Consulting the figure and naming the distance $a_2$, the following link parameters are apparent:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\theta_1$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$a_1 + a_2$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{\pi}{2}$</td>
<td>$a_2$</td>
<td>$d_3 + a_3$</td>
<td>0</td>
</tr>
</tbody>
</table>

Plugging into Equation 3.6, or the mathematica function gives the forward kinematics

$$\begin{pmatrix}
-a_1 \\ a_2 \\ 0 \\ 0
\end{pmatrix} =
\begin{bmatrix}
-\sin \theta_1 & 0 & \cos \theta_1 & (d_3 + a_3) \cos \theta_1 - a_2 \sin \theta_1 \\
\cos \theta_1 & 0 & \sin \theta_1 & a_2 \cos \theta_1 + (d_3 + a_3) \sin \theta_1 \\
0 & 1 & 0 & a_1 + d_2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
\theta_1 \\ \theta_2 \\ \theta_3 \\ 1
\end{pmatrix}
$$

6. (Craig, 3.20)
Following the usual rules for affixing frames to manipulators results in the frames illustrated in Figure 4.
Consulting the figure, and naming the distances $a_1$ and $a_2$, the following link parameters are apparent:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\alpha_{i-1}$</th>
<th>$a_{i-1}$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$d_1$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$a_1$</td>
<td>0</td>
<td>$\theta_2$</td>
</tr>
<tr>
<td>3</td>
<td>$a_2$</td>
<td>0</td>
<td>0</td>
<td>$\theta_3$</td>
</tr>
</tbody>
</table>

Determining $^0T_1$, $^1T_2$ and $^2T_3$ is now a simple matter of substituting these values into Equation 3.6, or, even simpler, using the Mathematica function from Problem 1.

```
<<forward.m
T[0,0,d1,0] . T[0,a1,0,t2] . T[0,a2,0,t3] //Simplify
```
gives

$$^3_0T =
\begin{bmatrix}
\cos(\theta_2 + \theta_3) & -\sin(\theta_2 + \theta_3) & 0 & a_1 + \cos(\theta_2) a_2 \\
\sin(\theta_2 + \theta_3) & \cos(\theta_2 + \theta_3) & 0 & \sin(\theta_2) a_2 \\
0 & 0 & 1 & d_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

To check to make sure it makes sense, take all the joint angles, $\theta_1$, $\theta_2$ and $\theta_3$ to be zero:

```
<</home/bill/courses/me469/math/forward.m
T[0,0,0,0] . T[Pi/2,0,0,0] . T[0,a2,0,0] //Simplify
```

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Figure 4. Frames for problem 6.
which gives the expected answer

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$