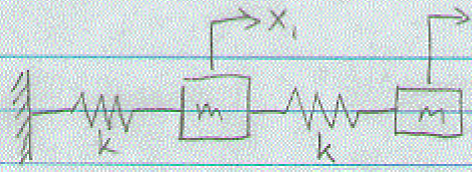
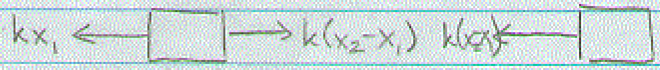
1. 

$(k=m=1)$

(a) FBD's:



$\Sigma F = m\ddot{x}$

$$-kx_1 + k(x_2-x_1) = m\ddot{x}_1 \qquad -k(x_2-x_1) = m\ddot{x}_2$$

With $k=m=1$, these simplify to:

$$\ddot{x}_1 = -2x_1 + x_2$$
$$\ddot{x}_2 = x_1 - x_2$$

(b) Assume $x_1(t) = a_1 \cos \omega t$, $x_2(t) = a_2 \cos \omega t$.
Then $\ddot{x}_1(t) = -\omega^2 a_1 \cos \omega t$, $\ddot{x}_2(t) = -\omega^2 a_2 \cos \omega t$.

$$-\omega^2 a_1 \cancel{\cos \omega t} = -2a_1 \cancel{\cos \omega t} + a_2 \cancel{\cos \omega t}$$
$$-\omega^2 a_2 \cancel{\cos \omega t} = a_1 \cancel{\cos \omega t} - a_2 \cancel{\cos \omega t}$$

$$-\omega^2 a_1 = -2a_1 + a_2 \quad \Rightarrow \omega^2 = 2 - \frac{a_2}{a_1}$$
$$-\omega^2 a_2 = a_1 - a_2 \quad \Rightarrow -\left(2-\frac{a_2}{a_1}\right)a_2 = a_1 - a_2$$
$$-\left(2-\frac{a_2}{a_1}\right) = \frac{a_1}{a_2} - 1$$

Let $R = \frac{a_2}{a_1}$; then $-2+R = \frac{1}{R} - 1$

$$R = \frac{1}{R} + 1$$
$$R^2 = 1 + R \Rightarrow R^2 - R - 1 = 0$$
$$\frac{a_2}{a_1} = R = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \underline{\frac{1}{2} \pm \frac{\sqrt{5}}{2}}$$

These are the two possible amplitude ratios. To get the corresponding frequencies (these must be >0!) use $\omega^2 = 2-R$

$$\omega = \sqrt{2-R} = \underline{\sqrt{\frac{3}{2} \mp \frac{\sqrt{5}}{2}}}.$$

(continued)

Numerically: $\left(\frac{a_2}{a_1}\right)_1 = 1.618$ or $\left(\frac{a_2}{a_1}\right)_2 = -0.618$

$\omega_1 = 0.618$ $\omega_2 = 1.618$.

(c) We have 2 independent solutions; in general, we can expect any particular I.C.s to yield a linear combination of them:

$$x_1(t) = a_1 \cos \omega_1 t + b_1 \cos \omega_2 t$$
$$x_2(t) = a_2 \cos \omega_1 t + b_2 \cos \omega_2 t$$

where $\frac{a_2}{a_1} = \left(\frac{a_2}{a_1}\right)_1$ where $\frac{b_2}{b_1} = \left(\frac{a_2}{a_1}\right)_2$

$$x_1(t) = a_1 \cos \sqrt{\tfrac{3-\sqrt5}{2}}\, t + b_1 \cos \sqrt{\tfrac{3+\sqrt5}{2}}\, t$$
$$x_2(t) = \tfrac{1+\sqrt5}{2} a_1 \cos \sqrt{\tfrac{3-\sqrt5}{2}}\, t + \tfrac{1-\sqrt5}{2} b_1 \cos \sqrt{\tfrac{3+\sqrt5}{2}}\, t$$

$x_1(0) = a_1 + b_1 = 1 \Rightarrow a_1 + b_1 = 1$

$x_2(0) = \frac{1+\sqrt5}{2} a_1 + \frac{1-\sqrt5}{2} b_1 = 1 \Rightarrow (1+\sqrt5)a_1 + (1-\sqrt5)b_1 = 2 \Rightarrow \sqrt5\, a_1 - \sqrt5\, b_1 = 1$

$a_1 - b_1 = \frac{1}{\sqrt5}$

Then add them: $2a_1 = 1 + \frac{\sqrt5}{5} \Rightarrow a_1 = \frac{5+\sqrt5}{10}$

