Hw7 Solution

1. $q_1=x_1$, $q_2=x_2$,the direction of y is upward

$$\bar{r}_1=\begin{bmatrix} x_1 \\ 0 \end{bmatrix} , \bar{r}_2=\begin{bmatrix} x_1+x_2\cos\theta \\ x_2\sin\theta \end{bmatrix} \Rightarrow \dot{\bar{r}}_1=\begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} , \dot{\bar{r}}_2=\begin{bmatrix} \dot{x}_1+\dot{x}_2\cos\theta \\ \dot{x}_2\sin\theta \end{bmatrix}$$

$$T_1=\frac{1}{2}\times m_1\times\dot{r}_1^{\,2}=\frac{1}{2}m_1\dot{x}_1^{\,2}$$

$$T_2=\frac{1}{2}\times m_2\times\dot{r}_2^{\,2}=\frac{1}{2}m_2(\dot{x}_1^{\,2}+\dot{x}_2^{\,2}+2\dot{x}_1\dot{x}_2\cos\theta)$$

$$T=T_1+T_2=\frac{1}{2}\left[m_1\dot{x}_1^{\,2}+m_2(\dot{x}_2^{\,2}+\dot{x}_1^{\,2}+2\dot{x}_1\dot{x}_1\cos\theta)\right]$$

$$V_1=\frac{1}{2}k_1x_1^2$$

$$V_2=\frac{1}{2}k_2x_2^2-m_2g\,x_2\cos\theta$$

$$V=V_1+V_2=\frac{1}{2}(k_1x_1^2+k_2x_2^2)-m_2g\,x_2\sin\theta$$

$$L=T-V=\frac{1}{2}\left[m_1\dot{x}_1^{\,2}+m_2(\dot{x}_1^{\,2}+\dot{x}_2^{\,2}+2\dot{x}_1\dot{x}_2\cos\theta)\right]$$
$$-\frac{1}{2}(k_1x_1^2+k_2x_2^2)+m_2g\,x_2\sin\theta$$

$\Rightarrow$ ***Euler − Lagarange Equation***

consider the gravity as the internal system force, so $Q_1=Q_2=0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=Q_1=0 \quad\Rightarrow\quad (m_1+m_2)\ddot{x}_1+m_2\ddot{x}_2\cos\theta+k_1x_1=0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=Q_2=0 \quad\Rightarrow\quad m_2\ddot{x}_2+m_2\ddot{x}_1\cos\theta+k_2x_2-m_2g\sin\theta=0$$

2. $q=\theta$, $\bar{r}=\begin{bmatrix} l\sin\theta\cos\omega t \\ -l\sin\theta\sin\omega t \\ -l\cos\theta \end{bmatrix}$, $\dot{\bar{r}}=\begin{bmatrix} l\dot{\theta}\cos\theta\cos\omega t-\omega l\sin\theta\sin\omega t \\ l\dot{\theta}\cos\theta\sin\omega t+\omega l\sin\theta\cos\omega t \\ l\dot{\theta}\sin\theta \end{bmatrix}$,here l is the radius of the hoop

$$T=\frac{1}{2}\times m\times\dot{\bar{r}}^2=\frac{1}{2}ml^2(\dot{\theta}^2+\omega^2\sin^2\theta)$$

Or you can just consider that $\bar{r}_1=\begin{bmatrix} l\sin\theta \\ l\cos\theta \end{bmatrix}$, $\dot{\bar{r}}_1=\begin{bmatrix} l\dot{\theta}\cos\theta \\ -l\dot{\theta}\sin\theta \end{bmatrix}$ which only consider the bead rotates the center point in the vertical circle. Then here are two parts of the kinetic energy: one is rotating the center point, the other is rotating the axis with the velocity $\omega l\sin\theta$.

$$T_1=\frac{1}{2}\times m\times\dot{\bar{r}}_1^{\,2}+\frac{1}{2}\times m\times(\omega\, l\sin\theta)^2=\frac{1}{2}ml^2\dot{\theta}^2+\frac{1}{2}m\omega^2 l^2\sin^2\theta$$

we can see that $T=T_1$ Take the horizontal line as the position of zero potential energy

$$V=-mgl\cos\theta$$

$$\Rightarrow\quad L=T-V=\frac{1}{2}ml^2\dot{\theta}^2+\frac{1}{2}m\omega^2 l^2\sin^2\theta+mgl\cos\theta$$

$$\Rightarrow\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=ml^2\ddot{\theta}-m\omega^2 l^2\sin\theta\cos\theta+mgl\sin\theta=Q=0$$

$$\Rightarrow\quad l^2\ddot{\theta}-\omega^2 l^2\sin\theta\cos\theta+gl\sin\theta=0$$

3. $q_1=\theta_1, q_2=\theta_2$,here θ_2 is the angle between l_1 and l_2 . And here the top hinge is the zero position of potential energy.

