University of Notre Dame
Aerospace and Mechanical Engineering

AME 302: Differential Equations, Vibrations and Control II
Homework 9 Solutions

1. Consider the penulum illustrated in Figure 1.

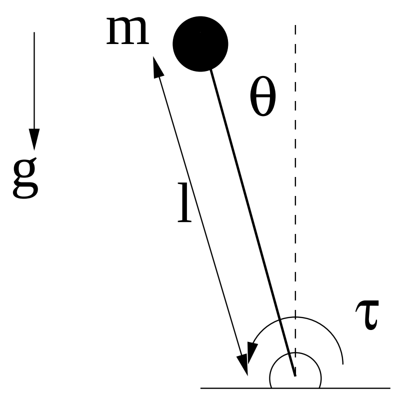

Figure 1. Pendulum for Problem 1.

(a) Simply using Newton's law gives

$$I\ddot{\theta} = \sum \text{torques}$$

or

$$ml^2\ddot{\theta} = mgl\sin\theta + \tau.$$

For small θ

$$ml^2\ddot{\theta} = mgl\theta + \tau. \qquad (1)$$

(b) Laplace transforming equation 1 and assuming zero initial conditions since none were specified gives

$$ml^2s^2\Theta(s) = mgl\Theta(s) + T(s)$$

or

$$\frac{\Theta(s)}{T(s)} = \frac{1}{ml^2s^2 - mgl}. \qquad (2)$$

(c) Since $\mathcal{L}(\dot{\theta}(t)) = s\Theta(s) = \Omega(s)$

$$\frac{\Omega(s)}{T(s)} = \frac{s}{ml^2s^2 - mgl}. \tag{3}$$

(d) For the circuit illustrated in Figure 2 we know that

$$e(t) = i(t)R + L\frac{di(t)}{dt} + k_e\dot{\theta}$$

and

$$\tau = i(t)k_t,$$

or

$$\begin{aligned} E(s) &= I(s)\,(R + Ls) + k_e s\Theta \\ T(s) &= I(s)k_t. \end{aligned}$$

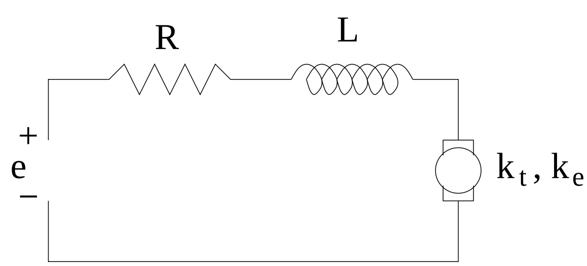

Figure 2. Motor circuit for Problem 1.

Eliminating $I(s)$ gives

$$E(s) = \frac{T(s)}{k_t}\,(R + Ls) + k_e s\Theta.$$

Solving for $T(s)$ gives

$$T(s) = \frac{k_t\,(E(s) - k_e s\Theta(s))}{Ls + R}. \tag{4}$$

The final answers are simply obtained by substituting equation 4 into equations 2 and 3.

i.

$$\frac{\Theta(s)}{E(s)} = \frac{k_t}{l^2Lms^2 + l^2mRs^2 + (k_ek_t - glLm)\,s - glmR}.$$

ii.

$$\frac{\Omega(s)}{E(s)} = \frac{sk_t}{l^2Lms^2 + l^2mRs^2 + (k_ek_t - glLm)\,s - glmR}.$$

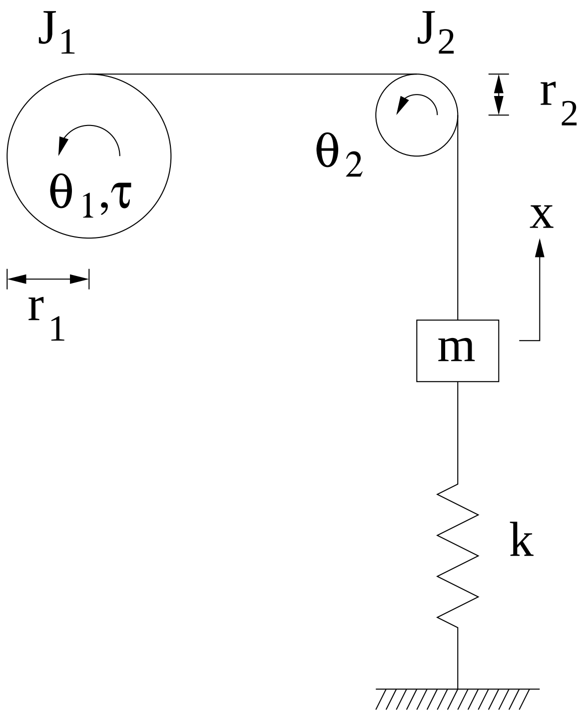

Figure 3. System for Problem 2.

