TA: Jingzhou Zhang AME 302 HW #5 solution

1. a. diagonalization.

Number 1: $\underset{\sim}{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

$$\begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0 \Rightarrow r_1 = 1 \qquad r_2 = -1$$

when $r_1 = 1$,

$$\begin{pmatrix} 2-r_1 & -1 \\ 3 & -2-r_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When $r_2 = -1$,

$$\begin{pmatrix} 2-r_2 & -1 \\ 3 & -2-r_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = 0$$

$$\Rightarrow \underset{\sim}{\xi}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The general solution of the homogeneous system is:

$$\underset{\sim}{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

Upon normalizing $\underset{\sim}{\xi}^{(1)}$ and $\underset{\sim}{\xi}^{(2)}$,

$$\underset{\approx}{T} = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt2} & \frac{3}{\sqrt{10}} \end{pmatrix} \Rightarrow \underset{\approx}{T}^{-1} = \begin{pmatrix} \frac{3\sqrt2}{2} & -\frac{\sqrt2}{2} \\ -\frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \end{pmatrix}$$

$$\underset{\approx}{D} = T^{-1} A T = \begin{pmatrix} \frac{3\sqrt2}{2} & -\frac{\sqrt2}{2} \\ -\frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt2} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$= \frac{1}{2\sqrt5} \begin{pmatrix} 3 & -1 \\ -\sqrt5 & \sqrt5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \sqrt5 & 1 \\ \sqrt5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

②

$$\underset{\sim}{T}^{-1}\underset{\sim}{g}(t) = \begin{pmatrix} \frac{3\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \end{pmatrix}\begin{pmatrix} e^t \\ t \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2}(3e^t - t) \\ \frac{\sqrt{10}}{2}(-e^t + t) \end{pmatrix}$$

$$\underset{\sim}{y}' = \underset{\sim}{D}\underset{\sim}{y} + \underset{\sim}{T}^{-1}\underset{\sim}{g}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underset{\sim}{y} + \frac{\sqrt{2}}{2}\begin{pmatrix} 3e^t - t \\ -\sqrt{5}e^t + \sqrt{5}t \end{pmatrix}$$

Thus,

$$\left.\begin{aligned} y_1' - y_1 &= \frac{3}{\sqrt{2}}e^t - \frac{t}{\sqrt{2}} \\ y_2' + y_2 &= -\frac{\sqrt{10}}{2}e^t + \frac{\sqrt{10}}{2}t \end{aligned}\right\}$$

$$\Rightarrow \begin{aligned} y_1 &= \frac{3}{\sqrt{2}}te^t + \frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}} + C_1e^{-t} \\ y_2 &= -\frac{\sqrt{10}}{4}e^t + \frac{\sqrt{10}}{2}t - \frac{\sqrt{10}}{2} + C_2e^{-t} \end{aligned}$$

$$\Rightarrow \underset{\sim}{x} = \underset{\sim}{T}\underset{\sim}{y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \frac{1}{\sqrt{5}} \\ 1 & \frac{3}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} y_1 + \frac{y_2}{\sqrt{5}} \\ y_1 + \frac{3y_2}{\sqrt{5}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{3}{\sqrt{2}}te^t + (C_1 - \frac{\sqrt{2}}{4})e^t + \sqrt{2}t + \frac{C_2}{\sqrt{5}}e^{-t} \\ \frac{3}{\sqrt{2}}te^t + (C_1 - \frac{3\sqrt{2}}{4})e^t + 2\sqrt{2}t - \sqrt{2} + \frac{3C_2}{\sqrt{5}}e^{-t} \end{pmatrix}$$

$$= K_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + K_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t + \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix}t$$

$$\left(K_1 = \frac{C_1}{\sqrt{2}}, \quad K_2 = \frac{C_2}{\sqrt{10}}\right).$$

3

Number 4:

$$\underset{\sim}{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$\begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = 0 \Rightarrow r_1 = 2 \quad r_2 = -3$$

When $r_1 = 2$,

$$\begin{pmatrix} 1-r_1 & 1 \\ 4 & -2-r_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

When $r_2 = -3$,

$$\begin{pmatrix} 1-r_2 & 1 \\ 4 & -2-r_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$\underset{\sim}{\xi}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