$b_1 = 1 - a_1 = \frac{5-\sqrt5}{10}$

Then the solution is:

$$x_1(t) = \tfrac{5+\sqrt5}{10} \cos \sqrt{\tfrac{3-\sqrt5}{2}}\, t + \tfrac{5-\sqrt5}{10} \cos \sqrt{\tfrac{3+\sqrt5}{2}}\, t$$
$$x_2(t) = \tfrac{5+3\sqrt5}{10} \cos \sqrt{\tfrac{3-\sqrt5}{2}}\, t + \tfrac{5-3\sqrt5}{10} \cos \sqrt{\tfrac{3+\sqrt5}{2}}\, t$$

Or numerically:

$$x_1(t) = 0.724 \cos 0.618t + 0.276 \cos 1.618t$$
$$x_2(t) = 1.170 \cos 0.618t - 0.171 \cos 1.618t$$

(d) To write the system as $\dot{\xi} = A\xi$, we need to deal with those second-derivative terms first.

Let $x_3 = \dot{x}_1$; then $\ddot{x}_1 = \dot{x}_3$.

$x_4 = \dot{x}_2$; then $\ddot{x}_2 = \dot{x}_4$.

The system then becomes:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -2x_1 + x_2 \\ \dot{x}_4 &= x_1 - x_2 \end{aligned} \implies \dot{\underline{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \underline{x}$$

(e)

Eigenvalues (from Mathematica):
+ Eigenvectors

$\mu_1 = 0,\ \omega_1 = 1.618$ $\qquad$ $\mu_2 = 0,\ \omega_2 = 0.618$

$$\lambda_1 = 1.618i \qquad \lambda_2 = -1.618i \qquad \lambda_3 = 0.618i \qquad \lambda_4 = -0.618i$$

$$\xi_1 = \begin{bmatrix} i \\ -0.618i \\ -1.618 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} -i \\ 0.618i \\ -1.618 \\ 1 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} -i \\ -1.618i \\ 0.618 \\ 1 \end{bmatrix} \quad \xi_4 = \begin{bmatrix} i \\ 1.618i \\ 0.618 \\ 1 \end{bmatrix}$$

(f)

$$\underline{a} = \begin{bmatrix} 0 \\ 0 \\ -1.618 \\ 1 \end{bmatrix}, \ \underline{b} = \begin{bmatrix} 1 \\ -0.618 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} 0 \\ 0 \\ 0.618 \\ 1 \end{bmatrix}, \ \underline{d} = \begin{bmatrix} 1 \\ 1.618 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{u}_1 = e^{\mu_1 t}(\underline{a}\cos\omega_1 t - \underline{b}\sin\omega_1 t) \qquad \underline{u}_2 = e^{\mu_2 t}(\underline{c}\cos\omega_2 t - \underline{d}\sin\omega_2 t)$$

$$\underline{v}_1 = e^{\mu_1 t}(\underline{a}\sin\omega_1 t + \underline{b}\cos\omega_1 t) \qquad \underline{v}_2 = e^{\mu_2 t}(\underline{c}\sin\omega_2 t + \underline{d}\cos\omega_2 t)$$

General solution is

$$\underline{x}(t) = c_1\underline{u}_1 + c_2\underline{v}_1 + c_3\underline{u}_2 + c_4\underline{v}_2$$

$$\underline{x}(t) = c_1\begin{bmatrix} -\sin 1.618t \\ 0.618\sin 1.618t \\ -1.618\cos 1.618t \\ \cos 1.618t \end{bmatrix} + c_2\begin{bmatrix} \cos 1.618t \\ -0.618\cos 1.618t \\ -1.618\sin 1.618t \\ \sin 1.618t \end{bmatrix} + c_3\begin{bmatrix} -\sin 0.618t \\ -1.618\sin 0.618t \\ 0.618\cos 0.618t \\ \cos 0.618t \end{bmatrix} + c_4\begin{bmatrix} \cos 0.618t \\ 1.618\cos 0.618t \\ 0.618\sin 0.618t \\ \sin 0.618t \end{bmatrix}$$

(g) We have initial conditions $x_1(0) = x_2(0) = 1$, and $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

Thus, $\underline{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

We have $\underline{u}_1(0) = \underline{a}$, $\underline{v}_1(0) = \underline{b}$, $\underline{u}_2(0) = \underline{c}$ and $\underline{v}_2(0) = \underline{d}$, so:

$$c_1 \underline{a} + c_2 \underline{b} + c_3 \underline{c} + c_4 \underline{d} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \Rightarrow \begin{cases} c_2 + c_4 = 1 \\ -0.618 c_2 + 1.618 c_4 = 1 \\ 1.618 c_1 + 0.618 c_3 = 0 \\ c_1 + c_3 = 0 \end{cases}$$

This yields the solution

$c_1 = 0$

$c_2 = 0.276$

$c_3 = 0$

$c_4 = 0.724$

and therefore:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.276 \cos 1.618t + 0.724 \cos 0.618t \\ -0.171 \cos 1.618t + 1.170 \cos 0.618t \end{bmatrix}$$

This confirms the answer obtained in part (c).

