Engineering Differential Equations: Theory and Applications

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Chapter 2

Second Order Systems

2.1 Introduction

2.2 Motivational Examples

2.3 Review of Complex Variable Theory

This section very briefly reviews some aspects of complex variable theory necessary for the rest of this chapter. It is far from complete. In fact, another review of complex variable theory containing additional material is included in Chapter **??**.

The usual way to express a complex number, z is to write it as the sum of its real and imagainary components, *i.e.,*

$$z = x + iy,$$

where x is the real part of z and y is the complex part. Denote the set of complex numbers by $\mathbb{C}$. Let $\mathrm{Re}\,(z) = x$ denote the real part of z and $\mathrm{Im}\,(z) = y$ denote the complex part. Convention also dictates that $i = \sqrt{-1}$ or $i^2 = -1$, which naturally leads to the normal rules for adding and multiplying complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i\,(y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)\,(x_2 + iy_2) \\ &= x_1x_2 + x_1iy_2 + iy_1x_2 + iy_1iy_2 \\ &= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_1x_2)\,. \end{aligned}$$

Clearly, since one must specify both the real and imaginary components of a complex number, it is also convenient to consider them as *ordered pairs* of real numbers, *i.e.,* $z = (x, y)$ where the normal rule for vector addition applies, *i.e.,*

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\,.$$

In this notation, the product may seem unusual, but is the same definition as above

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)\,.$$

When presented with an ordered pair of numbers, it is hard to resist plotting them, as illustrated in Figure 2.1. However, especially in a graphical presentation, it is clear that polar coordinates are an alternative means to specify a complex number. In particular, z may be represented by the magnitude or radius, r and angle or argument, θ. Clearly, the relationship between the Cartesian representation, $z = (x, y)$ and polar representation, $z = (r, \theta)$ is

$$\begin{aligned} x &= r\cos\theta & (2.1)\\ y &= r\sin\theta & (2.2)\\ r &= \sqrt{x^2+y^2} & (2.3)\\ \theta &= \arctan 2\,(y, x) & (2.4) \end{aligned}$$

where the $\arctan 2$ function is the arctan which keeps track of the quadrant of the complex number, so clearly

$$z = x + iy = r(\cos\theta + i\sin\theta)\,.$$

Other than simply manipulating complex numbers, the primary use of them in this chapter is exponentials of complex numbers. In particular, just as was the case in Chapter 1, exponential functions will play a fundamental role in the solution of second order differential equations. The starting point for this development is *Euler's formula*

$$e^{i\theta} = \cos\theta + i\sin theta. \qquad (2.5)$$

The approach for this development will be to take Euler's formula, Equation 2.5 as a definition and show that identities we expect to hold for exponentials naturally extend to the complex case when this is used.

Since it is natural to consider

$$e^z = e^{x+iy} = e^x e^{iy}$$

take as a definition for when $z = x + iy$

$$e^z = e^x(\cos y + i\sin y)\,.$$

It is clearly desirable that the usual identities hold such as

$$e^{z_1} e^{z_2} = e^{z_1+z_2} \qquad (2.6)$$

and

$$\frac{d}{dt}e^{zt} = ze^{zt}$$

hold in the complex case. The fact that they do is left as an exercise.

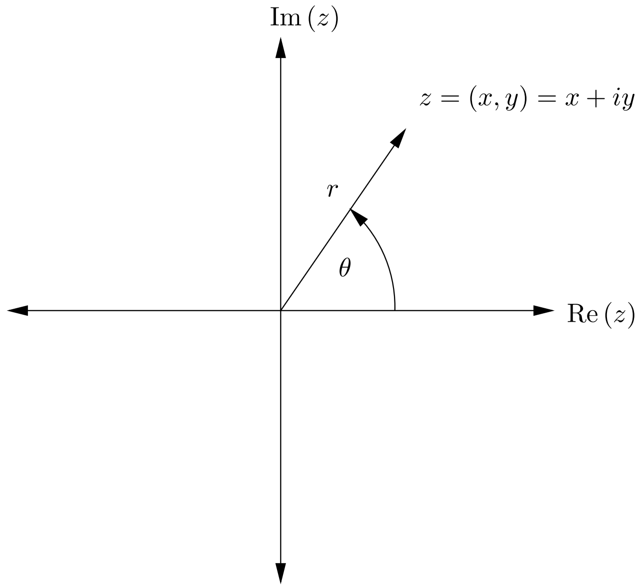

Figure 2.1. The complex plane.

2.4 Theory of Second Order Equations

2.4.1 Homogeneous Systems

Distinct Roots

Complex Roots

Repeated Roots

2.4.2 Nonhomogeneous Equations

2.5 The Method of Undetermined Coefficients

Method of Variation of Parameters

2.6 Applications

2.6.1 Free Vibrations

2.6.2 Forced Vibrations

Forced

Resonance

2.6.3 Vibrating Base

Exercises

1. Prove Equation 2.6, *i.e.,*

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

where z_1 and z_2 are complex numbers.

2. Using the fact that

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2},$$

prove the common additive trigonometric identities

$$\begin{aligned}
\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\
\sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2.
\end{aligned}$$

Chapter 3

Multiple-Degree of Freedom Systems

3.1 Introduction

So far this book has considered the theory and applications of first and second order differential equations. This chapter considers nth order differential equations, or equivalently, systems of n first order differential equations. As will become readily apparent, the theoretical basis for solving such systems relies heavily upon matrix algebra theory.

3.2 Motivational Example

Consider the mass-spring-damper system illustrated in Figure 3.1. While this is the simplified prototypical system that we will consider, it also is representative of a much larger class of useful engineering systems such as automobile suspensions and civil structures. As is the usual case, assume that x_1 and x_2 are absolute displacements of m_1 and m_2 respectively measured from the equilibrium configuration of the system. If there is no gravity, then x_1 and x_2 will be measured from the position of the masses when the springs are unstretched, and if there is gravity, then they will be measured from the position of the masses when the springs are statically compressed or extended by the weight of the masses.

Considering a free body diagram for each mass illustrated in Figure 3.2 and applying Newton's law gives

$$\begin{aligned} m_1\ddot{x}_1 &= -b_1\dot{x}_1 - k_1x_1 + k_2\left(x_2 - x_1\right) + b_2\left(\dot{x}_2 - \dot{x}_1\right) \\ m_2\ddot{x}_2 &= -k_2\left(x_2 - x_1\right) - b_2\left(\dot{x}_2 - \dot{x}_1\right) + F(t), \end{aligned}$$

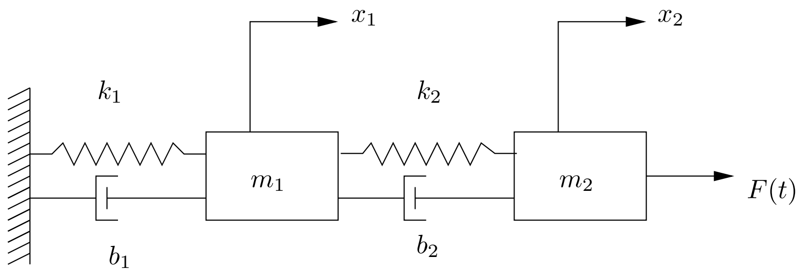

Figure 3.1. Two degree of freedom mass-spring-damper system.

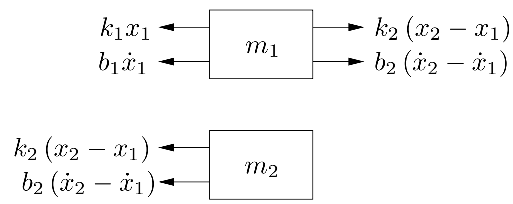

Figure 3.2. Free body diagrams for masses in Figure 3.1.

and rearranging into the standard form of descending order of derivatives gives

$$\begin{aligned} m_1\ddot{x}_1 + (b_1 + b_2)\,\dot{x}_1 - b_2\dot{x}_2 + (k_1 + k_2)\,x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - b_2\dot{x}_1 + b_2\dot{x}_2 - k_2x_1 + k_2x_2 &= F(t). \end{aligned} \tag{3.1}$$

These equations are *coupled* since x_1 appears in the x_2 equation and *vice-versa.* One's first inclination may be to try to solve one equation for one of either x_1 or x_2 and substitute into the other, but such an approach is impossible since the equations involve the derivatives of the variables as well.

An insightful extrapolation of the method considered in Chapter 2 might lead one to attempt to solve the homogeneous problem first followed by some method for the particular solutions; indeed, this is fundamentally the approach we will utilize. In fact, for the homogeneous case ($F(t) = 0$), *i.e.,*

$$\begin{aligned} m_1\ddot{x}_1 + (b_1 + b_2)\,\dot{x}_1 - b_2\dot{x}_2 + (k_1 + k_2)\,x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - b_2\dot{x}_1 + b_2\dot{x}_2 - k_2x_1 + k_2x_2 &= 0, \end{aligned}$$

a good guess may be assume

$$\begin{aligned} x_1(t) &= e^{\lambda_1 t} \\ x_2(t) &= e^{\lambda_2 t}, \end{aligned} \tag{3.2}$$

and substitute. This will actually work, but as we consider higher and higher order systems, *e.g.*, systems like in Figure 3.1 but with more masses, presenting algebra will become somewhat cumbersome. In order to consider the problem more concisely (and more elegantly) resorting to matrix algebra is the typical approach. For the mathematically inclined, the abstraction is nice because it still presents the essence of the problem; however, for those less mathematically inclined it can be problematic. The key concept to keep in mind is that behind all the matrix theory presented, the basic approach for the homogeneous problem is still to simply consider solutions of the form of Equation 3.3.

The following example illustrates the fact that assuming solutions of the form of Equation 3.3 actually works.

3.2.1 Example For computational simplicity, assume the following numerical parameter values

$$\begin{aligned} m_1 &= 1 \\ m_2 &= 1 \\ k_1 &= 1 \\ k_2 &= 1 \\ b_1 &= 4 \\ b_2 &= 4, \end{aligned}$$

which gives

$$\begin{aligned} \ddot{x}_1 + 8\dot{x}_1 - 4\dot{x}_2 + 2x_1 - x_2 &= 0 \\ \ddot{x}_2 - 4\dot{x}_1 - 4\dot{x}_2 - x_1 + x_2 &= 0. \end{aligned}$$

Substituting $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ into the differential equations gives

$$\begin{aligned} \lambda_1^2 e^{\lambda_1 t} + 8\lambda_1 e^{\lambda_1 t} - 4\lambda_2 e^{\lambda_2 t} + 2e^{\lambda_1 t} - e^{\lambda_2 t} &= 0 \\ \lambda_2^2 e^{\lambda_2 t} - 4\lambda_1 e^{\lambda_1 t} + 4\lambda_2 e^{\lambda_2 t} - e^{\lambda_1 t} + e^{\lambda_2 t} &= 0. \end{aligned}$$

Dividing the first equation by $e^{\lambda_1 t}$ and the second equation by $e^{\lambda_2 t}$

$$\begin{aligned} \lambda_1^2 + 8\lambda_1 - 4\lambda_2 e^{(\lambda_2-\lambda_1)t} + 2 - e^{(\lambda_2-\lambda_1)t} &= 0 \\ \lambda_2^2 - 4\lambda_1 e^{(\lambda_1-\lambda_2)t} + 4\lambda_2 - e^{(\lambda_1-\lambda_2)t} + 1 &= 0, \end{aligned}$$

gives

$$\begin{aligned} e^{(\lambda_2-\lambda_1)t} &= \frac{\lambda_1^2 + 8\lambda_1 + 2}{4\lambda_2 + 1} \\ e^{(\lambda_1-\lambda_2)t} &= \frac{\lambda_2^2 + 4\lambda_2 + 1}{4\lambda_4 + 1}. \end{aligned}$$

Since these are reciprocals, λ_1 and λ_2 satisfy

$$\left(\lambda_1^2 + 8\lambda_1 + 2\right)\left(\lambda_2^2 + 4\lambda_2 + 1\right) - (4\lambda_1 + 1)(4\lambda_2 + 1) = 0. \tag{3.3}$$

Resorting to a numerical calculation, we find that the pairs of values

$$(\lambda_1, \lambda_2) = (-10.2159, -0.2563) \qquad \text{and} \qquad (\lambda_1, \lambda_2) = (-1.2130, -0.3149)$$

satisfy Equation 3.3, so the solutions

$$x_1(t) = c_1 e^{-10.2159t} + c_2$$

finish ■

The general approach to solve systems of this type is to first convert the system into a system of first order equations. This is illustrated by the following example.