The general solution of the homogeneous system is:

$$\underset{\sim}{x} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

upon normalizing $\underset{\sim}{\xi}^{(1)}$ and $\underset{\sim}{\xi}^{(2)}$,

$$\underset{\sim}{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{2}} & -\frac{4}{\sqrt{17}} \end{pmatrix} \Rightarrow \underset{\sim}{T}^{-1} = \frac{1}{5} \begin{pmatrix} 4\sqrt{2} & \sqrt{2} \\ \sqrt{17} & -\sqrt{17} \end{pmatrix}$$

$$\underset{\sim}{D} = T^{-1} A T = \frac{1}{5} \begin{pmatrix} 4\sqrt{2} & \sqrt{2} \\ \sqrt{17} & -\sqrt{17} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{2}} & -\frac{4}{\sqrt{17}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\underset{\sim}{T}^{-1}\underset{\sim}{g}(t) = \frac{1}{5}\begin{pmatrix} 4\sqrt{2} & \sqrt{2} \\ \sqrt{17} & -\sqrt{17} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4\sqrt{2}e^{-2t} - 2\sqrt{2}e^{t} \\ \sqrt{17}e^{-2t} + 2\sqrt{17}e^{t} \end{pmatrix} \quad (4$$

$$\underset{\sim}{y}' = \underset{\sim}{D}\underset{\sim}{y} + \underset{\sim}{T}^{-1}\underset{\sim}{g}(t) = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\underset{\sim}{y} + \frac{1}{5}\begin{pmatrix} 4\sqrt{2}e^{-2t} - 2\sqrt{2}e^{t} \\ \sqrt{17}e^{-2t} + 2\sqrt{17}e^{t} \end{pmatrix}$$

Thus,
$$\begin{cases} y_1' - 2y_1 = \dfrac{2\sqrt{2}}{5}\left(2e^{-2t} - e^{t}\right) \\ y_2' + 3y_2 = \dfrac{\sqrt{17}}{5}\left(e^{-2t} + 2e^{t}\right) \end{cases}$$

$$\Rightarrow \quad y_1 = -\frac{\sqrt{2}}{5}e^{-2t} + \frac{2\sqrt{2}}{5}e^{t} + c_1 e^{2t}$$
$$y_2 = \frac{\sqrt{17}}{5}e^{-2t} + \frac{\sqrt{17}}{10}e^{t} + c_2 e^{-3t}$$

$$\Rightarrow \underset{\sim}{x} = \underset{\sim}{T}\underset{\sim}{y} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{17}} \\ \frac{1}{\sqrt{2}} & -\frac{4}{\sqrt{17}} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{y_1}{\sqrt{2}} + \frac{1}{\sqrt{17}}y_2 \\ \frac{y_1}{\sqrt{2}} - \frac{4}{\sqrt{17}}y_2 \end{pmatrix}$$

$$= \begin{bmatrix} \cancel{-\frac{1}{5}e^{-2t}} + \frac{2}{5}e^{t} + \frac{c_1}{\sqrt{2}}e^{2t} + \cancel{\frac{1}{5}e^{-2t}} + \frac{1}{10}e^{t} + \frac{c_2}{\sqrt{17}}e^{-3t} \\ -\frac{1}{5}e^{-2t} + \cancel{\frac{2}{5}e^{t}} + \frac{c_1}{\sqrt{2}}e^{2t} - \frac{4}{5}e^{-2t} \cancel{-\frac{2}{5}e^{t}} - \frac{4c_2}{\sqrt{17}}e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{t} + \frac{c_1}{\sqrt{2}}e^{2t} + \frac{c_2}{\sqrt{17}}e^{-3t} \\ -e^{-2t} + \frac{c_1}{\sqrt{2}}e^{2t} - \frac{4c_2}{\sqrt{17}}e^{-3t} \end{bmatrix}$$

$$= k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + k_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + \begin{pmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{pmatrix}$$

$$\left(k_1 = \frac{c_1}{\sqrt{2}}, \quad k_2 = \frac{c_2}{\sqrt{17}}\right)$$

b. undetermined coefficients.

Number 1: $\underset{\sim}{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

We write $\underset{\sim}{g}(t)$ in the form $\underset{\sim}{g}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$.

Then we assume that

$$\underset{\sim}{x} = \underset{\sim}{v}(t) = \underset{\sim}{a}te^t + \underset{\sim}{b}e^t + \underset{\sim}{c}t + \underset{\sim}{d}$$

Substituting above Eq into the differential equation,

$$\begin{cases} \underset{\sim}{A}\underset{\sim}{a} = \underset{\sim}{a} & \Rightarrow \underset{\sim}{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \cdots 1 \\ \underset{\sim}{A}\underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{b} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \cdots 2 \\ \underset{\sim}{A}\underset{\sim}{c} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \cdots 3 \\ \underset{\sim}{A}\underset{\sim}{d} = \underset{\sim}{c} & \cdots 4 \end{cases}$$

assume $\underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

2 $\Rightarrow \alpha = \frac{3}{2} \Rightarrow \underset{\sim}{a} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$.

