AME 301: Differential Equations, Control and Vibrations

Introduction, Overview and Motivation

Some boring catalog stuff:

- First of a two-course sequence, the course introduces methods of differential-equation solution together with common engineering applications in vibration analysis and controls. Second-order, linear differential equations, feedback control and numerical solutions to systems of ordinary differential equations.

- The objective of this course are for students to be able to solve certain classes of ordinary differential equations, analyze the stability of solutions of a system of differential equations and apply techniques from the theory of differential equations to design and analyze the stability of control systems and engineering vibrations problems.

What are Differential Equations?

- **Many** physical phenomena are governed by principles which involve the *rates*, *i.e.*, derivatives, at which they change (examples to follow).
- Thus, the governing equations may involve both the value of some quantity as well as its derivative.
- Simply put, a *differential equation* is an equation that contains derivatives.
- Some general examples — the underlying principles of the following fields are described by differential equations:
  - mechanics ($F = ma = m\ddot{x}$), (including fluid mechanics and aerodynamics),
  - thermodynamics,
  - heat transfer,
  - electrical circuits,
  - combustion,
  - nuclear reactions, *etc.*

Why Study Differential Equations?

- To be able to solve problems in all the important areas listed on the previous slide.
- To gain fundamental insight to the governing physical phenomena in those areas.
- To develop a level of mathematical sophistication that is appropriate and expected in the engineering profession.
- A famous quote:

  The burden of the lecture is just to emphasize the fact that it is impossible to explain honestly the beauty of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics. I am sorry, but this seems to be the case.
  Richard Feynman, *The Character of Physical Law*

Some Examples:

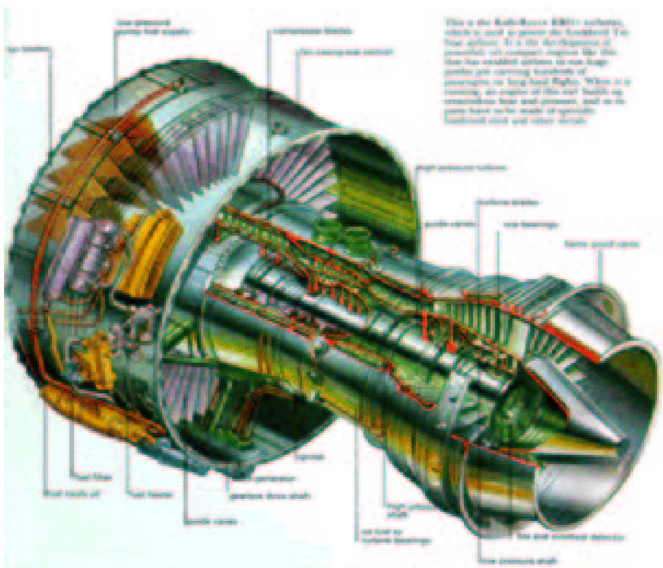
Question: what processes on an airplane are described by differential equations?

A few answers

Each of the following are obviously critical to the efficient and safe operation of an aircraft:

- the basic aerodynamics,
- the relationship between control surface positions (ailerons, rudder, elevator, *etc,* and the response of the aircraft,
- the dynamics of the hydraulic actuators,
- the functioning of the autopilot,
- the structural dynamics of the airplane (*e.g.,* wing vibrations),
- the heat transfer into and out of the airplane and the pressurization of the cabin (each in turn related to thermal expansion and contraction, which is related to the fatigue life of the airplane),
- wheel dynamics, braking dynamics and heat transfer,
- *etc.*

Consider only the engines:



Differential equations describe:

- compressible flow through the engine,
- the thermodynamic processes,
- the combustion process,
- heat transfer (cooling the turbine blades currently is a main performance limitation),
- mechanical vibration,
- overall engine dynamics,
- *etc.*

Other Applications

While we focused on an airplane example, differential equations also govern the fundamental operation of important areas such as automobile dynamics, automobile tire dynamics, automobile aerodynamics, automobile acoustics (wind, engine, exhaust, brake and tire noise), automobile active control systems (including speed control, engine performance and emissions control, climate control, ABS control systems, airbag deployment systems, *etc.*), structural dynamics of buildings, bridges and dams (*e.g.,* earthquake and wind engineering), industrial process control, control and operation of automation (robotic) systems, HVAC systems, the operation of the electric power grid, electric power generation (the generators as well as the process of energy creation via combustion, nuclear reactions, solar, wind, *etc.,*), orbital dynamics of satellite systems, heat transfer from electrical equipment (including computer chips), acoustics (accurate amplification and reproduction of music), highway traffic dynamics analyses, economic systems, biological systems, chemical systems *etc, etc, etc*

Solutions to Differential Equations

- "Solving" a differential equation means determining a **function** that satisfies an equation containing one or more derivatives of that function.