$$\bar{r}_1=\begin{bmatrix} l_1\sin\theta_1 \\ -l_1\cos\theta_1\end{bmatrix},\qquad \bar{r}_2=\begin{bmatrix} l_1\sin\theta_1+l_2\sin(\theta_1+\theta_2) \\ -l_1\cos\theta_1-l_2\cos(\theta_1+\theta_2)\end{bmatrix}$$

$$\Rightarrow\quad \dot{\bar{r}}_1=\begin{bmatrix} l_1\dot{\theta}_1\cos\theta_1 \\ l_1\dot{\theta}_1\sin\theta_1\end{bmatrix},\quad \dot{\bar{r}}_2=\begin{bmatrix} l_1\dot{\theta}_1\cos\theta_1+l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) \\ l_1\dot{\theta}_1\sin\theta_1+l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2)\end{bmatrix}$$

$$T_1=\frac{1}{2}\times m_1\times\dot{\bar{r}}_1^{\,2}=\frac{1}{2}m_1 l_1^2\dot{\theta}_1^2$$

$$\Rightarrow T_2=\frac{1}{2}\times m_2\times\dot{\bar{r}}_2^{\,2}=\frac{1}{2}m_2[l_1^2\dot{\theta}_1^2+l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2+2l_1 l_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\cos\theta_2]$$

$$V_1=-m_1 g l_1\cos\theta_1$$

$$V_2=-m_2 g[l_1\cos\theta_1+l_2\cos(\theta_1+\theta_2)]$$

$$\Rightarrow\quad \begin{aligned} &T=T_1+T_2=\frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^{\,2}+\frac{1}{2}m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2+m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\cos\theta_2 \\ &V=V_1+V_2=-(m_1+m_2)g l_1\cos\theta_1-m_2 g l_2\cos(\theta_1+\theta_2)\end{aligned}$$

$$\Rightarrow\quad \begin{aligned} L=T-V=&\frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^{\,2}+\frac{1}{2}m_2 l_2^2(\dot{\theta}_1+\dot{\theta}_2)^2+m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\cos\theta_2 \\ &+(m_1+m_2)g l_1\cos\theta_1+m_2 g l_2\cos(\theta_1+\theta_2)\end{aligned}$$

$$\Rightarrow\quad \begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\frac{\partial L}{\partial\theta_1}=\tau_1=\sin t \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=\tau_2=\sin 2t\end{aligned}$$

$$\Rightarrow\quad \begin{aligned} &(m_1+m_2)l_1^2\ddot{\theta}_1+m_2 l_2^2(\ddot{\theta}_1+\ddot{\theta}_2)+m_2 l_1 l_2(2\ddot{\theta}_1+\ddot{\theta}_2)\cos\theta_2-m_2 l_1 l_2\dot{\theta}_2(2\dot{\theta}_1+\dot{\theta}_2)\sin\theta_2 \\ &\qquad+(m_1+m_2)g l_1\sin\theta_1+m_2 g l_2\sin(\theta_1+\theta_2)=\sin t \\ &m_2 l_2^2(\ddot{\theta}_1+\ddot{\theta}_2)+m_2 l_1 l_2\ddot{\theta}_1\cos\theta_2+m_2 l_1 l_2\dot{\theta}_1^{\,2}\sin\theta_2+m_2 g l_2\sin(\theta_1+\theta_2)=\sin 2t\end{aligned}$$

If you use x to represent the angle of x to the vertical axis and the zero position of the potential energy is bottom when it is still.

$$L=\frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2+m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)+\frac{1}{2}m_2 l_2^2\dot{\theta}_2^2$$
$$-m_1 g[l_2+l_1(1-\cos\theta_1)]-m_2 g[l_2(1-\cos\theta_2)+l_1(1-\cos\theta_1)]$$

$$\Rightarrow \quad \begin{array}{c}(m_1+m_2)l_1^2\ddot{\theta}_1+m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+m_2 l_1 l_2\dot{\theta}_2^{\,2}\sin(\theta_1-\theta_2)+(m_1+m_2)g\,l_1\sin\theta_1=\sin t\\ m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2 l_1 l_2\dot{\theta}_1^{\,2}\sin(\theta_1-\theta_2)+m_2 l_2^2\ddot{\theta}_2+m_2 g\,l_2\sin\theta_2=\sin 2t\end{array}$$

4. $$q_1=x, q_2=\theta, \bar{r}=\begin{bmatrix}x\cos\theta\\ x\sin\theta\end{bmatrix} \quad \Rightarrow \quad \dot{\bar{r}}=\begin{bmatrix}\dot{x}\cos\theta-x\dot{\theta}\sin\theta\\ \dot{x}\sin\theta+x\dot{\theta}\cos\theta\end{bmatrix}$$

Here I consider the center as the origin, x is the coordinate in the radius direction, x_0 is the unstretched position.