3 $\Rightarrow \underset{\sim}{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

4 $\Rightarrow \underset{\sim}{d} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Finally, we obtain the particular solution:

$$\underset{\sim}{v}(t) = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The general solution is:

$$\underset{\sim}{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} + \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Number 4: $\underset{\sim}{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$. (6)

$$\underset{\sim}{g}(t) = \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2e^{t} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} e^{t}$$

~~$r_1 = -2$ is a eigenvalue of the coefficient matrix.~~

Assume that

$$\underset{\sim}{v} = \underset{\sim}{v}(t) = \underset{\sim}{a} e^{-2t} + \underset{\sim}{b} e^{t} + \underset{\sim}{c}$$

Substituting above equa. to differential eqa,

$$\begin{cases} \underset{\sim}{A}\underset{\sim}{a} = -2\underset{\sim}{a} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \underset{\sim}{A}\underset{\sim}{b} = \underset{\sim}{b} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \Rightarrow \underset{\sim}{b} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\ \underset{\sim}{A}\underset{\sim}{c} = 0. \Rightarrow \underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$$

Thus, particular solution is: $\underset{\sim}{x}^{*} = \underset{\sim}{v}(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{t}$

So, general solution:

$$\underset{\sim}{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{t}$$

C. Variation of parameters:

Number 1: $\underset{\sim}{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

From previous calculation,

$\underset{\sim}{\Psi}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$ is a fundamental matrix.

$\underset{\sim}{x} = \underset{\sim}{\Psi}(t)\underset{\sim}{u}(t)$ where $\underset{\sim}{u}(t)$ satisfies $\underset{\sim}{\Psi}(t)\underset{\sim}{u}'(t) = \underset{\sim}{g}(t)$,

$$\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\Rightarrow \begin{cases} u_1' = \frac{3}{2} - \frac{1}{2}te^{-t} \\ u_2' = \frac{1}{2}te^t - \frac{1}{2}e^{2t} \end{cases}$$

Hence,
$$u_1(t) = \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} + \frac{3}{2}t + C_1$$
$$u_2(t) = \frac{1}{2}te^t - \frac{1}{2}e^t - \frac{1}{4}e^{2t} + C_2$$

and
$$\underset{\sim}{x} = \underset{\sim}{\Psi}(t)\underset{\sim}{u}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t} + \frac{3}{2}t + C_1 \\ \frac{1}{2}te^t - \frac{1}{2}e^t - \frac{1}{4}e^{2t} + C_2 \end{pmatrix}$$

$$= C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + C_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} - \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t + \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^t$$
$$+ \begin{pmatrix} 1 \\ 2 \end{pmatrix}t.$$

Number 4:

$$\underset{\sim}{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$\underset{\approx}{\psi}(t) = \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix}$$

$$\underset{\approx}{\psi}(t)\,\underset{\sim}{u}'(t) = \underset{\sim}{g}(t) \Rightarrow \begin{pmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{pmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} u_1' = \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t} \\ u_2' = \frac{1}{5}e^{t} + \frac{2}{5}e^{4t} \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = -\frac{1}{5}e^{-4t} + \frac{2}{5}e^{-t} + C_1 \\ u_2 = \frac{1}{5}e^{t} + \frac{1}{10}e^{4t} + C_2 \end{cases}$$

and

$$\underset{\sim}{x} = \underset{\approx}{\psi}(t)\,\underset{\sim}{u}(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{-3t} + \begin{pmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{pmatrix}$$

2. numbers 3 by undetermined coefficients.

$$\underset{\sim}{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$\begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix} = 0 \Rightarrow r_{1,2} = \pm i \Rightarrow \underset{\sim}{\xi}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}, \ \underset{\sim}{\xi}^{(2)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}.$$

$$\underset{\sim}{g}(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t.$$

We assume that $\underset{\sim}{x} = \underset{\sim}{v}(t) = \underset{\sim}{a} t \cos t + \underset{\sim}{b} t \sin t + \underset{\sim}{c} \cos t + \underset{\sim}{d} \sin t.$