3.2.2 Example Let

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= \dot{x}_1 \\ \xi_3 &= x_2 \\ \xi_4 &= \dot{x}_2. \end{aligned}$$

Then

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \frac{-b_1\xi_2 - k_1\xi_1 + k_2\xi_3 - k_2\xi_1 + b_2\xi_4 - b_2\xi_2}{m_1} \\ \xi_3 \\ \frac{-k_2\xi_3 + k_2\xi_1 - b_2\xi_4 + b_4\xi_2}{m_2}. \end{bmatrix} \tag{3.4}$$

Since this equation is linear in the ξ_i's, it can be expressed as

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} \\ 0 & 0 & 1 & 0 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}.$$

If we let

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} \\ 0 & 0 & 1 & 0 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix}, \tag{3.5}$$

then this whole system can be expressed simply as

$$\dot{\xi} = A\xi. \tag{3.6}$$

Clearly, the way to solve this equation hinges on the property of the matrix A. Exploiting the properties of A to solve this equation is our task at hand.■

Now, considering a general first order matrix differential equation of the form

$$\dot{\xi} = A\xi \tag{3.7}$$

the question arises as to the nature of the solution. Motivated by the results from Chapters 1 and 2, consider the possibility of a solution of the form

$$\xi(t) = \hat{\xi}e^{\lambda t},$$

where $\hat{\xi}$ is a constant vector. In full detail,

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_n(t) \end{bmatrix} = \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \vdots \\ \hat{\xi}_n \end{bmatrix} e^{\lambda t} = \begin{bmatrix} \hat{\xi}_1 e^{\lambda t} \\ \hat{\xi}_2 e^{\lambda t} \\ \vdots \\ \hat{\xi}_n e^{\lambda t} \end{bmatrix}.$$

Substituting this into Equation 3.7 gives

$$\lambda \begin{bmatrix} \hat{\xi}_1 e^{\lambda t} \\ \hat{\xi}_2 e^{\lambda t} \\ \vdots \\ \hat{\xi}_n e^{\lambda t} \end{bmatrix} = A \begin{bmatrix} \hat{\xi}_1 e^{\lambda t} \\ \hat{\xi}_2 e^{\lambda t} \\ \vdots \\ \hat{\xi}_n e^{\lambda t} \end{bmatrix}.$$

Inserting an identity matrix gives

$$\lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 e^{\lambda t} \\ \hat{\xi}_2 e^{\lambda t} \\ \vdots \\ \hat{\xi}_n e^{\lambda t} \end{bmatrix} = A \begin{bmatrix} \hat{\xi}_1 e^{\lambda t} \\ \hat{\xi}_2 e^{\lambda t} \\ \vdots \\ \hat{\xi}_n e^{\lambda t} \end{bmatrix},$$

which can be rearranged to give

$$(A - \lambda I)\,\hat{\xi} = 0. \tag{3.8}$$

Recall from linear algebra, that the values for λ that satisfy Equation 3.8 are the eigenvalues of the matrix A and the $\hat{\xi}$ that satisfy it are the corresponding eigenvectors of A. More importantly, what this shows is that solutions to Equation 3.7 are the product of the eigenvectors and exponentials of the eigenvalues of A.

3.3 Review of Linear Algebra

In this section we review only topics from linear algebra necessary to solve differential equations of the form of Equation 3.6.

3.3.1 Linear independence

Consider the set of vectors $\{\xi_1, \ldots, \xi_k\} \in \mathbb{R}^n$, *i.e.,* k vectors that are n elements "tall" such as

$$\xi_i = \begin{bmatrix} \xi_{i,1} \\ \xi_{i,2} \\ \vdots \\ \xi_{i,n} \end{bmatrix}.$$

Definition 3.3.1 (Linear (in)dependence) *The set $\{\xi_1, \ldots, \xi_n\}$ is linearly independent if $\exists$ scalars $\alpha_1, \ldots, \alpha_k$, where at least one $\alpha_i \neq 0$ such that*

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \cdots + \alpha_k\xi_k = \sum_{i=1}^{k} \alpha_i\xi_i = 0.$$

If the set is non linearly dependent, then it is linearly independent.

A simple example is in order.

3.3.2 Example Let $n = 3$ and

$$\xi_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \xi_3 = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}.$$

Clearly, determining linear dependence or independence by inspection is not easy. So we try to solve

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, as three scalar equations

$$\begin{aligned} \alpha_1 + \alpha_2 + 5\alpha_3 &= 0 \\ 2\alpha_1 + \alpha_2 + 7\alpha_3 &= 0 \\ 3\alpha_1 + \alpha_2 + 9\alpha_3 &= 0. \end{aligned}$$

A tedious calculation gives

$$\begin{aligned} \alpha_1 &= 2 \\ \alpha_2 &= 3 \\ \alpha_3 &= 1, \end{aligned}$$

which determines that the set of vectors $\{\xi_1, \xi_2, \xi_3\}$ is linearly dependent.■

add reference for Cramer's rule

An easier approach is to recall Cramer's Rule and to utilize the following result.

Proposition 3.3.3 *If $A \in \mathbb{R}^{n\times n}$ and if $\det(A) = 0$ then the set of vectors that are the columns of A are linearly dependent. Also, the set of vectors that are the rows of A are linearly dependent. If $\det(A) \neq 0$ then the columns and rows are linearly independent.*

PROOF The proof of this result is beyond the scope of this book and relies upon an analysis of permutations. □

really skip the proof?

3.3.4 Example Considering the system in Example 3.3.2, an easy computation gives

$$\det\left(\begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 7 \\ 3 & 1 & 9 \end{bmatrix}\right) = 0$$

thus confirming the result from Example 3.3.2 that the vectors are linearly dependent. ■

The primary utility of the notion of linear independence is that in a n dimensional vector space, a set of n linearly independent vectors, $\{x_1, \ldots, x_n\}$, form a *basis* for the vector space. Thus any vector in that space can be written as a linear combination, *i.e.*, $x = \sum_{i=1}^{n} \alpha_i x_i$.

Remark 3.3.5 *Relationship with the Wronskian*

3.3.2 Eigenvalues and eigenvectors

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$, the product $y = Ax$ is simply another vector in $\mathbb{R}^n$. However, there are two classes of the vectors x that give a special result when multiplied into A. The first special case is then the resulting vector is all zeros and the second special case is when the resulting vector is just a scaled version of x. The following two definitions elaborate upon this.

Definition 3.3.6 (Null Space) *The null space of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $\mathcal{N}(A)$, is the set of all vectors $x \in \mathbb{R}^n$ such that*

$$Ax = 0.$$

In this case 0 is the vector in $\mathbb{R}^n$ full of n zeros.

Definition 3.3.7 (Eigenvectors and Eigenvalues) *An eigenvector of a matrix $A \in \mathbb{R}^{n\times n}$ is a non-zero vector, $\hat{\xi}$, such that*

$$A\hat{\xi} = \lambda\hat{\xi}.$$

The number λ, which may be real or complex, is the associated eigenvalue.

To compute eigenvalues and eigenvectors, note that

$$A\hat{\xi} = \lambda\hat{\xi} \qquad \Longrightarrow \qquad A\hat{\xi} - \lambda\hat{\xi} = (A - \lambda I)\,\hat{\xi} = 0, \tag{3.9}$$

where I is the $n \times n$ identity matrix. By Cramer's rule, Equation 3.9 has a solution if and only if

$$\det(A - \lambda I) = 0. \tag{3.10}$$

Equation 3.10 is an nth degree polynomial in λ and hence has n solutions. Thus, $A \in \mathbb{R}^{n\times n}$ has n eigenvalues. At this point, all we know is that there are n eigenvalues. Note that the eigenvalues may be all real and distinct, or some of them may be repeated and/or complex conjugate pairs.

To compute the eigenvalue associated with a particular eigenvalue λ, simply substitute the value for λ into Equation 3.9 and solve for each component of $\hat{\xi}$. As the following example illustrates, the eigenvector can only be determined up to a unique scaling factor.

3.3.8 Example Compute the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

First, to compute the eigenvalues,

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right) \\
&= (1-\lambda)(3-\lambda) - 2 \\
&= \lambda^2 - 4\lambda + 1 \\
&= 0.
\end{aligned}$$

Thus,

$$\lambda = 2 \pm \sqrt{3}.$$

To compute the eigenvectors, substituting the two values for λ into Equation 3.10 gives

$$\left(A - \left(2+\sqrt{3}\right) I\right) = \begin{bmatrix} 1-2-\sqrt{3} & 2 \\ 1 & 3-2-\sqrt{3} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

which gives

$$\begin{aligned}
\left(-1-\sqrt{3}\right)\xi_1 + 2\xi_2 &= 0 \\
\xi_1 + \left(1-\sqrt{3}\right)\xi_2 &= 0.
\end{aligned}$$

A quick computation will show that if we try to solve for one variable, say ξ_2, from one of the equations and substitute into the other equation, we will end up with the degenerate equation $0 = 0$. This is precisely due to the fact that we are trying to solve a system of linearly dependent equations. Thus there are an infinite number of solutions.

The most straightforward approach may be to simply set one of the variables equal to one and solve for the others. So, in this example, arbitrarily let $\xi_2 = 1$. Both equations then give $\xi_1 = \sqrt{3}-1$, and hence the eigenvector corresponding to the eigenvalue $\lambda = 2 + \sqrt{3}$ is

$$\hat{\xi} = \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix}.$$

Note that **any** vector of the form

$$\hat{\xi} = \alpha \begin{bmatrix} \sqrt{3}-1 \\ 1 \end{bmatrix},$$

where α is a real or complex number is also an eigenvector corresponding to the eigenvalue $\lambda = 2 + \sqrt{3}$.

A similar computation (and again arbitrarily setting $\xi_2 = 1$) gives

$$\hat{\xi} = \begin{bmatrix} -\sqrt{3}-1 \\ 1 \end{bmatrix}$$

as an eigenvector corresponding to the eigenvalue $\lambda = 2 - \sqrt{3}$. ■

3.4 Summary So Far

1. Systems of first order differential equations of the form

$$\dot{\xi} = A\xi \qquad \xi \in \mathbb{R}^n, \qquad A \in \mathbb{R}^{n\times n}$$

arise naturally in engineering problems with coupled elements.

2. The system is *homogeneous* since

$$\dot{\xi} = A\xi \qquad \dot{\xi} - A\xi = 0$$

and each homogeneous solution is of the form

$$\xi_h(t) = \hat{\xi}_i e^{\lambda_i t}$$

where $\hat{\xi}_i$ and λ_i is the ith eigenvector and eigenvalue of the matrix A.

3. In general A has n eigenvalue/eigenvector pairs $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\hat{\xi}_1, \ldots, \hat{\xi}_n\}$, (except possibly, as will be considered later, when A has repeated eigenvalues).

4. The general solution to $\dot{\xi} = A\xi$ is a linear combination of n homogeneous solutions

$$\xi(t) = c_1\hat{\xi}_1 e^{\lambda_1 t} + \cdots + c_n \xi_n e^{\lambda_n t},$$

and the coefficients c_i may be used to satisfy specified initial conditions.

3.5 Distinct Eigenvalues

The case where the matrix A has distinct eigenvalues is the easiest and will be considered first. It is basically a straight-forward application of what has been covered up to this point. First, a critically important theorem.

Theorem 3.5.1 *Let $A \in \mathbb{R}^{n\times n}$. If A has n distinct, real eigenvalues, then it has a set of n linearly independent eigenvectors.*

PROOF Let $\lambda_1, \ldots, \lambda_n$ denote the distinct eigenvalues of A, *i.e.*, $\lambda_i \neq \lambda_j$ if $i \neq j$ and let $\hat{\xi}_1, \ldots, \hat{\xi}_n$ denote the corresponding eigenvectors. To show that the eigenvectors are linearly independent it suffices to show that

$$\alpha_1\hat{\xi}_1 + \alpha_2\hat{\xi}_2 + \cdots + \alpha_n\hat{\xi}_n = 0 \qquad \Longleftrightarrow \qquad \alpha_i = 0 \quad \forall i,$$

that is there is no linear combination of the eigenvectors that is zero.