$$T_1=\frac{1}{2}\times m\times\dot{\bar{r}}^2=\frac{1}{2}m(\dot{x}^2+x^2\dot{\theta}^2)$$

$$T_2=\frac{1}{2}I\dot{\theta}^2$$

$$\Rightarrow \quad T=T_1+T_2=\frac{1}{2}(m\dot{x}^2+m x^2\dot{\theta}^2+I\dot{\theta}^2)$$

$$V=\frac{1}{2}k(x-x_0)^2$$

$$\Rightarrow \quad L=T-V=\frac{1}{2}(m\dot{x}^2+m x^2\dot{\theta}^2+I\dot{\theta}^2)-\frac{1}{2}k(x-x_0)^2$$

$$\Rightarrow \quad \begin{array}{c}\frac{d}{dt}(\frac{\partial L}{\partial\dot{x}})-\frac{\partial L}{\partial x}=0\\ \frac{d}{dt}(\frac{\partial L}{\partial\dot{\theta}})-\frac{\partial L}{\partial\theta}=0\end{array} \quad \Rightarrow \quad \begin{array}{c}m\ddot{x}-m x\dot{\theta}^2+k(x-x_0)=0\\ 2m x\dot{x}\dot{\theta}+m x^2\ddot{\theta}+I\ddot{\theta}=0\end{array}$$

If x is the coordinate from the unstretched position, x_0 is the length from the center.

$$\bar{r}=\begin{bmatrix}(x+x_0)\cos\theta\\ (x+x_0)\sin\theta\end{bmatrix}\Rightarrow \dot{\bar{r}}=\begin{bmatrix}\dot{x}\cos\theta-(x+x_0)\dot{\theta}\sin\theta\\ \dot{x}\sin\theta+(x+x_0)\dot{\theta}\cos\theta\end{bmatrix}$$

$$\Rightarrow \quad T=\frac{1}{2}[m\dot{x}^2+m(x+x_0)^2\dot{\theta}^2+I\dot{\theta}^2], \quad V=\frac{1}{2}k x^2$$

$$\Rightarrow \quad L=T-V=\frac{1}{2}[m\dot{x}^2+m(x+x_0)^2\dot{\theta}^2+I\dot{\theta}^2-k x^2]$$

$$\Rightarrow \quad \begin{array}{c}m\ddot{x}-m(x+x_0)\dot{\theta}^2+k x=0\\ 2m(x+x_0)\dot{x}\dot{\theta}+m(x+x_0)^2\ddot{\theta}+I\ddot{\theta}=0\end{array}$$

If using polar coordinates, $\dot{\bar{r}}=\begin{bmatrix}\dot{x}\\ (x+x_0)\dot{\theta}\end{bmatrix}$

$$\Rightarrow \quad T=\frac{1}{2}[m\dot{x}^2+m(x+x_0)^2\dot{\theta}^2+I\dot{\theta}^2], V=\frac{1}{2}k x^2\cdots$$

5. $$q_1=x, q_2=\theta, \quad \bar{r}=\begin{bmatrix}x+l\sin\theta\\ -l\cos\theta\end{bmatrix}\Rightarrow \quad \dot{\bar{r}}=\begin{bmatrix}\dot{x}+l\dot{\theta}\cos\theta\\ l\dot{\theta}\sin\theta\end{bmatrix}$$

$$T_1 = \frac{1}{2} m_1 \dot{x}^2$$

$$T_2 = \frac{1}{2} \times m_2 \times \dot{\vec{r}}^2 = \frac{1}{2} m_2 (\dot{x}^2 + l^2 \dot{\theta}^2 + 2 \dot{x} l \dot{\theta} \cos\theta)$$

$$\Rightarrow \quad T = T_1 + T_2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 \dot{x} l \dot{\theta} \cos\theta$$

$$V_1 = \frac{1}{2} k x^2$$

$$V_2 = -m_2 g l \cos\theta$$

$$\Rightarrow \quad V = V_1 + V_2 = \frac{1}{2} k x^2 - m_2 g l \cos\theta$$

$$\Rightarrow \quad L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 \dot{x} l \dot{\theta} \cos\theta - \frac{1}{2} k x^2 + m_2 g l \cos\theta$$

$$\Rightarrow \quad \begin{matrix} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\tau \end{matrix} \quad \Rightarrow \quad \begin{matrix} (m_1 + m_2)\ddot{x} + m_2 l (\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) + k x = 0 \\ m_2 l^2 \ddot{\theta} + m_2 l \ddot{x} \cos\theta + m_2 g l \sin\theta = -\cos t \end{matrix}$$

since $m_1 = m_2 = m$

$$\Rightarrow \quad \begin{matrix} 2 m \ddot{x} + m l (\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) + k x = 0 \\ m l^2 \ddot{\theta} + m l \ddot{x} \cos\theta + m g l \sin\theta = -\cos t \end{matrix}$$