2. Consider the system illustrated in Figure 3.

(a) Let $t_1(t)$ be the tension in the belt between the two pulleys. Let $t_2(t)$ be the tension between pulley 2 and the mass. Newton's law on pulley 1 gives

$$J_1\ddot{\theta}_1 = \tau - r_1 t_1,$$

on pulley 2 gives

$$J_2\ddot{\theta}_2 = r_2\left(t_1 - t_2\right),$$

and on the mass gives

$$m\ddot{x} = t_2 - kx.$$

Also assuming that the belt does not slip on the pulleys gives

$$\begin{aligned} r_1\theta_1 &= r_2\theta_2 \\ x &= r_2\theta_2. \end{aligned}$$

(b) Laplace transforming everything and assuming zero initial conditions gives

$$J_1 s^2 \Theta_1(s) = T(s) - r_1 T_1(s)$$

$$
\begin{aligned}
J_2 s^2 \Theta_2(s) &= r_2 \left(T_1(s) - T_2(s)\right) \\
m s^2 X(s) &= T_2(s) - k X(s) \\
r_1 \Theta_1(s) &= r_2 \Theta_2(s) \\
X(s) &= r_2 \Theta_2(s).
\end{aligned}
$$

We want $\frac{X(s)}{T(s)}$ so we need to solve for the Θ's and the tensions in terms of $X(s)$ and the torque, $T(s)$:

$$
\begin{aligned}
\Theta_1 &= \frac{X}{r_1} \\
\Theta_2 &= \frac{X}{r_2} \\
T_1(s) &= -\frac{J_1 s^2 \Theta_1(s) - T(s)}{r_1} = -\frac{J_1 s^2 X(s) - r_1 T(s)}{r_1^2} \\
T_2(s) &= -\frac{J_2 s^2 \Theta_2(s) - r_2 T_1(s)}{r_2}
\end{aligned}
$$

Substituting everything into

$$
m s^2 X(s) = T_2(s) - k X(s)
$$

and solving for $\frac{X(s)}{T(s)}$ gives the final answer,

$$
\frac{X(s)}{T(s)} = \frac{r_1 r_2^2}{\left(m r_1^2 r_2^2 + J_1 r_2^2 + J_2 r_1^2\right) s^2 + k r_1^2 r_2^2}.
$$

(c) This is easy. If $v(t) = \dot{x}(t)$, then $V(s) = sX(s)$ and

$$
\frac{V(s)}{T(s)} = \frac{r_1 r_2^2 s}{\left(m r_1^2 r_2^2 + J_1 r_2^2 + J_2 r_1^2\right) s^2 + k r_1^2 r_2^2}.
$$

(d) KVL on the circuit in Figure 4 gives

$$
e(t) = i(t) R + L \frac{di(t)}{dt} + k_e \dot{\theta}_1(t) + v_c(t)
$$

where v_c is the voltage drop across the capacitor. Laplace transforming and using $i(t) = C\frac{dv_c(t)}{dt}$ gives

$$
E(s) = I(s)\left(Ls + R\right) + k_e s \Theta_1(s) + \frac{I(s)}{Cs}.
$$

Also, using $\tau = i(t) k_t \Longleftrightarrow T(s) = I(s) k_t$ and $\Theta_1(s) = \frac{X(s)}{r_1}$ and substituting gives

$$
k_t E(s) = T\left(R + Ls + \frac{C}{s}\right) + k_t k_e s \frac{X}{r_1}.
$$

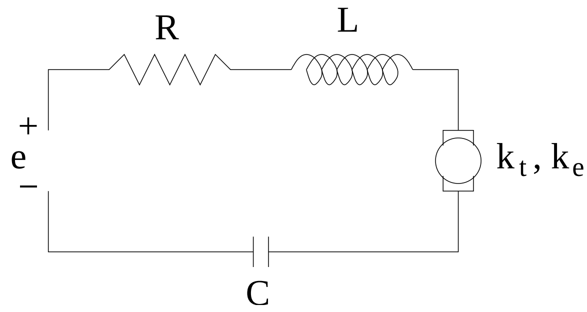

Figure 4. Motor circuit for Problem 2.

Solving for $T(s)$ and substituting into $\frac{X(s)}{T(s)}$ gives the final, messy, answers:

$$\frac{X(s)}{E(s)} = \frac{k_t r_1 r_2^2 s}{\left(Lmr_1^2 r_2^2 + J_1 L r_2^2 + J_2 L r_1^2\right) s^4 + \left(mRr_1^2 r_2^2 + J_1 R r_2^2 + J_2 R r_1^2\right) s^3 + \left(Cmr_1^2 r_2^2 + kLr_1^2 r_2^2 + k_e k_t r_2^2 + CJ_1 r_2^2 + CJ_2 r_1^2\right) s^2 + kRr_1^2 r_2^2 s + Ckr_1^2 r_2^2},$$

and

$$\frac{V(s)}{E(s)} = s\frac{X(s)}{E(s)}.$$