Assume that not all the α_i are zero, and without loss of generality, assume in particular that $\alpha_1 \neq 0$. Then

$$\hat{\xi}_1 = \frac{1}{\alpha_1}\sum_{i=2}^{n}\alpha_i\hat{\xi}_i.$$

Thus

$$\begin{aligned}
0 &= (A - \lambda_1 I)\,\hat{\xi}_1 \\
&= (A - \lambda_1 I)\,\frac{1}{\alpha_1}\sum_{i=2}^{n}\alpha_i\hat{\xi}_i \\
&= \sum_{i=2}^{m}\frac{\alpha_i}{\alpha_1}(A - \lambda_1 I)\,\hat{\xi}_i \\
&= \sum_{i=2}^{m}\frac{\alpha_i}{\alpha_1}\left(A\hat{\xi}_i - \lambda_1\hat{\xi}_i\right) \\
&= \sum_{i=2}^{m}\frac{\alpha_i}{\alpha_1}\left(\lambda_i\hat{\xi}_i - \lambda_1\hat{\xi}_i\right) \\
&= \sum_{i=2}^{m}\frac{\alpha_i}{\alpha_1}(\lambda_i - \lambda_1)\,\hat{\xi}_i.
\end{aligned}$$

If not all α_i are zero, then the only way for this sum to be zero is if $\lambda_i = \lambda_1$, which is a contradiction. □

3.5.1 Solution Technique for $\dot{\xi} = A\xi$

The general solution to $\dot{\xi} = A\xi$ is a linear combination of n homogeneous solutions

$$\xi(t) = c_1\hat{\xi}_1 e^{\lambda_1 t} + \cdots + c_n\xi_n e^{\lambda_n t},$$

and the coefficients c_i may be used to satisfy specified initial conditions. Since the eigenvectors are linearly independent, any initial condition may be satisfied with the appropriate coefficients, c_i's. In particular, for a specified $\xi(0)$

$$\begin{aligned}\xi(0) &= c_1\hat{\xi}_1 + \cdots + c_n\hat{\xi}_n \\ &= \begin{bmatrix} \hat{\xi}_1 & \cdots & \hat{\xi}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.\end{aligned}$$

Thus the coefficients can most concisely be expressed as

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \hat{\xi}_1 & \cdots & \hat{\xi}_n \end{bmatrix}^{-1} \xi(0).$$

3.5.2 Example Find the homogeneous solutions to

$$\dot{\xi} = A\xi \qquad \text{where} \qquad A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}. \tag{3.11}$$

Aside 3.5.3 *Note that the system in Equation 3.11 is exactly equivalent to the following two systems:*

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

and

$$\begin{aligned}\dot{\xi}_1 &= \xi_1 + 2\xi_2 \\ \dot{\xi}_2 &= \xi_1.\end{aligned}$$

If this is not readily apparent by inspection, some time should be invested in verifying this fact.

As determined previously, the homogeneous solutions of Equation 3.11 can be computed by determining the eigenvalues and eigenvectors of A. Thus

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)\lambda - 2 = \lambda^2 - \lambda - 2 - 0,$$

so the eigenvalues are

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -1.\end{aligned}$$

Substituting each eigenvalue into $(A - \lambda I)\xi = 0$ gives

$$\begin{aligned}\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad &\Longrightarrow \qquad \hat{\xi}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad &\Longrightarrow \qquad \hat{\xi}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.\end{aligned}$$

Thus

$$\begin{aligned}\xi_1(t) &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} \\ \xi_2(t) &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}\end{aligned}$$

both satisfy $\dot{\xi} = A\xi$. ■

From the above example, since each of the two solutions are homogeneous solutions, any linear combination of them also satisfies the differential equation, *i.e.,* the general solution,

$$\xi(t) = c_1 \hat{\xi}_1 e^{\lambda_1 t} + c_2 \hat{\xi}_2 e^{\lambda_2 t}$$

also satisfies $\dot{\xi} = A\xi$. If the problem were an initial value problem, then the coefficients c_1 and c_2 could be used to satisfy the initial condition.

3.5.4 Example Returning to Example 3.5.2 determine the solution to

$$\dot{\xi} = A\xi$$

where

$$\xi(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The general solution to Equation 3.11 is

$$\xi(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

Substituting $t = 0$ and the initial condition gives

$$\xi(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which gives

$$\begin{aligned}c_1 &= \frac{1}{3} \\ c_2 &= \frac{1}{3},\end{aligned}$$

so

$$\xi(t) = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} e^{2t} + \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} e^{-t}$$

is the solution to the initial value problem. ■

Next are a few useful theorems that sometimes allow for some computational shortcuts. It turns out that when the matrix A is symmetric, its eigenvalues and eigenvectors have especially nice properties. First, however, we generalize the notion of a *symmetric* matrix to the complex case and the corresponding properties of a *Hermitian* matrix.

Definition 3.5.5 *Hermitian Matrix Let* $A \in \mathbb{C}^{n\times n}$*. Let* $A^* = \overline{A}^T$*, i.e.,* A^* *denotes the matrix where* A *is transposed an all the elements are changed to their complex conjugates.* A *is Hermitian if* $A = A^*$*.*

Note the following:

1. the notation $A \in \mathbb{C}^{n\times n}$ simply means that A is n by n with complex numbers for elements; and,
2. in particular, if A is real and symmetric, *i.e.,* $A \in \mathbb{R}^{n\times n}$ and $A = A^T$ it is Hermitian.

Theorem 3.5.6 *If* $A \in \mathbb{C}^{n\times n}$ *is Hermitian, i.e.,* $A = A^*$*, then*

1. *all the eigenvalues of* A *are real;*
2. A *has* n *linearly independent eigenvectors, regardless of the multiplicity of any eigenvalue; and,*
3. *eigenvectors corresponding to different eigenvalues are orthogonal.*

PROOF

1. Assume $A = A^*$. Since
$$A\hat{\xi}_i = \lambda_i\hat{\xi}_i \quad \Longrightarrow \hat{\xi}_i^* A\hat{\xi}_i = \lambda_i\hat{\xi}_i^*\hat{\xi}_i$$
the eigenvalue may be expressed as
$$\lambda_i = \frac{\hat{\xi}_i^* A\hat{\xi}_i}{\hat{\xi}_i^*\hat{\xi}_i}.$$
Then
$$\lambda_i^* = \left(\frac{\hat{\xi}_i^* A\hat{\xi}_i}{\hat{\xi}_i^*\hat{\xi}_i}\right)^* = \frac{\left(\hat{\xi}_i^* A\hat{\xi}_i\right)^*}{\left(\hat{\xi}_i^*\hat{\xi}_i\right)^*} = \frac{\hat{\xi}_i^* A^*\hat{\xi}_i}{\hat{\xi}_i^*\hat{\xi}_i} = \frac{\hat{\xi}_i^* A\hat{\xi}_i}{\hat{\xi}_i^*\hat{\xi}_i} = \lambda_i.$$
Since $\lambda_i = \lambda_i^*$, it must be real.
2. finish
3. finish □

3.6 Complex Eigenvalues

3.6.1 Example Again consider the mass-spring-damper system illustrated in Figure 3.1. Let

$$\begin{aligned} m_1 &= 1 \\ m_2 &= 1 \\ k_1 &= 10 \\ k_2 &= 1 \\ b_1 &= 0.1 \\ b_2 &= 0.1. \end{aligned}$$

The damping has been decreased greatly compared to the example for distinct real roots in Section 3.5, so oscillatory solutions should be expected. Substituting these values into the A matrix in Equation 3.5 gives

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -11 & -0.2 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \\ 1 & 0.1 & -1 & -0.1 \end{bmatrix}$$

which has eigenvalues

$$\begin{aligned} \lambda_1 &= -0.1093 + 3.3285i \\ \lambda_2 &= -0.1093 - 3.3285i \\ \lambda_3 &= -0.0407 + 0.9487i \\ \lambda_4 &= -0.0407 - 0.9487i, \end{aligned}$$

and corresponding eigenvectors

$$\hat{\xi}_1 = \begin{bmatrix} -0.0094 - 0.2859i \\ 0.9527 \\ -0.0074 + 0.0287i \\ -0.0946 - 0.0278i \end{bmatrix} \qquad \hat{\xi}_2 = \begin{bmatrix} -0.0094 + 0.2859i \\ 0.9527 \\ -0.0074 - 0.0287i \\ -0.0946 + 0.0278i \end{bmatrix}$$

$$\hat{\xi}_3 = \begin{bmatrix} 0.0713 + 0.0060i \\ -0.0086 + 0.0674i \\ 0.7216 \\ -0.0294 + 0.6846i \end{bmatrix} \qquad \hat{\xi}_4 = \begin{bmatrix} 0.0713 - 0.0060i \\ -0.0086 - 0.0674i \\ 0.7216 \\ -0.0294 - 0.6846i \end{bmatrix}.$$

Observe that the eigenvalues occur in complex conjugate pairs. This should be obviously expected since eigenvalues are the roots of a polynomial. Less obvious, but probably not surprising is that the eigenvectors also occur in complex conjugate pairs. The reason this is true is given by the proof of the following. ■

Proposition 3.6.2 *If $A \in \mathbb{R}^{n\times n}$ and two eigenvalues of A are such that $\lambda_i = \overline{\lambda_j}$, then if $\hat{\xi}_i$ is the eigenvector corresponding to λ_i, $\overline{\hat{\lambda}_i}$ is an eigenvector corresponding to λ_j.*

Proof Eigenvector $\hat{\xi}_i$ satisfies

$$(A - \lambda_i I)\,\hat{\xi}_i = 0.$$

Taking the complex conjugate of both sides gives

$$\begin{aligned} \overline{(A - \lambda_i I)\,\hat{\xi}_i} &= \overline{0} \\ \left(A - \overline{\lambda_i} I\right)\overline{\hat{\xi}_i} &= 0 \\ (A - \lambda_j I)\,\overline{\hat{\xi}_i} &= 0. \end{aligned}$$

Thus we make take $\hat{\xi}_j = \overline{\hat{\xi}}_i$. □

To solve the initial value problem

$$\dot{\xi} = A\xi \qquad \xi(0) = \xi_0$$

we *may* to proceed as before and simply write the general solution

$$\xi(t) = c_1\hat{\xi}_1 e^{\lambda_1 t} + \cdots c_1\hat{\xi}_n e^{\lambda_n t},$$

substitute $t = 0$

$$\xi(0) = c_1\hat{\xi}_1 + \cdots c_1\hat{\xi}_n,$$

and solve for the unknown coefficients, c_i. The following example illustrates that fact. In order to make it computationally simple, however, a simple 2×2 system is considered rather than the 4×4 oscillation problem.

3.6.3 Example Solve

$$\dot{\xi} = A\xi \qquad \xi(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Computing the eigenvalues gives

$$\det(A - \lambda I) = (1-\lambda)^2 + 4 = 0 \qquad \Longrightarrow \qquad \lambda = 1 \pm 2i.$$

For $\lambda_1 = 1 + 2i$

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \hat{\xi}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

and for $\lambda_2 = 1 - 2i$

$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \hat{\xi}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So the general solution is

$$\xi(t) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1+2i)t} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1-2i)t}$$

and at $t = 0$,

$$\xi(0) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Either solving for c_1 and c_2 by inverting the matrix or by eliminating one coefficient from one equation and substituting into the other gives

$$\begin{aligned} c_1 &= \frac{1}{2} + \frac{1}{2}i \\ c_2 &= \frac{1}{2} - \frac{1}{2}i. \end{aligned}$$

Finally, substituting c_1 and c_2 into the general solution gives

$$\xi(t) = \left[\begin{array}{c} \frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i \end{array} \right] e^{(1+2i)t} + \left[\begin{array}{c} \frac{1}{2} - \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{2}i \end{array} \right] e^{(1-2i)t}.$$

This is the correct answer, however it is somewhat dissatisfy ing in that it is complex; whereas, the matrix A and the initial conditions were all real. Quite a bit more manipulation using Euler's formula eliminates this minor problem and yields

$$\xi(t) = \left[\begin{array}{c} \cos 2t - \sin 2t \\ \cos 2t + \sin 2t \end{array} \right] e^t.$$

■

The preceding example illustrates that the general solution may still be correctly expressed as a linear combination of the eigenvalues times the exponential of the corresponding eigenvectors. However,

1. the solution may not "naturally" result in a purely real expression for ξ, which is what is expected;

2. further, and perhaps arduous manipulation may be necessary to determine the form of the solution that is purely real;

3. many computations involving complex numbers, requiring four operations for multiplication and two operations for addition, are involved in computing the solution;

4. the fact that the eigenvalues and eigenvectors occur in complex conjugate pairs was not exploited at all.

In order to make the computations less burdensome, an alternative approach which is analogous to the approach in the case of second order system with complex roots is utilized. Fundamentally, the "shortcut" to this approach is based upon the conjugate nature of the eigenvalues and eigenvectors.

Consider a pair of complex conjugate eigenvalues and eigenvectors, denoted by

$$\begin{aligned} \lambda_1 &= \mu + i\omega \\ \lambda_2 &= \mu - i\omega \end{aligned}$$

and

$$\begin{aligned} \hat{\xi}_1 &= \mathbf{a} + i\mathbf{b} \\ \hat{\xi}_2 &= \mathbf{a} - i\mathbf{b}. \end{aligned}$$

Note that $\mathbf{a}$ and $\mathbf{b}$ are *vectors* in $\mathbb{R}^n$.

