7.6 #2. $\dot{\underline{x}} = \underbrace{\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}}_{A} \underline{x}$

Eigenvalues: $\det(A - \lambda I) = 0$

$\lambda^2 + 2\lambda + 5 = 0$

$\lambda_{1,2} = -1 \pm 2i$ so $\mu = -1,\ \omega = 2$

Eigenvectors: $(A - \lambda_1 I)\underline{\xi} = \underline{0}$

$-2i\,\xi_1 - 4\,\xi_2 = 0$ so $\underline{\xi}_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$, and $\underline{\xi}_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$.

Therefore, $\underline{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

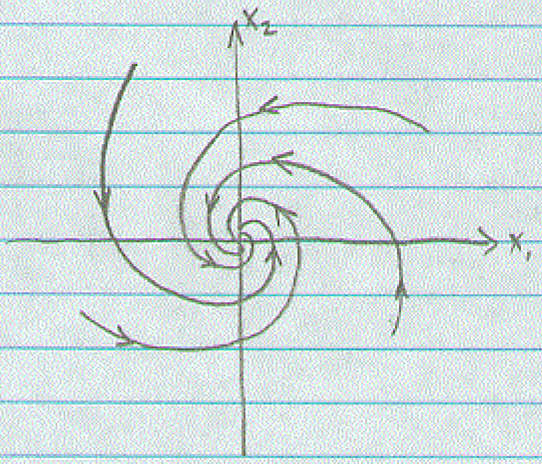
$\underline{u}(t) = e^{\mu t}(\underline{a} \cos \omega t - \underline{b} \sin \omega t)$

$\underline{v}(t) = e^{\mu t}(\underline{a} \sin \omega t + \underline{b} \cos \omega t)$

General solution:

$$\underline{x} = c_1 \underline{u}(t) + c_2 \underline{v}(t) = c_1 e^{-t} \begin{bmatrix} -2 \sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \cos 2t \\ \sin 2t \end{bmatrix}$$

$t \to \infty$: $\underline{x} \to \underline{0}$



7.6 #4. $\dot{\underline{x}} = \begin{bmatrix} 2 & -5/2 \\ 9/5 & -1 \end{bmatrix} \underline{x}$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\lambda^2 - \lambda + \tfrac{5}{2} = 0$$

$$\lambda_{1,2} = \tfrac{1}{2} \pm \tfrac{3}{2} i \quad \Rightarrow \quad \mu = \tfrac{1}{2},\ \omega = \tfrac{3}{2}$$

Eigenvectors: $(A - \lambda I)\underline{\xi} = \underline{0}$

$$\left(\tfrac{3}{2} - \tfrac{3}{2} i\right)\xi_1 - \tfrac{5}{2}\xi_2 = 0$$

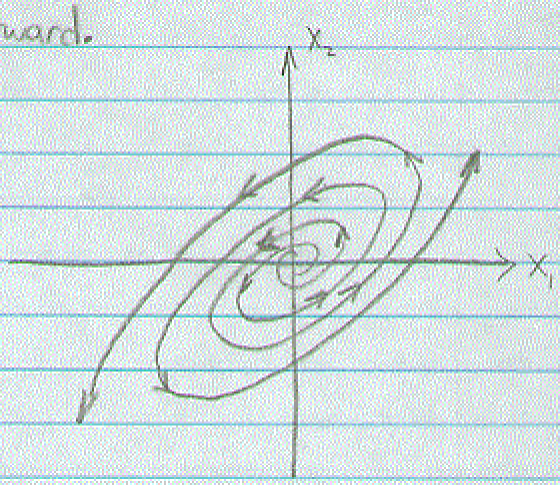
$$\underline{\xi}_1 = \begin{bmatrix} 5 \\ 3-3i \end{bmatrix} \text{ and } \underline{\xi}_2 = \begin{bmatrix} 5 \\ 3+3i \end{bmatrix} \Rightarrow \underline{a} = \begin{bmatrix} 5 \\ 3 \end{bmatrix},\ \underline{b} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$\underline{u}(t) = e^{\mu t}(\underline{a} \cos \omega t - \underline{b} \sin \omega t)$$
$$\underline{v}(t) = e^{\mu t}(\underline{a} \sin \omega t + \underline{b} \cos \omega t)$$

General solution:

$$\underline{x} = c_1 \underline{u}(t) + c_2 \underline{v}(t) = c_1 e^{\frac{1}{2}t} \begin{bmatrix} 5 \cos \frac{3}{2}t \\ 3 \cos \frac{3}{2}t + 3 \sin \frac{3}{2}t \end{bmatrix} + c_2 e^{\frac{1}{2}t} \begin{bmatrix} 5 \sin \frac{3}{2}t \\ 3 \sin \frac{3}{2}t + -3 \cos \frac{3}{2}t \end{bmatrix}$$

$t \to \infty$: $\underline{x} \to \underline{\infty}$, spiralling outward.