Substituted into differential eqn:

$$\underset{\sim}{a} \cos t - \underset{\sim}{a} t \sin t + \underset{\sim}{b} \sin t + \underset{\sim}{b} t \cos t - \underset{\sim}{c} \sin t + \underset{\sim}{d} \cos t$$

$$= \underset{\sim}{A} (\underset{\sim}{a} t \cos t + \underset{\sim}{b} t \sin t + \underset{\sim}{c} \cos t + \underset{\sim}{d} \sin t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t$$

$$\Rightarrow \begin{cases} \underset{\sim}{A}\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{d} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \leftarrow \cos t \\ \underset{\sim}{A}\underset{\sim}{d} = \underset{\sim}{b} - \underset{\sim}{c} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \leftarrow \sin t \\ \underset{\sim}{A}\underset{\sim}{a} = \underset{\sim}{b} & \leftarrow t \cos t. \\ \underset{\sim}{A}\underset{\sim}{b} = -\underset{\sim}{a} & \leftarrow t \sin t. \end{cases}$$

$$\Rightarrow \underset{\sim}{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \underset{\sim}{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \underset{\sim}{c} = \begin{pmatrix} -1-2d_1+5d_2 \\ -1-d_1+2d_2 \end{pmatrix}$$

Let $d_1 = 1$, $d_2 = 0$ $\Rightarrow$ $\underset{\sim}{c} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$, $\underset{\sim}{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

so,

$$\underset{\sim}{V}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t \cos t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} t \sin t + \begin{pmatrix} -3 \\ -2 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t.$$

general solution of the nonhomogeneous system is

$$\underset{\sim}{x} = c_1\begin{pmatrix}2+i\\1\end{pmatrix}e^{it} + c_2\begin{pmatrix}2-i\\1\end{pmatrix}e^{-it} +$$

$$\begin{pmatrix}2\\1\end{pmatrix}t\cos t + \begin{pmatrix}-1\\0\end{pmatrix}t\sin t + \begin{pmatrix}-3\\-2\end{pmatrix}\cos t +$$

$$\begin{pmatrix}1\\0\end{pmatrix}\sin t.$$

3. Number 5 by <u>variation of parameters.</u>

$$\underset{\sim}{x}' = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\underset{\sim}{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0$$

$$\begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = 0 \Rightarrow r_{1,2} = 0. \quad \Rightarrow \underset{\sim}{\xi} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus one solution of the system is

$$\underset{\sim}{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{0\cdot t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The second solution is

$$\underset{\sim}{x}^{(2)}(t) = \underset{\sim}{\xi}\, t + \underset{\sim}{\eta}.$$

$$\underset{\sim}{A}\,\underset{\sim}{\eta} = \underset{\sim}{\xi} \Rightarrow \underset{\sim}{\eta} = \begin{pmatrix} \frac{1+2k}{4} \\ k \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + k\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}.$$ we get.

$$\underset{\sim}{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + k\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

the last term is merely a multiple of the first solution $\underset{\sim}{x}^{(1)}(t)$ and may be ignored. so

$$\underset{\sim}{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$$

The general solution of the homogeneous system is

$$\underset{\sim}{x}_h = c_1 \underset{\sim}{x}^{(1)}(t) + c_2 \underset{\sim}{x}^{(2)}(t).$$

$$= c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + c_2\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}.$$

Fundamental matrix is:

$$\underset{\sim}{\Psi}(t) = \begin{pmatrix} 1 & t+\frac{1}{4} \\ 2 & 2t \end{pmatrix}.$$

(12)

$$\underset{\sim}{\Psi}(t)\underset{\sim}{u}'(t)=\underset{\sim}{g}(t) \Rightarrow \begin{pmatrix} 1 & t+\frac{1}{4} \\ 2 & 2t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}=\begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}.$$

$$\Rightarrow u_1' = -\frac{9}{2}t^{-2}-2t^{-1} \Rightarrow u_1 = \frac{9}{2}t^{-1}-2\ln t + C_1.$$

$$u_2' = 4t^{-3}+2t^{-2} \Rightarrow u_2 = -2t^{-2}-2t^{-1}+C_2$$

$$\Rightarrow \underset{\sim}{x} = \underset{\sim}{\Psi}(t)\underset{\sim}{u}(t)$$

$$= \begin{pmatrix} 1 & t+\frac{1}{4} \\ 2 & 2t \end{pmatrix}\begin{pmatrix} \frac{9}{2}t^{-1}-2\ln t+C_1 \\ -2t^{-2}-2t^{-1}+C_2 \end{pmatrix}$$

$$= C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}t + C_2\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}.$$

$$+\begin{pmatrix} 2 \\ 5 \end{pmatrix}t^{-1} - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}\ln t - 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}t^{-2}.$$