The general solution is of the form

$$\xi(t) = c_1 \hat{\xi}_1 e^{\lambda_1 t} + c_2 \hat{\xi}_2 e^{\lambda_2 t} + \cdots .$$

Substituting for the components of λ_1, λ_2, $\hat{\xi}_1$ and $\hat{\xi}_2$ and using Euler's formula gives

$$\begin{aligned}
\xi(t) &= c_1\hat{\xi}_1 e^{\lambda_1 t} + c_2\hat{\xi}_2 e^{\lambda_2 t} + \cdots \\
&= c_1\left(\mathbf{a}+i\mathbf{b}\right)e^{(\mu+i\omega)t} + c_2\left(\mathbf{a}-i\mathbf{b}\right)e^{(\mu-i\omega)t} + \cdots \\
&= c_1\left(\mathbf{a}+i\mathbf{b}\right)e^{\mu t}\left(\cos\omega t + i\sin\omega t\right) + c_2\left(\mathbf{a}-i\mathbf{b}\right)e^{\mu t}\left(\cos\omega t - i\sin\omega t\right) + \cdots \\
&= e^{\mu t}\left[c_1\mathbf{a}\cos\omega t - c_1\mathbf{b}\sin\omega t + ic_1\mathbf{a}\sin\omega t + ic_1\mathbf{b}\cos\omega t + \right. \\
&\quad \left. c_2\mathbf{a}\cos\omega t - c_2\mathbf{b}\sin\omega t - ic_2\mathbf{b}\cos\omega t - ic_2\mathbf{a}\sin\omega t\right] + \cdots \\
&= e^{\mu t}\left[\left(c_1+c_2\right)\mathbf{a}\cos\omega t - \left(c_1+c_2\right)\mathbf{b}\sin\omega t\right] + \\
&\quad e^{\mu t} i\left[\left(c_1-c_2\right)\mathbf{a}\sin\omega t + \left(c_1-c_2\right)\mathbf{b}\cos\omega t\right] + \cdots .
\end{aligned}$$

Let

$$\begin{aligned}
k_1 &= c_1 + c_2 \\
k_2 &= i\left(c_1 - c_2\right)
\end{aligned}$$

and substituting into $\xi(t)$ gives

$$\xi(t) = k_1 e^{\mu t}\left(\mathbf{a}\cos\omega t - \mathbf{b}\sin\omega t\right) + k_2 e^{\mu t}\left(\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t\right) + \cdots . \tag{3.12}$$

3.6.4 Example Returning to the mass-spring-damper system in Example 3.6.1, observe that we have

$$\begin{aligned}
\mu_1 &= -0.1093 \\
\omega_1 &= 3.3285 \\
\mu_2 &= -0.0407 \\
\omega_2 &= 0.9487
\end{aligned}$$

and

$$\mathbf{a}_1 = \begin{bmatrix} -0.0094 \\ 0.9527 \\ -0.0074 \\ -0.0946 \end{bmatrix} \qquad \mathbf{b}_1 = \begin{bmatrix} -0.2859 \\ 0 \\ 0.0287 \\ -0.0278 \end{bmatrix}$$

$$\mathbf{a}_2 = \begin{bmatrix} 0.0713 \\ -0.0086 \\ 0.7216 \\ -0.0294 \end{bmatrix} \qquad \mathbf{b}_2 = \begin{bmatrix} 0.0060 \\ 0.0674 \\ 0 \\ 0.6846 \end{bmatrix}.$$

The general solution is of the form

$$\begin{aligned}
\xi(t) &= k_1 e^{\mu_1 t}\left(\mathbf{a}_1\cos\omega_1 t - \mathbf{b}_1\sin\omega_1 t\right) + k_2 e^{\mu_1 t}\left(\mathbf{a}_1\sin\omega_1 t + \mathbf{b}_1\cos\omega_1 t\right) \\
&+ k_3 e^{\mu_2 t}\left(\mathbf{a}_2\cos\omega_2 t - \mathbf{b}_2\sin\omega_2 t\right) + k_4 e^{\mu_2 t}\left(\mathbf{a}_2\sin\omega_2 t + \mathbf{b}_2\cos\omega_2 t\right),
\end{aligned}$$

or substituting all the numerical values

$$\begin{aligned}
\xi(t) &= k_1 e^{-0.1093t}\left(\begin{bmatrix} -0.0094 \\ 0.9527 \\ -0.0074 \\ -0.0946 \end{bmatrix}\cos 3.3285t - \begin{bmatrix} -0.2859 \\ 0 \\ 0.0287 \\ -0.0278 \end{bmatrix}\sin 3.3285t\right) \\
&+ k_2 e^{-0.1093t}\left(\begin{bmatrix} -0.0094 \\ 0.9527 \\ -0.0074 \\ -0.0946 \end{bmatrix}\sin 3.3285t + \begin{bmatrix} -0.2859 \\ 0 \\ 0.0287 \\ -0.0278 \end{bmatrix}\cos 3.3285t\right) \\
&+ k_3 e^{-0.0407t}\left(\begin{bmatrix} 0.0713 \\ -0.0086 \\ 0.7216 \\ -0.0294 \end{bmatrix}\cos 0.9487t - \begin{bmatrix} 0.0060 \\ 0.0674 \\ 0 \\ 0.6846 \end{bmatrix}\sin 0.9487t\right) \\
&+ k_4 e^{-0.0407t}\left(\begin{bmatrix} 0.0713 \\ -0.0086 \\ 0.7216 \\ -0.0294 \end{bmatrix}\sin 0.9487t + \begin{bmatrix} 0.0060 \\ 0.0674 \\ 0 \\ 0.6846 \end{bmatrix}\cos 0.9487t\right). \blacksquare
\end{aligned}$$

3.7 Repeated Eigenvalues

3.7.1 Example Consider $\dot{\xi} = A\xi$ where

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Computing the eigenvalues gives

$$(2-\lambda)^2 = 0 \qquad \Longrightarrow \qquad \lambda = 2.$$

Computing the eigenvectors,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \hat{\xi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\square$ $\blacksquare$

In the preceding example, the eigenvalue $\lambda = 2$ was repeated. It may not be surprising that there also is only one eigenvector, $\hat{\xi}$ as well. However, things are not so simple. Consider the following example.

3.7.2 Example Consider $\dot{\xi} = A\xi$ where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Computing the eigenvalues gives

$$(2-\lambda)^2 = 0 \qquad \Longrightarrow \qquad \lambda = 2,$$

which is exactly the same as before. Now computing the eigenvectors,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this case, however, we have that

$$\hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \hat{\xi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

both satisfy the eigenvector equation and are linearly independent. □ ■

These two examples illustrate the fact that when there are n repeated eigenvalues, there may or may not be n linearly independent eigenvectors. This is problematic in that to use the approach utilized so far to solve $\dot{\xi} = A\xi$ we need n linearly independent eigenvectors in order to obtain a general solution.

First we address the practical computational matter of determining how many linearly independent eigenvectors are associated with a repeated eigenvalue. Then we delineate the solution techniques for each case.

3.7.1 Geometric and Algebraic Multiplicities

The number of times that an eigenvalue is repeated is called its *algebraic multiplicity*. Similarly, the number of linearly independent eigenvectors associated with an eigenvalue is called its *geometric multiplicity*. Clearly, the former is an algebraic concept and the latter a geometric one as is clear from the following more general mathematical definitions of the two terms.

Definition 3.7.3 (Algebraic Multiplicity) *Let $A \in \mathbb{R}^{n\times n}$ and let*

$$\det(A-\lambda I) = \sum_{i=1}^{m} (\lambda - \lambda_i)^{k_i}$$

where each λ_i is distinct. Note that $\sum_{i=1}^{m} k_i = n$. The number k_i is the algebraic multiplicity of eigenvalue λ_i.

3.7.4 Example Add example to illustrate the general characteristic equation formula. □ ■

Definition 3.7.5 (Geometric Multiplicity) *Let $A \in \mathbb{R}^{n\times n}$. The dimension of the null space of $(A - \lambda_i I)$ is the geometric multiplicity of eigenvalue λ_i.*

The definition of geometric multiplicity should make sense. Since the definition of an eigenvector is a nonzero vector, $\hat{\xi}$ satisfying

$$(A-\lambda I)\,\hat{\xi} = 0,$$

and the null space of a matrix is simply all the vectors that, when multiplied into the matrix produce the zero vector, the number of linearly independent vectors that produce the zero vector is simply the dimension of the null space.

First we will consider a matrix with distinct eigenvalues to illustrate the concept of the dimension of the null space of $(A - \lambda I)$ being the number of linearly independent eigenvectors associated with an eigenvalue as well as the simple procedural aspect of computing it.

3.7.6 Example Determine all the linearly independent eigenvectors of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}.$$

The characteristic equation is

$$\begin{vmatrix} (1-\lambda) & 0 & 1 \\ 0 & (1-\lambda) & 1 \\ 0 & -2 & (4-\lambda) \end{vmatrix} = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0,$$

so the eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3. \end{aligned}$$

Since the eigenvalues are distinct, by Theorem 3.5.1, each should have one linearly independent eigenvector associated with it and $\dim\left(\mathcal{N}\left(A - \lambda_i I\right)\right) = 1$ for each λ_i.

In detail, for $\lambda_1 = 1$ the associated eigenvalue satisfies

$$\left(A - \lambda_1 I\right)\hat{\xi}_1 = \left(A - I\right)\hat{\xi}_1 = 0.$$

Recall, to solve a set of linear equations

$$Ax = b,$$

where $A \in \mathbb{R}^{n \times n}$, $b, x \in \mathbb{R}^n$ where A and b are given and x is to be determined, one approach is to construct the augmented matrix

$$\left[\begin{array}{c|c} A & b \end{array}\right]$$

and use row reduction operations to convert the left part of the augmented matrix to a convenient form (typically either the identity or a triangular form). Somewhat arbitrarily, we will use upper triangular form.

Hence, in this example, the augmented matrix is

$$\left[\begin{array}{ccc|c} 1-\lambda & 0 & 1 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & -2 & 4-\lambda & 0 \end{array}\right]. \tag{3.13}$$

Substituting $\lambda_1 = 1$ and making a couple elementary row manipulations yields

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right] \iff \left[\begin{array}{ccc|c} 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \iff \left[\begin{array}{ccc|c} 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The last augmented matrix has one row of zeros, indicating that the dimension of its null space is one, so there is one linearly independent eigenvector associated with $\lambda_1 = 1$.. From the second row, the third component of $\hat{\xi}_1$ clearly must be zero. Using this fact and noting the first row indicates that the second component must also be zero. Finally, the first component of $\hat{\xi}_1$ is clearly arbitrary. Thus, the eigenvector must be

$$\hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Similarly, substituting $\lambda_2 = 2$ into Equation 3.13 gives

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \iff \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Picking the third component of $\hat{\xi}_2$ to be one, we have

$$\hat{\xi}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Finally, for $\lambda_3 = 3$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \iff \left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This time picking the third component of $\hat{\xi}_3$ to be 2 gives

$$\hat{\xi}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

□ ■

Now consider an example with repeated eigenvalues.

3.7.7 Example Determine the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -4 & 5 & 1 \\ -5 & 1 & 5 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^3 - 10\lambda^2 + 32\lambda - 32 = 0,$$

so the eigenvalues are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 4\\ \lambda_3 &= 4.\end{aligned}$$

For $\lambda_1 = 2$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0\\ -4 & 3 & 1 & 0\\ -4 & 1 & 3 & 0\end{array}\right] \iff \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\end{array}\right] \iff \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

Since there is one row of zeros, there is one linearly independent eigenvalue associated with $\lambda_1 = 2$, which is expected since it is not repeated. Picking the third component of $\hat{\xi}_1$ to be one,

$$\hat{\xi}_1 = \begin{bmatrix} 1\\ 1\\ 1\end{bmatrix}.$$

Now, for $\lambda_2 = 4$

$$\left[\begin{array}{ccc|c} -4 & 1 & 1 & 0\\ -4 & 1 & 1 & 0\\ -4 & 1 & 1 & 0\end{array}\right] \iff \left[\begin{array}{ccc|c} -4 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right].$$

Since there are two rows of zeros, there are two linearly independent eigenvectors associated with $\lambda_2 = 4$. Picking the third component of $\hat{\xi}_2$ to be 4 and the second component to be zero, we have

$$\hat{\xi}_2 = \begin{bmatrix} 1\\ 0\\ 4\end{bmatrix}.$$

Since there are two rows of zeros, we can find another solution to the equations. To determine one, we pick another combination of variables with the only restriction that it cannot be a scaled version of two of the components of $\hat{\xi}_2$. Picking the third component to be zero and the second component to be 4 gives

$$\hat{\xi}_3 = \begin{bmatrix} 1\\ 4\\ 0\end{bmatrix}.$$

The fact that there were two rows of zeros in upper triangular form of the augmented matrix indicates that the dimension of the null space of $(A - 4I)$ was two. Thus, we were able to determine two linearly independent eigenvectors associated with the repeated eigenvalue. $\square$ $\blacksquare$

Finally, just to complete the picture, the following is an example of an eigenvalue with algebraic multiplicity two but a geometric multiplicity of one.