2. $$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad \dot{\mathbf{x}} = A\mathbf{x}$$

(a) First, let's find all the "regular" eigenvectors we can for $\lambda = 2$:

$$(A-2I)\,\underline{v} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ v_4 \\ 0 \\ -4v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So the null space of $A-2I$ consists of all vectors of the form

$\underline{v} = \begin{bmatrix} r \\ s \\ t \\ 0 \\ 0 \end{bmatrix}$; these can be spanned by, say, $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\underline{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Generalized eigenvector:

Find $\underline{w}$ such that $(A-2I)\underline{w} = \underline{v}_3$; then $(A-2I)^2\underline{w} = (A-2I)\underline{v}_3 = \underline{0}$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ w_4 \\ 0 \\ -4w_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{w} = \begin{bmatrix} r \\ s \\ t \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ to avoid redundancy.

So the 4 vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ and $\underline{w}$ lie in the 4-dimensional null space of $(A-2I)^4$; since there are also 4 of them, and they are linearly independent, they span the null space.

(b) For $\underline{v}_1, \underline{v}_2, \underline{v}_3$, $(A-2I)\underline{v}_i = \underline{0}$, so

$$\mathbf{x}_{h_i}(t) = e^{2t}\left(\underline{v}_i + t\cancel{(A-2I)\underline{v}_i}^{\,0} + \tfrac{t^2}{2}\cancel{(A-2I)^2\underline{v}_i}^{\,0} + \cdots\right)$$

$$\mathbf{x}_{h_i}(t) = e^{2t}\,\underline{v}_i \qquad (i = 1, 2, 3)$$

For $\underline{w}$, $(A-2I)\underline{w} \neq \underline{0}$, but $(A-2I)^2\underline{w} = \underline{0}$, so:

$$\mathbf{x}_{h_4}(t) = e^{2t}\left(\underline{w} + t(A-2I)\underline{w}\right) = e^{2t}\left(\underline{w} + t\underline{v}_3\right) = e^{2t}\begin{bmatrix} 0 \\ 0 \\ t \\ 1 \\ 0 \end{bmatrix}$$

(c) Again, let's start with the null space of just $(A-2\underline{I})$, which corresponds to the regular eigenvectors of A associated with $\lambda=2$.

As before, $(A-2\underline{I})\underline{v}=\underline{0} \Rightarrow \underline{v}=\begin{bmatrix} r \\ s \\ t \\ 0 \\ 0 \end{bmatrix}$, where r,s,t can be independently chosen.

Let's take a different basis for this null space:

$$\underline{v}_1=\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \underline{v}_2=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \underline{v}_3=\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We still need one generalized eigenvalue. Take any $\underline{v}$ in the null space of $A-2\underline{I}$:

$$(A-2\underline{I})\underline{w}=\underline{v} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ w_4 \\ 0 \\ -4w_5 \end{bmatrix}=\begin{bmatrix} r \\ s \\ t \\ 0 \\ 0 \end{bmatrix} \Rightarrow (r=0, s=0) \text{ and } w_4=t,\ w_5=0.$$

Let $t=1$; this gives us: $\underline{w}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. (corresponds to $\underline{v}=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$)

(d) Just as in (b), the regular eigenvectors $\underline{v}_i$ $(i=1,2,3)$ have homogeneous solutions

$$\underline{\xi}_{h_i}(t)=e^{2t}\left(\underline{v}_i+t\cancelto{0}{(A-2\underline{I})\underline{v}_i}+\frac{t^2}{2}\cancelto{0}{(A-2\underline{I})^2\underline{v}_i}+\cdots\right)$$
$$=e^{2t}\underline{v}_i$$

(Different $\underline{v}_i$'s than in part (b), though!)

And the generalized eigenvalue $\underline{w}$ has

$$\underline{\xi}_{h_4}(t)=e^{2t}\left(\underline{w}+t(A-2\underline{I})\underline{w}+\frac{t^2}{2}\cancelto{0}{(A-2\underline{I})^2\underline{w}}+\cdots\right)$$
$$=e^{2t}(\underline{w}+t\underline{v})$$
$$=e^{2t}\begin{bmatrix} 0 \\ 0 \\ t \\ 1 \\ 0 \end{bmatrix}.$$