7.6 #7.

$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}}_{A} \underline{x}$$

Eigenvalues: $\det(A-\lambda I) = 0$

$$(1-\lambda)^3 + 4(1-\lambda) = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 + 4\right] = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 1+2i, \quad \lambda_3 = 1-2i \qquad (\mu = 1, \omega = 2)$$

Eigenvectors:

For $\lambda_1 = 1$: $(A - \lambda_1 I)\underline{x} = \underline{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \left.\begin{array}{l} \Rightarrow 0 = 0 \\ \Rightarrow x_1 = x_3 \\ \Rightarrow 3x_1 = -2x_2 \end{array}\right\} \Rightarrow \underline{\xi}_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 1+2i$:

$$\begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \left.\begin{array}{l} \Rightarrow x_1 = 0 \\ \Rightarrow x_1 - ix_2 - x_3 = 0 \\ \Rightarrow 3x_1 + 2x_2 - 2ix_3 = 0 \end{array}\right\} \Rightarrow \underline{\xi}_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

Since $\lambda_3 = \overline{\lambda_2}$, $\underline{\xi}_3 = \overline{\underline{\xi}_2} = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$.

Therefore, $\underline{a} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$\underline{u}(t) = e^t\left(\underline{a}\cos 2t - \underline{b}\sin 2t\right) = e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix}$$

$$\underline{v}(t) = e^t\left(\underline{a}\sin 2t + \underline{b}\cos 2t\right) = e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}$$

(continued)

7.6 #7 (continued)

General solution:

$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{\xi}_1 + c_2 \underline{u} + c_3 \underline{v}$$

$$= c_1 e^{t} \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}$$

As $t \to \infty$, all three of these terms grow unbounded, due to positive exponents.

7.6 #8.

$$\dot{x} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} x$$

Eigenvalues: $\det(A-\lambda I)=0$

$$\det \begin{bmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{bmatrix} = (-3-\lambda)(-1-\lambda)(-\lambda) + (2)(1)(-1) - (-2)(-1-\lambda)(2) = 0$$

$$-\lambda^3 - 4\lambda^2 - 7\lambda - 6 = 0$$

$$-(\lambda+2)(\lambda^2+2\lambda+3) = 0$$

$$\Rightarrow \lambda_1 = -2,\ \lambda_2 = -1+\sqrt{2}i,\ \lambda_3 = -1-\sqrt{2}i$$

$$(\text{so } \mu = -1, \omega = \sqrt{2})$$

Eigenvectors: $(A-\lambda I)\xi = 0$

For $\lambda_1 = -2$:

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \xi_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -1+\sqrt{2}i$:

$$\begin{bmatrix} -2-\sqrt{2}i & 0 & 2 \\ 1 & -\sqrt{2}i & 0 \\ -2 & -1 & 1-\sqrt{2}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \xi_2 = \begin{bmatrix} 2 \\ -\sqrt{2}i \\ 2+\sqrt{2}i \end{bmatrix}$$

So $\xi_3 = \overline{\xi_2} = \begin{bmatrix} 2 \\ \sqrt{2}i \\ 2-\sqrt{2}i \end{bmatrix}$, $\underline{a} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 0 \\ -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$.

Then $$\underline{u} = e^{\mu t}(\underline{a}\cos\omega t - \underline{b}\sin\omega t) = e^{-t}\begin{bmatrix} 2\cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \\ 2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix}$$

$$\underline{v} = e^{\mu t}(\underline{a}\sin\omega t + \underline{b}\cos\omega t) = e^{-t}\begin{bmatrix} 2\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \\ 2\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

(continued)

7.6 #8. General solution:
(continued)

$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{\xi}_1 + c_2 \underline{u} + c_3 \underline{v}$$

$$= c_1 e^{-2t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2\cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \\ 2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 2\sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t \\ 2\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t \end{bmatrix}$$

As $t \to \infty$, all of these terms decay, so the trajectories spiral in toward the origin.