4. Number 7 by und. method.

$$\underset{\sim}{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \underset{\sim}{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

using ~~diagonalization~~ undetermined coefficients

$\underset{\sim}{g}(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t \qquad \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \Rightarrow r_1 = -1,\ r_2 = 3$

~~Assuming that~~ when $r_1 = -1$,

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

when $r_2 = 3$,

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^{(2)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

general solution of the homogeneous equation is:

$$\underset{\sim}{x} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$

Assuming that

$\underset{\sim}{x}^*(t) = \underset{\sim}{a} e^t + \underset{\sim}{b}$ substituted into differential eqn,

$$\underset{\sim}{a} e^t = \underset{\sim}{A}(\underset{\sim}{a} e^t + \underset{\sim}{b}) + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$

$$\Rightarrow \begin{cases} \underset{\sim}{a} = A\underset{\sim}{a} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ 0 = \underset{\sim}{A}\underset{\sim}{b} \end{cases}$$

$\Rightarrow \underset{\sim}{a} = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ particular solution is

$$\underset{\sim}{x}^* = \begin{pmatrix} \frac{1}{4} \\ -2 \end{pmatrix} e^t$$

hence,

the general solution ~~of~~ the non homogeneous system is:

$$\underset{\sim}{v} = c_1\begin{pmatrix}-1\\2\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + \begin{pmatrix}\frac{1}{4}\\-2\end{pmatrix}e^{t}$$

5. Using your general solution for section 7.9 numbers, determin the solution if $\underset{\sim}{x}(0) = \begin{bmatrix}4\\0\end{bmatrix}$.

$$\underset{\sim}{x}(0) = c_1\begin{pmatrix}2+i\\1\end{pmatrix} + c_2\begin{pmatrix}2-i\\1\end{pmatrix} + \begin{pmatrix}-3\\-2\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}.$$

$$\Rightarrow \begin{cases} 2c_1+2c_2-3+i(c_1-c_2) = 4. \\ c_1+c_2-2 = 0 \end{cases}$$

$$\Rightarrow \cancel{c_1 = 1,\ c_2} \begin{cases} c_1 = 1-\frac{3}{2}i \\ c_2 = 1+\frac{3}{2}i. \end{cases}$$

$\Rightarrow$ solution:

$$\underset{\sim}{x} = \left(1-\frac{3}{2}i\right)\begin{pmatrix}2+i\\1\end{pmatrix}e^{it} + \left(1+\frac{3}{2}i\right)\begin{pmatrix}2-i\\1\end{pmatrix}e^{-it}$$

$$+\begin{pmatrix}2\\1\end{pmatrix}t\cos t + \begin{pmatrix}-1\\0\end{pmatrix}t\sin t + \begin{pmatrix}-3\\-2\end{pmatrix}\cos t +$$

$$\begin{pmatrix}1\\0\end{pmatrix}\sin t.$$

$$\frac{d^2T}{dx^2} - m^2(T - T_\infty) = 0 \Rightarrow \frac{d^2T}{dx^2} - m^2 T = -m^2 T_\infty .$$

7. characteristic eqn is:

$$r^2 - m^2 = 0. \Rightarrow r = \pm m.$$

general solution of the homogeneous ~~syst~~ eqn is:

$$T_h(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Assuming $T^* = a$ (a is a constant) is the particular solution of the nonhomogeneous eqn. substituting it into $\frac{d^2T}{dx^2} - m^2 T = -m^2 T_\infty$, we obtain,

$$a = +T_\infty \Rightarrow T^* = T_\infty .$$

Hence, the general solution of the nonhomogeneous eqn is

$$T(x) = T_h(x) + T^* = C_1 e^{mx} + C_2 e^{-mx} + T_\infty .$$

$$\begin{cases} T(0) = T_b \Rightarrow T_b = C_1 + C_2 + T_\infty \Rightarrow C_1 + C_2 = T_b - T_\infty \ \& \\ \lim_{x \to \infty} T(x) = T_\infty \Rightarrow C_1 = 0 \ (\text{suppose } m > 0.) \end{cases}$$

$$\Rightarrow C_2 = T_b - T_\infty$$

$$\Rightarrow T(x) = (T_b - T_\infty) e^{-mx} + T_\infty \quad (m > 0).$$

$$\left.\frac{dT}{dx}\right|_{x=0} = -m(T_b - T_\infty) e^{-mx} \Big|_{x=0} = -m(T_b - T_\infty)$$