3.7.8 Example Returning to the matrix from Example 3.7.1 with

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

we computed previously that $\lambda = 2$ was the only eigenvalue and that it had an algebraic multiplicity of two. Constructing the augmented matrix for $A - 2I$ gives

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is one row of zeros, the geometric multiplicity is one. Clearly the first component of the eigenvector is arbitrary and the second component must be zero. Thus, for example

$$\hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\square$ $\blacksquare$

Finally, after this rather extensive detour into the realm of the nature of repeated eigenvalues and the computational details of computing the associated eigenvectors, we return to the main task at hand which is to solve $\dot{\xi} = A\xi$.

3.7.2 Homogeneous Solutions with Repeated Eigenvalues

Equal Algebraic and Geometric Multiplicities

This is the case for which to hope because the solution technique is identical to the case of distinct eigenvalues. Even if there are repeated eigenvalues, the general solution is simply

$$\xi(t) = c_1\hat{\xi}_1 e^{\lambda_1 t} + c_2\hat{\xi}_2 e^{\lambda_2 t} + \cdots + c_n\hat{\xi}_n e^{\lambda_n t}.$$

This is, in fact, the general solution. Since the set of eigenvectors is linearly independent, it will always be possible to solve for the coefficients for a specified initial condition regardless of the fact that some of the eigenvalues are repeated.

Repeated Complex Eigenvalues

The statement immediately preceding this is still correct, even if there are complex conjugate eigenvalues and even if some of the repeated eigenvalues are complex conjugates. In the first case where the repeated eigenvalues are real, the more convenient form of the solution will be to simply convert the complex conjugate eigenvalue and eigenvector pairs to the real and imaginary components and express the two homogeneous solutions corresponding to the complex conjugate pair in terms of the real functions given in Equation 3.12.

Algebraic Multiplicity Greater than the Geometric Multiplicity

The case where the geometric multiplicity of an eigenvalue is less than its algebraic multiplicity is much more interesting, but unfortunately, requires a bit more work. In this case, if we simply compute eigenvectors, we will have a set of homogeneous solutions of the form

$$\xi_h(t) = \hat{\xi}_i e^{\lambda_i t},$$

but we will not have n linearly independent eigenvalues, so the partial general solution will be of the form

$$\xi(t) = c_1\hat{\xi}_1 e^{\lambda_1 t} + c_2\hat{\xi}_2 e^{\lambda_2 t} + \cdots + c_m\hat{\xi}_m e^{\lambda_m t},$$

where $m < n$. In this case, it will not be possible to compute coefficients, c_i to satisfy any set of initial conditions since there is not a full set of linearly independent eigenvectors.

Recall from Chapter 2 that in the case of repeated roots, the approach was to multiply the one homogeneous solution by the independent variable, t and add it to the first solution. The following two examples illustrate that fact, but also then goes to make a connection to the matrix approach that is the subject of this chapter.

3.7.9 Example Find the general solution to

$$\ddot{x} + 4\dot{x} + 4x = 0. \tag{3.14}$$

Assuming $x(t) = e^{\lambda t}$ and substituting gives

$$\begin{aligned} \lambda^2 + 4\lambda + 4 &= 0 \qquad (3.15)\\ (\lambda + 2)^2 &= 0. \end{aligned}$$

So, $\lambda = 2$ is the solution. Hence, $x_h(t) = e^{-2t}$ is a homogeneous solution. Since there is no other root to the characteristic equation, the approach (which was fully detailed in Chapter 2) is to assume a second homogeneous solution of the form $x_h(t) = te^{-2t}$. The fact that this a second homogeneous solution can be verified by substituting it into Equation 3.14 and the fact that it is linearly independent can be verified by computing the Wronskian. Thus the general solution to Equation 3.14 is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}. \tag{3.16}$$

□ ■

3.7.10 Example Consider the same equation as in Equation 3.14, but first convert it into a system of two first order equations. The equivalent system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} x \qquad \text{where} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Computing the eigenvalues for the matrix in the preceding equation gives

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0.$$

It is no coincidence that the characteristic equation for the eigenvalue problem is exactly the same as Equation 3.15. Thus, the only distinction is one of nomenclature: there are "repeated eigenvalues" instead of "repeated roots." Now computing the eigenvectors corresponding to $\lambda_1 = -2$ gives

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right] \iff \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus, there is one linearly independent eigenvector,

$$\hat{\xi}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The goal is to obviously construct a solution that is equivalent to the general solution in Equation 3.16. Differentiating Equation 3.16 gives

$$\dot{x}(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t},$$

or in vector form

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t} \right) \\ &= c_1 \hat{\xi}_1^1 e^{\lambda_1 t} + c_2 \left(\hat{\xi}_1^2 e^{\lambda_1 t} + t \hat{\xi}_1^1 e^{\lambda_1 t} \right). \end{aligned}$$

Clearly, in the notation of the last line in the above equation, $\hat{\xi}_1^1$ is simply the eigenvector that we already computed. The question is how to compute the other vector, $\hat{\xi}_1^2$. Note that the superscripts for the $\hat{\xi}$'s are indices, not powers.

Recall, that the whole business regarding eigenvalues and eigenvectors came about by simply *assuming* solutions of the form $\xi(t) = \hat{\xi} e^{\lambda t}$. Substituting this into $\dot{\xi} = A\xi$ then indicated that $\hat{\xi}$ had to be an eigenvector and λ had to be an eigenvalue. The approach now is pretty obvious: substitute the assumed form of the second homogeneous solution

$$\xi_h(t) = \left(\hat{\xi}_1^2 + t\hat{\xi}_1^1 \right) e^{\lambda t}$$

to verify first, that $\hat{\xi}_1^1$ indeed satisfies the eigenvector equation (so that the fact that they are the same in this example is not a coincidence) and second, to determine what sort of equation $\hat{\xi}_1^2$ must satisfy. Differentiating and substituting gives

$$\lambda \left(\hat{\xi}_1^2 + t\hat{\xi}_1^1 \right) e^{\lambda t} + \hat{\xi}_1^1 e^{\lambda t} = A \left(\hat{\xi}_1^2 + t\hat{\xi}_1^1 \right) e^{\lambda t}.$$

Since this must hold for all t, the coefficients of the different powers of t must be equal. Therefore, collecting terms multiplying the same powers of t gives

$$\begin{aligned} t^0 &: \lambda\left(\hat{\xi}_1^2 + \hat{\xi}_1^1\right) e^{\lambda t} = A\hat{\xi}_1^2 e^{\lambda t} \\ t^1 &: \lambda\hat{\xi}_1^1 e^{\lambda t} = A\hat{\xi}_1^1 e^{\lambda t}. \end{aligned}$$

Since $e^{\lambda t}$ is never zero we have the following two equations

$$\begin{aligned} (A - \lambda I)\,\hat{\xi}_1^1 &= 0 \\ (A - \lambda I)\,\hat{\xi}_1^2 &= \hat{\xi}_1^1. \end{aligned}$$

The first equation has already been solved, so

$$\hat{\xi}_1^1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

For the second equation we have

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right] \iff \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

Clearly, as with eigenvectors, the solution is determined only up to an arbitrary scaling constant. In this case, clearly, the vector

$$\hat{\xi}_1^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

satisfies the equation for $\hat{\xi}_1^2$. ■

The task now is to generalize the approach used in the above example to systems of n equations where the multiplicity of a repeated eigenvalue may be greater than 2.

Now consider the general case of

$$\dot{\xi} = A\xi \qquad A \in \mathbb{R}^{n\times n},$$

and assume that the algebraic multiplicity of eigenvalue λ_i is m but that the geometric multiplicity is less than m. Motivated by the above example, clearly the approach is to multiply exponential solutions by t to obtain additional linearly independent solutions. In the example, since the system was second order, the highest power of t in the general solution was 1; however, in the case where the algebraic multiplicity is greater than 2, additional powers of t may be necessary. Therefore, let us propose the following sequence of homogeneous solutions corresponding to eigenvalue λ_i with algebraic multiplicity m

$$\begin{aligned}
\xi_{h_1}(t) &= \hat{\xi}_i^1 e^{\lambda_i t} \\
\xi_{h_2}(t) &= \hat{\xi}_i^2 e^{\lambda_i t} + t\hat{\xi}_i^1 e^{\lambda_i t} \\
\xi_{h_3}(t) &= \hat{\xi}_i^3 e^{\lambda_i t} + t\hat{\xi}_i^2 e^{\lambda_i t} + \frac{t^2}{2}\hat{\xi}_i^1 e^{\lambda_i t} \\
&\vdots \\
\xi_{h_m}(t) &= \hat{\xi}_i^m e^{\lambda_i t} + t\hat{\xi}_i^{m-1} e^{\lambda_i t} + \frac{t^2}{2}\hat{\xi}_i^{m-2} e^{\lambda_i t} + \cdots + \frac{t^{m-1}}{m-1}\hat{\xi}_i^1 e^{\lambda_i t}.
\end{aligned}$$

Differentiating the $\xi_{h_1}(t)$ and substituting into $\dot{\xi} = A\xi$ gives the expected equation

$$\lambda_i \hat{\xi}_i^1 = A\hat{\xi}_i^1.$$

Differentiating the second equation and substituting into $\dot{\xi} = A\xi$ gives

$$\lambda_i \hat{\xi}_i^2 e^{\lambda_i t} + \hat{\xi}_i^1 e^{\lambda_i t} + \lambda_i t\hat{\xi}_i^1 e^{\lambda_i t} = A\left(\hat{\xi}_i^2 e^{\lambda_i t} + t\hat{\xi}_i^1 e^{\lambda_i t}\right).$$

Since this must be true for all t, the coefficients for each power of t must be equal, so

$$\begin{aligned}
t^0 &: \lambda_i \hat{\xi}_i^2 + \hat{\xi}_i^1 = A\hat{\xi}_i^2 \\
t^1 &: \lambda_i \hat{\xi}_i^1 = A\hat{\xi}_i^1,
\end{aligned}$$

where the term $e^{\lambda_i t}$ has been canceled since it is never equal to zero. Proceeding in this manner and differentiating the mth proposed solution gives

$$\begin{aligned}
\dot{\xi}_{h_m}(t) &= \lambda_i \hat{\xi}^m e^{\lambda_i t} + \hat{\xi}_i^{m-1} e^{\lambda_i t} + \lambda_i t\hat{\xi}_i^{m-1} + t\hat{\xi}_i^{m-2} e^{\lambda_i t}\lambda_i t^2\hat{\xi}_i^{m-2} e^{\lambda_i t} + \\
&\quad \cdots + t^{m-2}\hat{\xi}_i^1 e^{\lambda_i t} + \lambda_i t^{m-1}\hat{\xi}_i^t e^{\lambda_i t}. \qquad (3.17)
\end{aligned}$$

Also,

$$A\xi_{h_m}(t) = A\left(\hat{\xi}_i^m e^{\lambda_i t} + t\hat{\xi}_i^{m-1} e^{\lambda_i t} + t^2\hat{\xi}_i^{m-2} e^{\lambda_i t} + \cdots + t^{m-1}\hat{\xi}_i^1 e^{\lambda_i t}\right) \qquad (3.18)$$

Since $e^{\lambda_i t}$ is never zero it can be canceled from both equations and since $\dot{\xi} = At$ must hold for all t, each the terms for each power of t in Equations 3.17 and 3.18, which gives

$$\begin{aligned}
t^0 &: \lambda_i \hat{\xi}_i^m + \hat{\xi}_i^{m-1} = A\hat{\xi}_i^m \\
t^1 &: \lambda_i \hat{\xi}_i^{m-1} + \hat{\xi}_i^{m-2} = A\hat{\xi}_i^{m-1} \\
t^2 &: \lambda_i \hat{\xi}_i^{m-2} + \hat{\xi}_i^{m-2} = A\hat{\xi}_i^{m-2} \\
\vdots &\quad \vdots \\
t^{m-1} &: \lambda_i \hat{\xi}_i^1 = A\hat{\xi}_i^1.
\end{aligned}$$

Note that in every case, the following sequence of equations is obtained

$$\begin{aligned}(A-\lambda_i I)\,\hat{\xi}_i^1 &= 0 \\ (A-\lambda_i I)\,\hat{\xi}_i^2 &= \hat{\xi}_1^1 \\ (A-\lambda_i I)\,\hat{\xi}_i^3 &= \hat{\xi}_1^2 \\ &\vdots \\ (A-\lambda_i I)\,\hat{\xi}_i^m &= \hat{\xi}_1^{m-1}\end{aligned} \tag{3.19}$$

The first equation is simply the equation for a regular eigenvalue. The vectors $\hat{\xi}_i^2$ through $\hat{\xi}_i^m$ are called *generalized eigenvectors* and are determined by sequentially solving the second through mth equations.