7.6 #10. $\dot{\underline{x}} = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \underline{x}$ and $\underline{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda) + 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\Rightarrow \lambda_1 = -2+i, \quad \lambda_2 = -2-i \qquad (\mu = -2, \omega = 1)$$

Eigenvectors: $(A - \lambda I)\underline{\xi} = \underline{0}$

For $\lambda_1 = -2+i$:

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \underline{\xi}_1 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

Then $\underline{\xi}_2 = \overline{\underline{\xi}_1}$, $\underline{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$$\underline{u} = e^{\mu t}(\underline{a} \cos \omega t - \underline{b} \sin \omega t) = e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix}$$

$$\underline{v} = e^{\mu t}(\underline{a} \sin \omega t + \underline{b} \cos \omega t) = e^{-2t} \begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix}$$

General solution:

$$\underline{x}(t) = c_1 \underline{u} + c_2 \underline{v} = c_1 e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix}$$

Particular solution:

$$\underline{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad c_1 = -2 \text{ and } c_2 = -3$$

$$\underline{x}(t) = -2e^{-2t} \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} - 3e^{-2t} \begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} = e^{-2t} \begin{bmatrix} \cos t - 5 \sin t \\ -2\cos t - 3 \sin t \end{bmatrix}.$$

2. $\dot{\xi} = A\xi$, $\quad A = \begin{bmatrix} 1 & -3 \\ 6 & 1 \end{bmatrix}$ $\quad \xi(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenvectors: $\det(A - \lambda I) = 0$

$$(1-\lambda)^2 + 18 = 0$$

$$\lambda = 1 \pm \sqrt{18}\, i = 1 \pm 3\sqrt{2}\, i$$

$$\lambda_1 = 1 + 3\sqrt{2}\, i, \quad \lambda_2 = 1 - 3\sqrt{2}\, i$$

Eigenvectors: $(A - \lambda_i I)\underline{x}_i = \underline{0}$

For $\lambda_1 = 1 + 3\sqrt{2}i$: $\begin{bmatrix} -3\sqrt{2}i & -3 \\ 6 & -3\sqrt{2}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{x}_1 = \begin{bmatrix} i \\ \sqrt{2} \end{bmatrix}$

Since $\lambda_2 = \overline{\lambda_1}$, $\underline{x}_2 = \overline{\underline{x}_1} = \begin{bmatrix} -i \\ \sqrt{2} \end{bmatrix}$.

(a) $\underline{a} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$ and $\underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

General solution is

$\xi(t) = c_1 \underline{u} + c_2 \underline{v}$, where

$$\underline{u} = e^{\mu t}(\underline{a}\cos\omega t - \underline{b}\sin\omega t) = e^{t}\begin{bmatrix} -\sin 3\sqrt{2}t \\ \sqrt{2}\cos 3\sqrt{2}t \end{bmatrix}$$

$$\underline{v} = e^{\mu t}(\underline{a}\sin\omega t + \underline{b}\cos\omega t) = e^{t}\begin{bmatrix} \cos 3\sqrt{2}t \\ \sqrt{2}\sin 3\sqrt{2}t \end{bmatrix}$$

So $\xi(t) = e^{t}\left(c_1 \begin{bmatrix} -\sin 3\sqrt{2}t \\ \sqrt{2}\cos 3\sqrt{2}t \end{bmatrix} + c_2 \begin{bmatrix} \cos 3\sqrt{2}t \\ \sqrt{2}\sin 3\sqrt{2}t \end{bmatrix} \right)$

Initial conditions: $\underline{u}(0) = \underline{a}$ and $\underline{v}(0) = \underline{b}$, so

$$\xi(0) = c_1 \underline{a} + c_2 \underline{b} = \begin{bmatrix} c_2 \\ \sqrt{2}c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{\sqrt{2}}{2} \text{ and } c_2 = 1.$$

Hence the solution is $\underline{\xi(t) = e^{t}\begin{bmatrix} \cos 3\sqrt{2}t - \frac{\sqrt{2}}{2}\sin 3\sqrt{2}t \\ \cos 3\sqrt{2}t + \sqrt{2}\sin 3\sqrt{2}t \end{bmatrix}}$.

(b) Complex-eigenvector method.

