Solution HW # 6 AME 302

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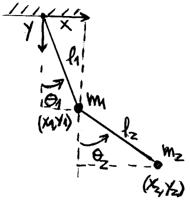

$g = 0$

Lagrangian L of the system is: $L = T \leftarrow$ Kinetic Energy

$$L = (T_1 + T_2)$$

$$T_1 = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) \quad , \; m_1 \text{ has coordinates } (x_1, y_1)$$

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) \quad , \; m_2 \text{ has coordinates } (x_2, y_2)$$

$$x_1 = l_1 \sin\theta_1$$

$$y_1 = l_1 \cos\theta_1$$

$$x_2 = l_2 \sin\theta_2 + x_1 = l_2 \sin\theta_2 + l_1 \sin\theta_1$$

$$y_2 = l_2 \cos\theta_2 + y_1 = l_2 \cos\theta_2 + l_1 \cos\theta_1$$

therefore

$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1$$
$$\dot{y}_1 = -l_1 \dot{\theta}_1 \sin\theta_1$$
$$\dot{x}_2 = l_2 \dot{\theta}_2 \cos\theta_2 + l_1 \dot{\theta}_1 \cos\theta_1$$
$$\dot{y}_2 = -l_2 \dot{\theta}_2 \sin\theta_2 - l_1 \dot{\theta}_1 \sin\theta_1$$

then

$$\dot{x}_1^2 = l_1^2 \dot{\theta}_1^2 \cos^2\theta_1$$
$$\dot{y}_1^2 = l_1^2 \dot{\theta}_1^2 \sin^2\theta_1$$
$$\dot{x}_2^2 = l_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + l_1^2 \dot{\theta}_1^2 \cos^2\theta_1$$
$$\dot{y}_2^2 = l_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2 + l_1^2 \dot{\theta}_1^2 \sin^2\theta_1$$

$$T_1 = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{m_1}{2}\left(l_1^2 \dot{\theta}_1^2 \left(\cos^2\theta_1 + \sin^2\theta_1\right)\right)$$

$$\Rightarrow T_1 = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2$$

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{m_2}{2}\Big(l_2^2 \dot{\theta}_2^2 \left(\cos^2\theta_2 + \sin^2\theta_2\right) + l_1^2 \dot{\theta}_1^2 \left(\cos^2\theta_1 + \sin^2\theta_1\right)$$
$$+ l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left(2\cos\theta_1 \cos\theta_2 + 2\sin\theta_1 \sin\theta_2\right)\Big)$$

$$\Rightarrow T_2 = \frac{m_2}{2}\left(l_2^2 \dot{\theta}_2^2 + l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)$$

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$$L = T_1 + T_2$$

$$\boxed{L = \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} l_2^2 \dot{\theta}_2^2 + \frac{m_2}{2} l_1^2 \dot{\theta}_1^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)}$$

Euler - Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

q_i are the generalized coordinates of the system (θ_1 & θ_2)

$Q_i = 0$ since there are no forces ($g=0$)

θ_1: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \quad (1)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - (\dot{\theta}_1 - \dot{\theta}_2) m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$

P1| substituting into equation (1) we get

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - (\dot{\theta}_1-\dot{\theta}_2) m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)$$

$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) = 0 \quad / \frac{1}{l_1}$$

$$\Rightarrow \ddot{\theta}_1 (m_1 l_1 + m_2 l_1) + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - (\dot{\theta}_1-\dot{\theta}_2) m_2 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)$$

$$+ m_2 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) = 0$$

$$\Rightarrow \boxed{\ddot{\theta}_1 = \frac{-m_2 l_2 \left(\ddot{\theta}_2 \cos(\theta_1-\theta_2) + \dot{\theta}_2^2 \sin(\theta_1-\theta_2)\right)}{l_1(m_1+m_2)}} \quad (2)$$

$$\theta_2 : \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \quad (3)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - (\dot{\theta}_1-\dot{\theta}_2) m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2)$$

Substituting into equation (3) we get

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - (\dot{\theta}_1-\dot{\theta}_2) m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2)$$

$$- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) = 0 \quad \Big/ \frac{1}{m_2 l_2}$$

$$\Rightarrow l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) = 0$$

$$\Rightarrow \boxed{\ddot{\theta}_2 = \frac{l_1\left(\dot{\theta}_1^2 \sin(\theta_1-\theta_2) - \ddot{\theta}_1 \cos(\theta_1-\theta_2)\right)}{l_2}} \quad (4)$$

Equations (2) and (4) depend upon each other.

Substituting (4) into (2) $\Rightarrow$ we get $\ddot{\theta}_1$

$$\Rightarrow \boxed{\ddot{\theta}_1 = \frac{-m_2 \cos(\theta_1-\theta_2) l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)}{l_1(m_1+m_2) - m_2 \cos^2(\theta_1-\theta_2)}} \quad (5)$$

Substituting (2) into (4) $\Rightarrow$ we get $\ddot{\theta}_2$

$$\Rightarrow \boxed{\ddot{\theta}_2 = \frac{(m_1+m_2) l_1 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{l_2(m_1+m_2\sin^2(\theta_1-\theta_2))}} \quad (6)$$

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Since $m_1 = m_2 = 2$

$l_1 = l_2 = 1$

$$\Rightarrow \boxed{\ddot{\theta}_1 = \frac{-\dot{\theta}_1^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2) - \dot{\theta}_2^2 \sin(\theta_1-\theta_2)}{2 - \cos^2(\theta_1-\theta_2)}} \quad (7)$$

$$\Rightarrow \boxed{\ddot{\theta}_2 = \frac{2\dot{\theta}_1^2 \sin(\theta_1-\theta_2) + \dot{\theta}_2^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{1 + \sin^2(\theta_1-\theta_2)}} \quad (8)$$