Note that if the second line of Equation 3.19 is multiplied on the left by $(A-\lambda_i I)$ then

$$(A-\lambda_i I)\,(A-\lambda_i I)\,\hat{\xi}_i^2 = (A-\lambda_i I)\,\hat{\xi}_i^1,$$

but since

$$(A-\lambda_i I)\,\hat{\xi}_i^1 = 0$$

then

$$(A-\lambda_i I)^2\,\hat{\xi}_i^2 = 0.$$

Similarly, multiplying the jth line in Equation 3.19 by $(A-\lambda_i I)^j$ where $1 < j < m$ gives

$$(A-\lambda_i I)^j\,\hat{\xi}_i^j = 0.$$

Further note that

$$(A-\lambda_i I)^m\,\hat{\xi}_i^j = (A-\lambda_i I)^{m-j}\,(A-\lambda_i I)^j\,\hat{\xi}_i^j = 0.$$

Hence, *all* the eigenvectors and generalized eigenvectors associated with λ_i are in the null space of $(A-\lambda_i I)^m$, which motivates the following definition.

Definition 3.7.11 (Generalized Eigenspace) *The null space of $(A-\lambda_i I)^m$ is the generalized eigenspace of A associated with λ_i.*

The following theorem assures us that the dimension of the generalized eigenspace associated with λ_i is the same as the algebraic multiplicity of λ_i. This fact is necessary in order to ensure that enough generalized eigenvectors exist to generate a full set of linearly independent homogeneous solutions to construct a general solution.

Theorem 3.7.12 *The dimension of the generalized eigenspace of A associated with λ_i is equal to the algebraic multiplicity of the eigenvalue λ_i, i.e., if the algebraic multiplicity of the eigenvalue λ_i is m, then*

$$\dim\left(\mathcal{N}\,(A-\lambda_i I)^m\right) = m.$$

PROOF The reader is referred to [1] and [2]. □

The following theorem gives the form of the homogeneous solution for *any* vector in generalized eigenspace of λ_i.

Theorem 3.7.13 *For $A \in \mathbb{R}^{n\times n}$ and λ_i an eigenvector of A with algebraic multiplicity m, if*

$$(A - \lambda_i I)^m \hat{\xi} = 0,$$

then

$$\xi_h(t) = \left(\hat{\xi}_i + t\,(A-\lambda_i I)\,\hat{\xi}_i + \frac{t^2}{2}\,(A-\lambda_i I)^2\,\hat{\xi}_i + \cdots + \frac{t^{m-1}}{m-1}\,(A-\lambda_i I)^{m-1}\,\hat{\xi}_i\right) e^{\lambda_i t}$$

satisfies

$$\dot{\xi} = A\xi.$$

Proof This is by direct computation. Simply differentiate $\xi_h(t)$ and substitute into $\dot{\xi} = A\xi$. Differentiating $\xi_h(t)$ gives

$$\begin{aligned} \dot{\xi}_h(t) \quad &= \quad \lambda_i \left(\hat{\xi}_i + t\,(A-\lambda_i I)\,\hat{\xi}_i + \frac{t^2}{2}\,(A-\lambda_i I)^2\,\hat{\xi}_i + \cdots + \frac{t^{m-1}}{m-1}\,(A-\lambda_i I)\,\hat{\xi}_i\right) e^{\lambda_i t} + \\ & \qquad \left((A-\lambda_i I)\,\hat{\xi}_i + t\,(A-\lambda_i I)^2\,\hat{\xi} + \cdots + \frac{t^{m-2}}{m-1}\,(A-\lambda_i I)\,\hat{\xi}_i\right) e^{\lambda_i t}. \end{aligned}$$

Equating $\dot{\xi}_h(t)$ with $A\xi_h(t)$ and equating powers of t gives

$$\begin{aligned} \lambda_i\hat{\xi}_i + (A-\lambda_i I)\,\hat{\xi}_i \quad &= \quad A\hat{\xi}_i \\ \lambda_i\,(A-\lambda_i I)\,\hat{\xi}_i + (A-\lambda_i I)^2\,\hat{\xi}_i \quad &= \quad A\,(A-\lambda_i I)\,\hat{\xi}_i \end{aligned}$$

□

So, finally we have the following solution technique for $\dot{\xi} = A\xi$, for $A \in \mathbb{R}^{n\times n}$ where λ_i has an algebraic multiplicity of m.

1. For the nonrepeat ed eigenvalues, λ_j, the corresponding homogeneous solution is $\xi_h(t) = \hat{\xi}_j e^{\lambda_j t}$. If two of these eigenvalues are a complex conjugate pair, then converting the homogeneous solution to sines and cosines as outlined in Section 3.6 is preferable.
2. For each repeated λ_i
   (a) Determine the smallest power, p, such that

   $$\dim \mathcal{N}\,(A-\lambda_i I)^p = m.$$

   (b) Find all m $\hat{\xi}_i$ in the generalized eigenspace of λ_i, *i.e.*,

   $$(A-\lambda_i I)^p\,\hat{\xi} = 0.$$

   These $\hat{\xi}_i$ may be regular eigenvectors, generalized eigenvectors or linear combinations thereof.

(c) The homogeneous solution corresponding to each $\hat{\xi}_i$ is

$$\xi_h(t) = \left(\hat{\xi}_i + t\,(A - \lambda_i I)\,\hat{\xi}_i + \frac{t}{2}\,(A - \lambda_i I)^2\,\hat{\xi} + \cdots + \frac{t^{m-1}}{m-1}\,(A - \lambda_i I)\,\hat{\xi}_i\right) e^{\lambda_i t}.$$

A few examples will help illustrate the approach.

3.7.14 Example Determine the general solution to $\dot{\xi} = A\xi$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Since the matrix is triangular, the eigenvalues are the values along the diagonal. Thus

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 2 \\ \lambda_3 &= 2 \\ \lambda_4 &= 2. \end{aligned}$$

Thus, $\lambda = 2$ is an eigenvalue with algebraic multiplicity of 4. For $\lambda_1 = 1$, the eigenvector is

$$(A - \lambda_1 I)\,\hat{\xi}_1 = 0 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \quad \Longleftrightarrow \quad \hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_2 = \lambda_3 = \lambda_4 = 2$

$$(A - \lambda_1 I)\,\hat{\xi}_2 = 0 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is only one row of zeros, the dimension of the null space of $(A - \lambda_2 I)$ is one, and hence there is only one regular eigenvector, and hence we must also compute two generalized eigenvectors. Clearly

$$\hat{\xi}_2 = \hat{\xi}_2^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

To compute the generalized eigenvectors

$$(A - \lambda_2 I)\,\hat{\xi}_2^2 = \hat{\xi}_2^1 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Longleftrightarrow \quad \hat{\xi}_2^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

and

$$(A - \lambda_2 I)\,\hat{\xi}_2^3 = \hat{\xi}_2^2 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Longleftrightarrow \quad \hat{\xi}_2^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the general solution is

$$\begin{aligned} \xi(t) &= c_1\hat{\xi}_1 e^{\lambda_1 t} + c_2\hat{\xi}_2 e^{\lambda_2 t} + c_3\left(\hat{\xi}_2^2 + t\hat{\xi}_2^1\right)e^{\lambda_2 t} + c_4\left(\hat{\xi}_2^3 + t\hat{\xi}_2^2 + \frac{t^2}{2}\hat{\xi}_2^1\right)e^{\lambda_2 t} \\ &= c_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^t + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) e^{2t} + \\ &\quad c_4\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) e^{2t}. \end{aligned}$$

That this is a solution may be verified by directly substituting this into the original differential equation. ■

The previous example illustrates that generalized eigenvectors are also not unique. Note that in both $\hat{\xi}_2^2$ and $\hat{\xi}_2^3$ the second component could be *any real number.* However, since this would simply be adding a scale multiple of $\hat{\xi}_2 = \hat{\xi}_2^1$ to the generalized eigenvector, the impact on the solution to the differential equation would only be to alter the values of the coefficients, c_i if some initial conditions were specified.

The following is an example of a 4×4 system where there are two eigenvalues with algebraic multiplicity of two, but for one of them there are two linearly independent eigenvectors and for the other there is only one, so a generalized eigenvector must be computed.

3.7.15 Example Determine the general solution to $\dot{\xi} = A\xi$ where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Since the matrix is triangular, the eigenvalues are the values along the diagonal. Thus

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 2 \\ \lambda_3 &= 3 \\ \lambda_4 &= 3. \end{aligned}$$

For $\lambda_1 = \lambda_2 = 2$

$$(A - \lambda_1 I)\,\hat{\xi}_1 = 0 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Since there are two rows of zeros, the dimension of the null space of $(A - \lambda_1 I)$ is two, and hence there are two regular eigenvectors. Thus the geometric multiplicity is equal to the algebraic multiplicity, and hence it suffices to compute the two regular eigenvectors, which are

$$\hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \hat{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda_3 = \lambda_4 = 3$

$$(A - \lambda_3 I)\,\hat{\xi}_1 = 0 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is only one row of zeros, the dimension of the null space of $(A - \lambda_3 I)$ is one, and hence there is only one regular eigenvector. In particular,

$$\hat{\xi}_3 = \hat{\xi}_3^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Computing the generalized eigenvector

$$(A - \lambda_3 I)\,\hat{\xi}_3^2 = \hat{\xi}_3^1 \quad \Longleftrightarrow \quad \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Longleftrightarrow \quad \hat{\xi}_3^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence a general solution is

$$\begin{aligned} \xi(t) &= c_1\hat{\xi}_1 e^{\lambda_1 t} + c_2\hat{\xi}_2 e^{\lambda_1 t} + c_3\hat{\xi}_3^1 e^{\lambda_3 t} + c_4\left(\hat{\xi}_3^2 + t\hat{\xi}_3^1\right)e^{\lambda_3 t} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} + \\ &\quad c_4 \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) e^{3t}. \end{aligned}$$

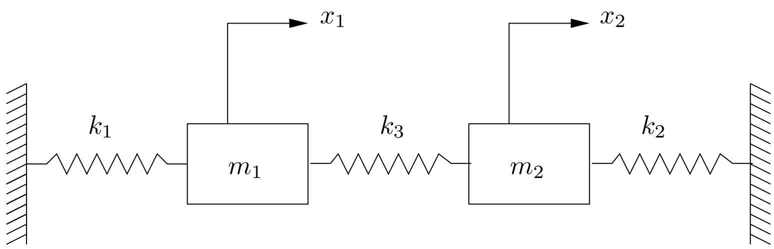

Figure 3.3. Two degree of freedom mass-spring-damper system.

Again, as in the previous example, the generalized eigenvector can only be uniquely determined up to the addition of a scale multiple of its associated regular eigenvectors. ■

3.8 Diagonalization and Jordan Normal Form

3.9 Applications of Homogeneous Systems of First Order Equations

3.9.1 Classical Normal Modes of Vibration

Consider the system illustrated in Figure 6.1. We will first analyze this system using the approach from classical vibrations theory and then relate it to the material covered previously in this chapter.