General solution is

$$\vec{x} = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2$$

$$= c_1 e^{(1+3\sqrt{2}i)t} \begin{bmatrix} i \\ \sqrt{2} \end{bmatrix} + c_2 e^{(1-3\sqrt{2}i)t} \begin{bmatrix} -i \\ \sqrt{2} \end{bmatrix}$$

$$= e^t \left(c_1 e^{3\sqrt{2}it} \begin{bmatrix} i \\ \sqrt{2} \end{bmatrix} + c_2 e^{-3\sqrt{2}it} \begin{bmatrix} -i \\ \sqrt{2} \end{bmatrix} \right)$$

Apply Euler's formula:

$$= e^t \left(c_1 (\cos 3\sqrt{2}t + i \sin 3\sqrt{2}t) \begin{bmatrix} i \\ \sqrt{2} \end{bmatrix} + c_2 (\cos 3\sqrt{2}t - i \sin 3\sqrt{2}t) \begin{bmatrix} -i \\ \sqrt{2} \end{bmatrix} \right)$$

$$= e^t \left(\cos 3\sqrt{2}t \left(c_1 \begin{bmatrix} i \\ \sqrt{2} \end{bmatrix} + c_2 \begin{bmatrix} -i \\ \sqrt{2} \end{bmatrix} \right) + \sin 3\sqrt{2}t \left(c_1 i \begin{bmatrix} i \\ \sqrt{2} \end{bmatrix} - c_2 i \begin{bmatrix} -i \\ \sqrt{2} \end{bmatrix} \right) \right)$$

$$= e^t \left(\cos 3\sqrt{2}t \begin{bmatrix} i(c_1 - c_2) \\ \sqrt{2}(c_1 + c_2) \end{bmatrix} + \sin 3\sqrt{2}t \begin{bmatrix} -(c_1 + c_2) \\ \sqrt{2}i(c_1 - c_2) \end{bmatrix} \right)$$

Let $d_1 = i(c_1 - c_2)$ and $d_2 = c_1 + c_2$:

$$= e^t \left(\cos 3\sqrt{2}t \begin{bmatrix} d_1 \\ \sqrt{2} d_2 \end{bmatrix} + \sin 3\sqrt{2}t \begin{bmatrix} -d_2 \\ \sqrt{2} d_1 \end{bmatrix} \right)$$

$$\vec{x}(t) = e^t \left(d_1 \begin{bmatrix} \cos 3\sqrt{2}t \\ \sqrt{2} \sin 3\sqrt{2}t \end{bmatrix} + d_2 \begin{bmatrix} -\sin 3\sqrt{2}t \\ \sqrt{2} \cos 3\sqrt{2}t \end{bmatrix} \right)$$

This is the same as the general solution from part (a), just with different names for the arbitrary constants.

So when we plug in the I.C., we get the same result:

$d_1 = 1$, $d_2 = \frac{\sqrt{2}}{2}$

$$\vec{x}(t) = e^t \begin{bmatrix} \cos 3\sqrt{2}t - \sqrt{2}/2 \sin 3\sqrt{2}t \\ \cos 3\sqrt{2}t + \sqrt{2} \sin 3\sqrt{2}t \end{bmatrix}.$$

3. (a) Prove that $\overline{ab} = \bar{a}\,\bar{b}$, for a and b complex numbers.

Proof: Let $a = w + xi$ and $b = y + zi$, where w, x, y, z are real numbers. (Every complex number can be expressed this way, so we don't lose generality.)

Then $\bar{a} = w - xi$ and $\bar{b} = y - zi$.

$$\begin{aligned}
\bar{a}\,\bar{b} &= (w - xi)(y - zi) \\
&= wy - xyi - wzi + xzi^2 \\
&= (wy - xz) + (-xy - wz)i \\
\overline{ab} &= \overline{(w + xi)(y + zi)} \\
&= \overline{wy + xyi + wzi + xzi^2} \\
&= \overline{(wy - xz) + (xy + wz)i} \\
&= (wy - xz) + (-xy - wz)i
\end{aligned}$$

equal to each other.

(b) Prove $\overline{a + b} = \bar{a} + \bar{b}$, for a and b complex numbers.

Proof: As above, let $a = w + xi$ and $b = y + zi$.

Then $\bar{a} = w - xi$ and $\bar{b} = y - zi$.

$$\begin{aligned}
\bar{a} + \bar{b} &= w - xi + y - zi \\
&= (w + y) + (-x - z)i \\
\overline{a + b} &= \overline{w + xi + y + zi} \\
&= \overline{(w + y) + (x + z)i} \\
&= (w + y) + (-x - z)i
\end{aligned}$$

equal to each other.