(e) For the eigenvalue -2,

$$(A-(-2)\underline{I})\,\underline{v}_5 = \underline{0} \Rightarrow \begin{bmatrix} 4&0&0&0&0\\ 0&4&0&0&0\\ 0&0&4&1&0\\ 0&0&0&4&0\\ 0&0&0&0&0 \end{bmatrix}\begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix} \Rightarrow \underline{v}_5 = \begin{bmatrix} 0\\0\\0\\0\\1 \end{bmatrix}.$$

This has the homogeneous solution

$$\underline{x}_{h_5}(t) = e^{-2t}\,\underline{v}_5$$

So the general solution is:

$$\underline{x} = c_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}e^{2t} + c_2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}e^{2t} + c_3\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}e^{2t} + c_4\begin{bmatrix}0\\0\\t\\1\\0\end{bmatrix}e^{2t} + c_5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}e^{-2t}.$$

(f) The eigenvector for $\lambda = -2$ is the same as in (e), so we get:

$$\underline{x} = c_1\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}e^{2t} + c_2\begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}e^{2t} + c_3\begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}e^{2t} + c_4\begin{bmatrix}0\\0\\t\\1\\0\end{bmatrix}e^{2t} + c_5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}e^{-2t}$$

This is different from in part (e)!

(g) For the general solution from (e):

$$\underline{x}(0) = \begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \quad \text{so} \quad \underline{x}(t) = \begin{bmatrix} e^{2t}\\ 2e^{2t}\\ 3e^{2t}+4te^{2t}\\ 4e^{2t}\\ 5e^{-2t} \end{bmatrix}$$

For the general solution from (f):

$$\underline{x}(0) = \begin{bmatrix}c_1+c_3\\c_1+c_2\\c_2+c_3\\c_4\\c_5\end{bmatrix} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\\c_4\\c_5\end{bmatrix} = \begin{bmatrix}0\\2\\1\\4\\5\end{bmatrix} \quad \text{so} \quad \underline{x}(t) = \begin{bmatrix} e^{2t}\\ 2e^{2t}\\ 3e^{2t}+4te^{2t}\\ 4e^{2t}\\ 5e^{-2t} \end{bmatrix}$$

These solutions are the same. They always will be, since they're just different representations of the same IVP.
By reorganizing constants ($\tilde{c}_1 = \frac{c_1-c_2+c_3}{2}$, $\tilde{c}_2 = \frac{c_1+c_2-c_3}{2}$, $\tilde{c}_3 = \frac{-c_1+c_2+c_3}{2}$) we can show that the two general solutions are actually equivalent.

3. $$A = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad \dot{\underline{\xi}} = A\underline{\xi}$$

Eigenvalues: $\lambda = -4, -4, -4, -2, -2.$

Eigenvectors:

For $\lambda = -4$: $(A+4I)\underline{v} = \underline{0} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v} = \begin{bmatrix} r \\ 0 \\ s \\ 0 \\ 0 \end{bmatrix}$, r and s independent

This gives us the 2 eigenvectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\underline{v}_3$, we need a generalized eigenvector:

$$(A+4I)\underline{v}_3 = \begin{bmatrix} r & 0 & s & 0 & 0 \end{bmatrix}^T$$

$$\underbrace{\begin{bmatrix} v_2 & 0 & 0 & 2v_4 & 2v_5 \end{bmatrix}^T}_{} = \begin{bmatrix} r & 0 & s & 0 & 0 \end{bmatrix}^T \Longrightarrow \underline{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = -2$: $(A+2I)\underline{v} = \underline{0} \Rightarrow \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -2v_1+v_2 \\ -2v_2 \\ -2v_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \underline{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ r \\ s \end{bmatrix}.$

So we get the 2 eigenvectors

$$\underline{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \underline{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Homogeneous solutions:

$\underline{\xi}_{h_1}(t) = e^{-4t}\,\underline{v}_1$

$\underline{\xi}_{h_2}(t) = e^{-4t}\,\underline{v}_2$

$\underline{\xi}_{h_3}(t) = e^{-4t}(\underline{v}_3 + t(A+4I)\underline{v}_3) = e^{-4t}\begin{bmatrix} t \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\underline{\xi}_{h_4}(t) = e^{-2t}\,\underline{v}_4$

$\underline{\xi}_{h_5}(t) = e^{-2t}\,\underline{v}_5$

General solution:

$$\underline{\xi}(t) = c_1 \underline{\xi}_{h_1}(t) + c_2 \underline{\xi}_{h_2}(t) + c_3 \underline{\xi}_{h_3}(t) + c_4 \underline{\xi}_{h_4}(t) + c_5 \underline{\xi}_{h_5}(t)$$

$$\underline{\xi}(t) = e^{-4t} \begin{bmatrix} c_1 + tc_3 \\ c_3 \\ c_2 \\ 0 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_4 \\ c_5 \end{bmatrix}$$