7.1.2 $u'' + 0.5u' + 2u = 3\sin t$

$x_1 = u$

$x_2 = \dot{u} = \dot{x}_1$

$$\dot{x}_2 + 0.5x_2 + 2x_1 = 3\sin t$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 3\sin t - 0.5x_2 - 2x_1 \end{cases}$$

7.1.17 $F = ma = m\frac{\partial^2 x}{\partial t^2}$

$m_1 \frac{\partial^2 x_1}{\partial t^2} = k_2(x_2 - x_1) - k_1 x_1 + F_1(t)$

$m_1 \frac{\partial^2 x_1}{\partial t^2} = -(k_1 + k_2)x_1 + k_2 x_2 + F_1(t)$ } Eq I ✓

$m_2 \frac{\partial^2 x_2}{\partial t^2} = F_2(t) - k_3 x_3 - k_2(x_2 - x_1)$

$m_2 \frac{\partial^2 x_2}{\partial t^2} = k_2 x_1 - (k_2 + k_3)x_2 + F_2(t)$ } Eq I ✓

7.2.4 $A = \begin{bmatrix} 3-2i & 1+i \\ 2-i & -2+3i \end{bmatrix}$

a) $A^T = \begin{bmatrix} 3-2i & 2-i \\ 1+i & -2+3i \end{bmatrix}$

b) $\bar{A} - \begin{bmatrix} 3+2i & 1-i \\ 2+i & -2-3i \end{bmatrix}$

c) $A^* = \bar{A}^T = \begin{bmatrix} 3+2i & 2+i \\ 1-i & -2-3i \end{bmatrix}$

7.3.19 $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} = A$

$$\det(A-\lambda I) = \begin{bmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2$$

$\lambda = 2$

$$Ax = \lambda x$$

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x + \sqrt{3}y = 2x$$

$$\sqrt{3}\,\underset{\underset{1}{\uparrow}}{y} = \underset{\underset{\sqrt{3}}{\uparrow}}{x} \qquad \xi = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -2\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x + \sqrt{3}y = -2x$$

$$\sqrt{3}\,\underset{\underset{-3}{\uparrow}}{y} = -3\underset{\underset{\sqrt{3}}{\uparrow}}{x} \qquad \xi = \begin{bmatrix} \sqrt{3} \\ -3 \end{bmatrix}$$

7.3.21

$$A=\begin{bmatrix}1&0&0\\2&1&-2\\3&2&1\end{bmatrix}$$

$\det(A-\lambda I)=0$

$$\det\begin{bmatrix}1-\lambda&0&0\\2&1-\lambda&-2\\3&2&1-\lambda\end{bmatrix}=0$$

$(1-\lambda)^3-2(-2)(1-\lambda)=0$

$-\lambda^3+3\lambda^2-7\lambda+5=0$

$(\lambda-1)(-\lambda^2+2\lambda-5)=0$

$\lambda=1,\ 1\pm 2i$

$\lambda=1$

$$\underset{A}{\begin{bmatrix}1&0&0\\2&1&-2\\3&2&1\end{bmatrix}}\underset{x}{\begin{bmatrix}x\\y\\z\end{bmatrix}}=\underset{\lambda x}{\begin{bmatrix}x\\y\\z\end{bmatrix}}$$

$$\left.\begin{array}{l}x=x\\2x+y-2z=y\ \rightarrow 2x-2z=0\\3x+2y+z=z\ \rightarrow 3x+2y=0\end{array}\right\}\ \xi_1=\begin{bmatrix}2\\-3\\2\end{bmatrix}$$

$\lambda_2=1+2i$

$$\underset{(A-\lambda)}{\begin{bmatrix}-2i&0&0\\2&-2i&-2\\3&2&-2i\end{bmatrix}}\underset{x}{\begin{bmatrix}x\\y\\z\end{bmatrix}}=\underset{0}{\begin{bmatrix}0\\0\\0\end{bmatrix}}$$

$-2ix=0\ \Rightarrow x=0$

$-2iy-2z=0\ \rightarrow\ z=-iy$

$2y-2iz=0\qquad i\quad -1$

$$\xi_2=\begin{bmatrix}0\\-1\\i\end{bmatrix}$$

$\lambda_3=1-2i$

Since $\lambda_3=\overline{\lambda_2}$

$\xi_3=\overline{\xi_2}$

$$\xi_3=\begin{bmatrix}0\\1\\i\end{bmatrix}$$

7.3.26

Show $(Ax, y) = (x, A^* y)$

$$(Ax, y) = (Ax)^T \bar{y} = x^T A^T \bar{y} \qquad \text{but } A^T = \bar{A}^*$$

$$= x^T \bar{A}^* \bar{y}$$

$$= x^T \overline{A^* y}$$

$$= (x, A^* y)$$

$$\therefore \; (Ax, y) = (x, A^* y)$$

7.5.4 $\quad x' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} x$

Find λ's + ξ's.