A simple analysis of the free body diagrams for the two masses yields the following equations of motion

$$\begin{aligned} m\ddot{x}_1 + (k_1 + k_3)\, x_1 - k_3 x_2 &= 0 \\ m\ddot{x}_2 + (k_2 + k_3)\, x_2 - k_3 x_2 &= 0. \end{aligned} \tag{3.20}$$

Classical Approach

The classical approach is simply to assume (perhaps based upon some intuitive insight into the problem) the form of the solutions for masses one and two. For present purposes, assume

$$\begin{aligned} x_1(t) &= a_1 \cos \omega t \\ x_2(t) &= a_2 \cos \omega t. \end{aligned}$$

Note the assume form of the solution is very restrictive; in particular, it will at best only be valid when $\dot{x}_1(0) = \dot{x}_2(0) = 0$; furthermore, it assumes the frequency of oscillation of the two masses must be the same. Regardless, let us proceed to substitute these solutions into the equations of motion. Upon doing so we obtain

$$\begin{aligned}\left[-m_1 a_1 \omega^2 + (k_1 + k_3)\, a_1 - k_3 a_2\right] \sin \omega t &= 0 \\ \left[-m_2 a_2 \omega^2 + (k_2 + k_3)\, a_2 - k_3 a_1\right] \sin \omega t &= 0.\end{aligned}$$

Since this must be true for all t, the terms in brackets must be zero, which gives

$$\begin{aligned}\frac{a_1}{a_2} &= \frac{-k_3}{m_1 \omega^2 - k_1 - k_3} \\ \frac{a_1}{a_2} &= \frac{m_2 \omega^2 - k_2 - k_3}{-k_3}.\end{aligned}$$

Since these must be equal

$$\frac{-k_3}{m_1 \omega^2 - k_1 - k_3} = \frac{m_2 \omega^2 - k_2 - k_3}{-k_3},$$

which gives

$$\omega^4 + \left(\frac{k_1 + k_3}{m_1} + \frac{k_2 + k_3}{m_2}\right) \omega^2 + \frac{k_1 k_2 + k_2 k_3 + k_1 k_3}{m_1 m_2} = 0.$$

Note this is a quartic equation in ω but due to the absence of the odd powers of ω it may be considered a quadratic equation in ω^2. Although it is not necessary, to simplify things a bit, assume

$$\begin{aligned}k_1 = k_2 &= k \\ m_1 = m_2 &= m.\end{aligned} \tag{3.21}$$

Using these values

$$\begin{aligned}\frac{a_1}{a_2} &= \frac{-k_3}{m \omega^2 - k - k_3} \\ \frac{a_1}{a_2} &= \frac{m \omega^2 - k - k_3}{-k_3},\end{aligned} \tag{3.22}$$

and

$$\omega^4 + \left(2 \frac{k + k_3}{m}\right) \omega^2 + \frac{k\,(k + 2k_3)}{m^2} = 0.$$

This has roots

$$\omega^2 = \frac{k + k_3}{m} \pm \sqrt{\left(\frac{k + k_3}{m}\right)^2 - \frac{k\,(k + 2k_3)}{m^2}},$$

so

$$\begin{aligned}\omega^2 &= \frac{k}{m} \quad \text{or} \\ &= \frac{k + 2k_3}{m}.\end{aligned}$$

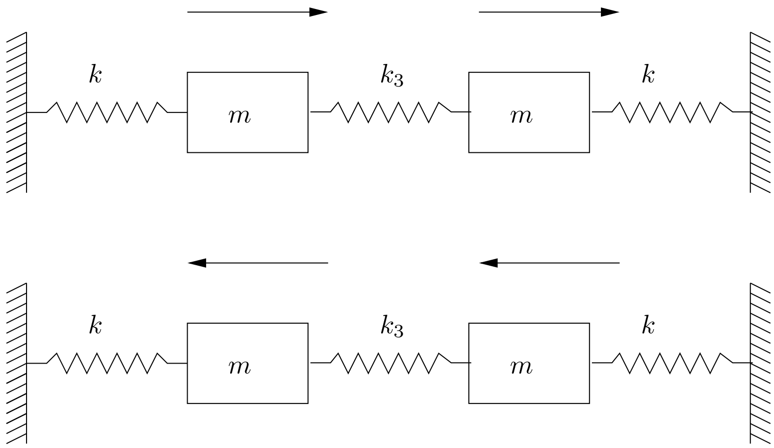

Figure 3.4. Mode one oscillations.

Substituting these values into Equation 3.22 gives

$$\begin{aligned} \frac{a_1}{a_2} &= 1 \\ \frac{a_1}{a_2} &= -1, \end{aligned}$$

for each of the two values of ω^2 respectively.

The interpretation of these two pairs of values for ω^2 and $\frac{a_1}{a_2}$ is straightforward. Considering

$$\begin{aligned} \omega^2 &= \frac{k}{m} \\ \frac{a_1}{a_2} &= 1 \end{aligned}$$

the two solutions are

$$\begin{aligned} x_1(t) &= a\cos\sqrt{\frac{k}{m}}t \\ x_2(t) &= a\cos\sqrt{\frac{k}{m}}t \end{aligned}$$

where $a_1 = a_2 = a$. Thus, the two masses move with the same frequency, in the same direction with the same magnitude of oscillation, as is schematically illustrated in Figure 3.4.

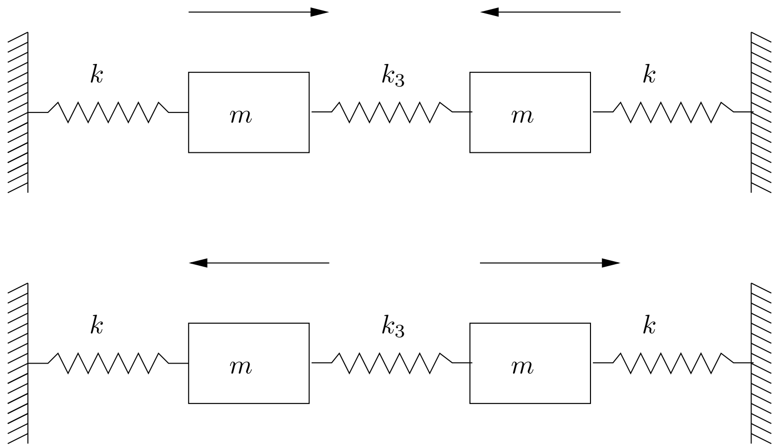

Figure 3.5. Mode two oscillations.

A similarly straight-forward analysis for the second solution shows that

$$\begin{aligned} x_1(t) &= a\cos\sqrt{\frac{k+2k_3}{m}}t \\ x_2(t) &= -a\cos\sqrt{\frac{k+2k_3}{m}}t, \end{aligned}$$

where the masses move in opposite directions, as is illustrated in Figure 3.5.

Since the system is linear, the principle of superposition applies; hence, any solution starting with zero initial velocities may be written as a combination of the two modes of oscillation

$$\begin{aligned} x_1(t) &= a\cos\sqrt{k}mt + b\cos\sqrt{k+2k_3}mt \\ x_2(t) &= a\cos\sqrt{k}mt - b\cos\sqrt{k+2k_3}mt. \end{aligned}$$

A similarly straightforward analysis starting with assumed solutions of the form

$$\begin{aligned} x_1(t) &= a_1\cos\omega t + c_1\sin\omega t \\ x_2(t) &= a_2\cos\omega t + c_2\sin\omega t \end{aligned}$$

would yield the same solutions for ω^2 and the same conditions on the relationship between the coefficients b_1 and b_2. Since the same conditions apply for b_1 and b_2, the same interpretation of the two modes applies for systems with initial velocities.

Hence any solution, including solutions with nonzero initial velocities, may be represented as

$$\begin{aligned}
x_1(t) &= a\cos\sqrt{\frac{k}{m}}t + b\cos\sqrt{\frac{k+2k_3}{m}}t + c\sin\sqrt{\frac{k}{m}}t + d\sin\sqrt{\frac{k+2k_3}{m}}t \\
x_2(t) &= a\cos\sqrt{\frac{k}{m}}t - b\cos\sqrt{\frac{k+2k_3}{m}}t + c\sin\sqrt{\frac{k}{m}}t - d\sin\sqrt{\frac{k+2k_3}{m}}t,
\end{aligned}$$

where the coefficients a, b, c and d depend upon the initial conditions.

Eigenvalue/Eigenvector Approach

Considering the equations of motion for the system illustrated in Figure 6.1, which are given by Equation 3.20, if

$$\begin{aligned}
\xi_1 &= x_1 \\
\xi_2 &= \dot{x}_1 \\
\xi_3 &= x_2 \\
\xi_4 &= \dot{x}_2,
\end{aligned}$$

and the simplifications given in Equation 3.21 hold, then

$$\dot{\xi} = \frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k+k_3}{m} & 0 & \frac{k_3}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_3}{m} & 0 & -\frac{k+k_3}{m} & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = A\xi.$$

The eigenvalues of A are determined by the cofactor expansion

$$\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -\frac{k+k_3}{m} & -\lambda & \frac{k_3}{m} & 0 \\ 0 & 0 & -\lambda & 1 \\ \frac{k_3}{m} & 0 & -\frac{k+k_3}{m} & -\lambda \end{vmatrix} \\
&= -\lambda\begin{vmatrix} -\lambda & \frac{k_3}{m} & 0 \\ 0 & -\lambda & 1 \\ 0 & -\frac{k+k_3}{m} & -\lambda \end{vmatrix} + (-1)\begin{vmatrix} -\frac{k+k_3}{m} & \frac{k_3}{m} & 0 \\ 0 & -\lambda & 1 \\ \frac{k_3}{m} & -\frac{k+k_3}{m} & -\lambda \end{vmatrix} \\
&= \lambda^4 + 2\frac{k+k_3}{m}\lambda^2 + \left(\frac{k+k_3}{m}\right)^2 - \left(\frac{k_3}{m}\right)^2 \\
&= 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= i\sqrt{\frac{k}{m}} \\
\lambda_2 &= -i\sqrt{\frac{k}{m}} \\
\lambda_3 &= i\sqrt{\frac{k+2k_3}{m}} \\
\lambda_4 &= -i\sqrt{\frac{k+2k_3}{m}}.
\end{aligned}$$

Now computing the eigenvectors gives

$$(A-\lambda_1 I)\,\hat{\xi}_1 = 0 \iff \left[\begin{array}{cccc|c} -i\sqrt{\frac{k}{m}} & 1 & 0 & 0 & 0 \\ -\frac{k+k_3}{m} & -i\sqrt{\frac{k}{m}} & \frac{k_3}{m} & 0 & 0 \\ 0 & 0 & -i\sqrt{\frac{k}{m}} & 1 & 0 \\ \frac{k_3}{m} & 0 & -\frac{k+k_3}{m} & -i\sqrt{\frac{k}{m}} & 0 \end{array}\right]$$

multiply first row by $-\frac{\frac{k+k_3}{m}}{i\sqrt{\frac{k}{m}}}$ and add to second row $\Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{\frac{k}{m}} & 1 & 0 & 0 & 0 \\ 0 & i\frac{k_3}{\sqrt{km}} & \frac{k_3}{m} & 0 & 0 \\ 0 & 0 & -i\sqrt{\frac{k}{m}} & 1 & 0 \\ \frac{k_3}{m} & 0 & -\frac{k+k_3}{m} & -i\sqrt{\frac{k}{m}} & 0 \end{array}\right]$

multiply first row by $\frac{\frac{k_3}{m}}{i\sqrt{\frac{k}{m}}}$ and add to fourth row $\Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{\frac{k}{m}} & 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\frac{k_3}{km}} & \frac{k_3}{m} & 0 & 0 \\ 0 & 0 & -i\sqrt{\frac{k}{m}} & 1 & 0 \\ 0 & -i\sqrt{\frac{k_3}{km}} & -\frac{k+k_3}{m} & -i\sqrt{\frac{k}{m}} & 0 \end{array}\right]$

add second row to fourth row $\Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{\frac{k}{m}} & 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\frac{k_3}{km}} & \frac{k_3}{m} & 0 & 0 \\ 0 & 0 & -i\sqrt{\frac{k}{m}} & 1 & 0 \\ 0 & 0 & -\frac{k}{m} & -i\sqrt{\frac{k}{m}} & 0 \end{array}\right]$

multiply third row by $-\frac{\frac{k}{m}}{i\sqrt{\frac{k}{m}}}$ and add to fourth row $\Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{\frac{k}{m}} & 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\frac{k_3}{km}} & \frac{k_3}{m} & 0 & 0 \\ 0 & 0 & -i\sqrt{\frac{k}{m}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$

Thus,

$$\hat{\xi}_1 = \begin{bmatrix} -i \\ \sqrt{\frac{k}{m}} \\ -i \\ \sqrt{\frac{k}{m}} \end{bmatrix}.$$

Similar computations show that

$$\hat{\xi}_2 = \begin{bmatrix} i \\ \sqrt{\frac{k}{m}} \\ i \\ \sqrt{\frac{k}{m}} \end{bmatrix} \qquad \hat{\xi}_3 = \begin{bmatrix} 1 \\ i\sqrt{\frac{k+2k_3}{m}} \\ -1 \\ -i\sqrt{\frac{k+2k_3}{m}} \end{bmatrix} \qquad \hat{\xi}_4 = \begin{bmatrix} 1 \\ -i\sqrt{\frac{k+2k_3}{m}} \\ -1 \\ i\sqrt{\frac{k+2k_3}{m}} \end{bmatrix}.$$

The important point of this example is two fold:

1. the eigenvalues are exactly the same as the frequencies computed using the classical method; and,

2. the eigenvectors reflect the relative magnitude conditions as well; *i.e.,* in particular

   (a) the first and third components of $\hat{\xi}_1$ and $\hat{\xi}_2$ are identical, which is a consequence of the fact that $\frac{a_1}{a_2} = 1$ in the case where the frequency is $\sqrt{\frac{k}{m}}$; and,

   (b) the first and third components of $\hat{\xi}_3$ and $\hat{\xi}_4$ have the same magnitude but opposite sign, which is a consequence of the fact that $\frac{a_1}{a_2} = -1$ in the case where the frequency is $\sqrt{\frac{k+2k_3}{m}}$.