- Example:

$$\frac{dx(t)}{dt} = x(t), \qquad x(0) = 1.$$

  - t is the *independent variable*,
  - $x(t)$ is the solution,
  - $x(0) = 1$ is the *initial condition*,
  - in this case $x(t) = e^t$ since it satisfies both the differential equation as well as the initial condition.

- Alternative representations: $\dot{x} = x$, $y'(x) = y(x)$, $\frac{d\xi(\theta)}{d\theta} = \xi(\theta)$ are all the same equation.

Categorization of Differential Equations

- **Ordinary** versus **partial** differential equations:
  - A differential equation is ordinary (o.d.e.) if it only has one independent variable.
  - A differential equation is a partial differential equation (p.d.e.) if it has more than one independent variable (so there are partial derivatives).
  - Examples:
    * o.d.e.: $\ddot{x} + \dot{x} + x = \sin(t)$ (harmonic oscillation)
    * p.d.e.: $a^2 \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{\partial^2 y(x,t)}{\partial t^2}$ (the "wave equation" (vibrating string).
- If more than one unknown function need to be determined, then a **system** of equations is needed. Example:

$$\begin{aligned} \dot{h} &= ah - \alpha hp \\ \dot{p} &= -cp + \gamma hp. \end{aligned}$$

Need two equations to determine both $h(t)$ and $p(t)$ (the Lotka-Volterra predator-prey equations)

Linear versus Nonlinear Equations

- This is a *critical* distinction since the former are much easier to deal with than the latter.
- Mathematically, the ordinary differential equation

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2} \dots, \frac{d^nx}{dt^n}\right) = 0$$

is linear if it it is a linear function in the variables $x, \dots, \frac{d^nx}{dt^n}$ (note: it doesn't have to be linear in t).

- An equation is linear if it can be converted to the general form

$$a_nf_n(t)\frac{d^nx}{dt^n} + a_{n-1}f_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_0f_0(t)x = g(t),$$

where the a_i are constants and the $f_i(t)$ and $g(t)$ are functions of t.

Linear versus Nonlinear

- Examples:

Linear	Nonlinear
$\ddot{x} + \dot{x} + x = 0$	$5\ddot{x} + \sin x = 0$
$5t^3\ddot{x} + \sin(t)x = 0$	$x\dot{x} + x = 0$
$\sin(t^2)\ddot{x} + 10x = e^t$	$\ddot{x} + \dot{x} + x^3 = 0$
$\frac{dt}{dx} + t = x^2$	$(1 + y^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^x$

- Simply check that the unknown function and its derivatives are only multiplied by functions of the independent variable or constants.

The Order of a Differential Equation

- One more definition: the **order** of a differential equation is the order of the highest derivative.

- Examples:

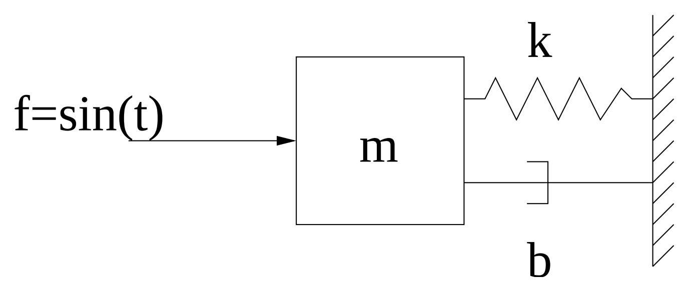
$\ddot{x} + \dot{x} + x = 0$	second order
$5t^3\ddot{x} + \sin(t)x = 0$	second order
$\sin(t^2)\dddot{x} + 10x = e^t$	third order
$\frac{dt}{dx} + t = x^2$	first order

Vibration Analysis

- Vibration analysis is an application of differential equations (typically involving second order, ordinary differential equations with constant coefficients, *e.g.*,

$$m\ddot{x} + b\dot{x} + kx = \sin(t).$$

- Earthquake and wind engineering,
- vibration of machines with rotating components (motors, jet engines, etc.)
- automotive and aircraft engineering (engine vibration, wing and structure vibration, suspension desing, *etc.*).



Concepts in Vibrations

- Forced response:

$$m\ddot{x} + b\dot{x} + kx = \sin t,$$

- unforced response:

$$m\ddot{x}+b\dot{x}+kx = 0 \qquad x(0) = 1,$$

- resonance,
- damping,
- logrithmic decrement,
- *etc.*