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} = 0 \quad = -(1-\lambda)(2+\lambda) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0 \qquad \lambda_1 = -3, \; \lambda_2 = 2$$

$\lambda_1 | -3$

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad 4x + y = 0 \qquad \xi_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$-4x = y$$

$$\uparrow \quad \uparrow$$

$$1 \quad -4$$

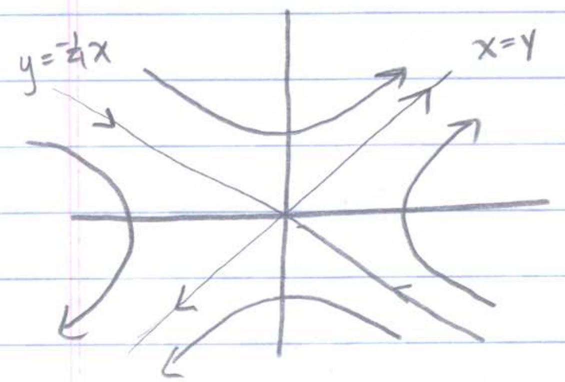
$\lambda_2 | 2$

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad -x + y = 0 \qquad \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y = x$$

$$x(t) = c_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

Directed Plot



As $t \to \infty$ the solution is asymptotic to the line $x=y$. It goes to ∞, 0, or $-\infty$, depending on initial conditions

7.5.6 $x' = \begin{bmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{bmatrix} x$

Find λ's and ξ's

$\det \begin{bmatrix} 5/4-\lambda & 3/4 \\ 3/4 & 5/4-\lambda \end{bmatrix} = 0$

$(5/4-\lambda)^2 - \frac{9}{16} = 0$

$\lambda^2 - \frac{5}{2}\lambda + 1 = 0$

$2\lambda^2 - 5\lambda + 2 = 0$

$(2\lambda-1)(\lambda-2) = 0$ $\qquad \lambda = 2, \frac{1}{2}$

$\lambda_1 = \frac{1}{2}$

$\begin{bmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$3/4 x + 3/4 y = 0 \Rightarrow x = -y \qquad \xi_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 2$

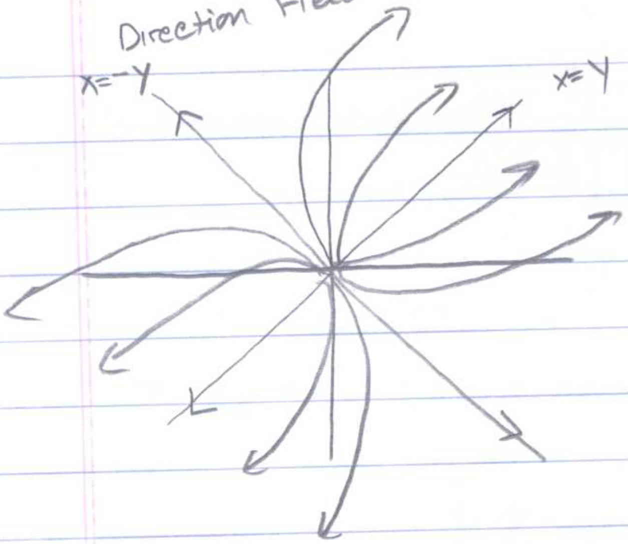
$\begin{bmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-3/4 x + 3/4 y = 0 \Rightarrow y = x \qquad \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Gen Soln

$x(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

Direction Field



As $t \to \infty$ the solution approaches ∞ or $-\infty$ depending on initial conditions

7.5.16 $\quad x = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} x \qquad x_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find λ's and ξ's.

$$\det \begin{bmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{bmatrix} = 0 \qquad (-2-\lambda)(4-\lambda)+5=0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = 3, -1$$

$\underline{\lambda_1 = 3}$

$$\begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow -5x + y = 0$$

$$5x = y \qquad \xi_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$\underline{\lambda_2 = -1}$

$$\begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow -x + y = 0 \qquad \xi_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = y$$

Gen Soln: $x(t) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$

$$x(0) = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$c_1 + c_2 = 1$$

$$\underline{5c_1 + c_2 = 3}$$

$$-4c_1 = -2 \qquad c_1 = \tfrac{1}{2} \Rightarrow c_2 = 1/2$$

Particular Solution:

$$x(t) = \frac{1}{2}\begin{bmatrix}1\\5\end{bmatrix}e^{3t} + \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$$

As $t \to \infty$ the solution tends towards ∞ or $-\infty$ depending on initial conditions.

7.5.13 $$x' = \begin{bmatrix}1 & 1 & 1\\2 & 1 & -1\\-8 & -5 & -3\end{bmatrix} x$$

On MATLAB:

```
>> A = [1 1 1; 2 1 -1; -8 -5 -3]

A =

     1     1     1
     2     1    -1
    -8    -5    -3

>> [v,d]=eig(A)

v =

    0.0000    0.5571   -0.4216
    0.7071   -0.7428    0.5270
   -0.7071   -0.3714    0.7379


d =

    2.0000         0         0
         0   -1.0000         0
         0         0   -2.0000
```

So gen solution is: $$x(t) = C_1\begin{bmatrix}0\\1\\-1\end{bmatrix}e^{2t} + C_2\begin{bmatrix}0.5571\\-0.7428\\-0.3714\end{bmatrix}e^{-t} + C_3\begin{bmatrix}-0.4216\\0.5270\\0.7379\end{bmatrix}e^{-2t}$$