3.9.2 Finite Element and Finite Difference Methods

3.10 Nonhomogeneous Systems of First Order Equations

Now we consider how to solve systems of the type

$$\dot{\xi} = A\xi + g(t),$$

where

$$\begin{aligned} A &\in \mathbb{R}^{n\times n} \\ \xi &\in \mathbb{R}^n \\ g(t) &\in \mathbb{R}^n, \end{aligned}$$

or in detail

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}. \tag{3.23}$$

First consider a mechanical example that gives rise to equations of this nature.

3.10.1 Example As an example of a type of system that is modeled by such a set of equations, consider again the system illustrated in Figure 3.1, but unlike before we will not assume that $F(t) = 0$. As before, if

$$\begin{aligned} \xi_1 &= x_1 \\ \xi_2 &= \dot{x}_1 \\ \xi_3 &= x_2 \\ \xi_4 &= \dot{x}_2 \end{aligned}$$

then the equations of motion given in Equation 3.1 are equivalent to

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} \\ 0 & 0 & 1 & 0 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{F(t)}{m_2} \end{bmatrix}.$$

■

The following three methods are appropriate for solving nonhomogeneous systems of first order linear ordinary differential equations.

3.10.1 Diagonalization and Jordan Canonical Form

The fundamental idea underlying this approach is to convert the system of coupled first order equations into decoupled equations. What this means mathematically will be apparent shortly, but the consequence of this approach is unlike the system in Equation 3.23 where the entire system must be solved at once, each equation (or row) can be solved individually, or one at a time. First we need to investigate the concept of converting a matrix to *diagonal* form.

For a system of the form

$$\dot{\xi} = A\xi + g(t) \tag{3.24}$$

we first consider the easier case where A has a full set of n linearly independent eigenvectors, $\hat{\xi}_1, \dots, \hat{\xi}_n$, and define the matrix T as the matrix with the eigenvectors of A as its columns, *i.e.*,

$$T = \begin{bmatrix} \hat{\xi}_1 & \hat{\xi}_2 * \cdots & \hat{\xi}_n \end{bmatrix}.$$

Since the definition of an eigenvector is

$$A\hat{\xi}_i = \lambda_i \hat{\xi}_i$$

then

$$AT = \begin{bmatrix} \lambda_1 \hat{\xi}_1 & \lambda_2 \hat{\xi}_2 & \cdots & \lambda_n \hat{\xi}_n \end{bmatrix}.$$

Now, since we assumed that $\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n$ were linearly independent, then T is invertible. Note that by definition

$$T^{-1}T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ ) & 0 & \cdots & 1 \end{bmatrix}.$$

Considering this equation column by column, we have

$$T^{-1}\hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad T^{-1}\hat{\xi}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad T^{-1}\hat{\xi}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Also, since $A\hat{\xi}_i = \lambda_i \hat{\xi}_i$

$$T^{-1}A\hat{\xi}_1 = T^{-1}\lambda_1\hat{\xi}_1 = \lambda_i T^{-1}\hat{\xi}_1 = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$T^{-1}A\hat{\xi}_2 = T^{-1}\lambda_2\hat{\xi}_2 = \lambda_i T^{-1}\hat{\xi}_2 = \lambda_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2 1 \\ \vdots \\ 0 \end{bmatrix},$$

and so forth until

$$T^{-1}A\hat{\xi}_n = T^{-1}\lambda_n\hat{\xi}_n = \lambda_i T^{-1}\hat{\xi}_n = \lambda_n \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

Finally, putting it all together gives the important relation

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Now, using this, again consider Equation 3.24 and let

$$\xi = T\psi$$

where the columns of T are the eigenvectors of A as before. Note that since T is a constant matrix,

$$\dot{\xi} = T\dot{\psi}.$$

Substituting into Equation 3.24 gives

$$T\dot{\psi} = AT\psi + g(t),$$

or

$$\dot{\psi} = T^{-1}AT\psi + T^{-1}g(t).$$

In detail, this looks like

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_n \end{bmatrix} &= \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_n \end{bmatrix} + T^{-1}\begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \\ \vdots \\ g_n(t) \end{bmatrix} \qquad (3.25) \\
&= \begin{bmatrix} \lambda_1\psi_1 \\ \lambda_2\psi_2 \\ \lambda_3\psi_3 \\ \vdots \\ \lambda_n\psi_n \end{bmatrix} + T^{-1}\begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \\ \vdots \\ g_n(t) \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1\psi_1 \\ \lambda_2\psi_2 \\ \lambda_3\psi_3 \\ \vdots \\ \lambda_n\psi_n \end{bmatrix} + \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \\ \vdots \\ h_n(t) \end{bmatrix}
\end{aligned}$$

where

$$h(t) = T^{-1}g(t).$$

The significance of Equation 3.25 is that each of the ψ_i equations are *decoupled* and in the form of

$$\dot{\psi}_i = \lambda_i\psi + h_i(t).$$

Hence, each can be solved independently using the appropriate method from Chapter 1. For example, using an integrating factor

$$\begin{aligned}
\frac{d}{dt}\psi_i - \lambda_i\psi_i &= h_i(t) \\
e^{-\lambda_i t}\left(\frac{d}{dt}\psi_i - \lambda_i\psi_i\right) &= e^{-\lambda_i t}h_i(t) \\
\frac{d}{dt}\left(e^{-\lambda_i t}\psi_i\right) &= e^{-\lambda_i t}h_i(t).
\end{aligned}$$

Hence, integrating both sides gives

$$\begin{aligned}\int_0^t \frac{d}{d\tau}\left(e^{-\lambda_i \tau}\psi_i(\tau)\right) &= \int_0^t e^{-\lambda_i\tau}h_i(\tau)d\tau \\ e^{-\lambda_i t}\psi_i(t) - \psi_i(0) &= \int_0^t e^{-\lambda_i\tau}h_i(\tau)d\tau.\end{aligned}$$

Hence

$$\psi_i(t) = e^{\lambda_i t}\int_0^t e^{-\lambda_i\tau}h(\tau)d\tau + \psi_i(0)e^{\lambda_i t},$$

if the initial condition is specified or

$$\psi_i(t) = e^{\lambda_i t}\int_0^t e^{-\lambda_i\tau}h(\tau)d\tau + ce^{\lambda_i t},$$

if the general solution is desired.

After solving all the $\psi_i(t)$ equations, the solution for the ξ variables is simply computed using the original equation

$$\xi = T\psi.$$

3.10.2 Example Determine the general solution to

$$\frac{d}{dt}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ -8 & -5 & -3\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \cos t\end{bmatrix}.$$

Computing the eigenvalues and eigenvectors gives

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -1\\ \lambda_3 &= 2\end{aligned}$$

and

$$\hat{\xi}_1 = \begin{bmatrix}-4\\ 5\\ 7\end{bmatrix}, \quad \hat{\xi}_2 = \begin{bmatrix}-3\\ 4\\ 2\end{bmatrix}, \quad \hat{\xi}_3 = \begin{bmatrix}0\\ -1\\ 1\end{bmatrix}.$$

Thus

$$T = \begin{bmatrix}-4 & -3 & 0\\ 5 & 4 & -1\\ 7 & 2 & 1\end{bmatrix}$$

and

$$T^{-1} = \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\\ -1 & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{3}{2} & -\frac{13}{12} & -\frac{1}{12}\end{bmatrix}.$$

Computing $T^{-1}AT$ and $T^{-1}g(t)$ gives the following equations for ψ

$$\frac{d}{dt}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{bmatrix} + \begin{bmatrix}\frac{1}{4}\cos t\\ -\frac{1}{3}\cos t\\ -\frac{1}{12}\cos t\end{bmatrix}$$

or as individual equations

$$\begin{aligned}\dot{\psi}_1 &= -2\psi_1 + \frac{1}{4}\cos t\\ \dot{\psi}_2 &= -\psi_2 - \frac{1}{3}\cos t\\ \dot{\psi}_3 &= 2\psi_3 - \frac{1}{12}\cos t.\end{aligned}$$

The solutions to these equations are

$$\begin{aligned}\psi_1 &= e^{-2t}\int_0^t e^{2t}\frac{1}{4}\cos\tau d\tau + \psi_1(0)e^{-2t}\\ \psi_2 &= -e^{-t}\int_0^t e^{t}\frac{1}{3}\cos\tau d\tau + \psi_2(0)e^{-t}\\ \psi_3 &= -e^{2t}\int_0^t e^{-2t}\frac{1}{12}\cos\tau d\tau + \psi_3(0)e^{2t},\end{aligned}$$

or

$$\begin{aligned}\psi_1(t) &= c_1e^{-2t} + \frac{1}{10}\cos t + \frac{1}{20}\sin t\\ \psi_2(t) &= c_2e^{-t} - \frac{1}{6}\cos t - \frac{1}{6}\sin t\\ \psi_3(t) &= c_3e^{2t} + \frac{1}{30}\cos t - \frac{1}{60}\sin t.\end{aligned}$$

The final solution is computed by determining

$$\xi = T\psi,$$

which is a bit too messy to write out in detail. ∎

3.10.2 Undetermined Coefficients

Recall that the method of undetermined coefficients from Section 2.5 was based upon the fact that derivatives of functions of the form

1. $\sin\omega t$ and $\cos\omega t$,
2. $e^{\alpha t}$,
3. $\alpha_0t^n + \alpha_1t^{n-1} + \alpha_2t^{n-2} + \cdots + \alpha_{n-1}t + \alpha_n$, and

4. products and sums of them,

are exactly the same set of functions. Thus when the nonhomogeneous term contains function of this type, the particular solution of an ordinary differential equation will be a general combination of the same type of functions. There are two slight complications or variations that are necessary distinguish the approach for systems of first order equations from one scalar second order system.

General Form of Particular Solution

The first complication is that even though the nonhomogeneous term may appear in one component of the differential equation, the form of the solution must have undetermined coefficients for all of the components. In a general functional description, if the all the nonhomogeneous terms that appear in the vector $g(t)$ would require a particular solution of the form

$$x_p(t) = af_1(t) + bf_2(t) + cf_3(t) + \cdots$$

in the scalar (first or second order) case, then in the case of

$$\dot{\xi} = A\xi + g(t), \qquad \xi \in \mathbb{R}^n,$$

then the assumed form of the solution will be

$$\xi_p(t) = af_1(t) + bf_2(t) + cf_3(t) + \cdots$$

where $a, b, c, \ldots \in \mathbb{R}^n$, *i.e.,* the coefficients are *vectors.* The following example illustrates this point.

3.10.3 Example Find the general solution to

$$\frac{d}{dt}\left[\begin{array}{c} \xi_1 \\ \xi_2 \end{array}\right] = \left[\begin{array}{cc} 2 & 1 \\ 0 & 3 \end{array}\right]\left[\begin{array}{c} \xi_1 \\ xi_2 \end{array}\right] + \left[\begin{array}{c} 0 \\ \cos 4t \end{array}\right].$$

In the scalar case, the assumed form of the solution would simply be $x_p(t) = a\cos 4t + b\sin 4t$, so for this problem we assume

$$\xi_p(t) = a\cos 4t + b\sin 4t = \left[\begin{array}{c} a_1 \\ a_2 \end{array}\right]\cos 4t + \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]\sin 4t.$$

The rest of the procedure is exactly as before. Substitute the assumed form of the particular solution into the differential equations and equate the coefficients of different functions of t. Thus,

$$\dot{\xi}_p(t) = -4a\sin 4t + 4b\cos 4t,$$

and substituting gives

$$\left[\begin{array}{c} -4a_1\sin 4t + 4b_1\cos 4t \\ -4a_2\sin 4t + 4b_2\cos 4t \end{array}\right] = \left[\begin{array}{cc} 2 & 1 \\ 0 & 3 \end{array}\right]\left[\begin{array}{c} a_1\cos 4t + b_1\sin 4t \\ a_2\cos 4t + b_2\sin 4t \end{array}\right] + \left[\begin{array}{c} 0 \\ \cos 4t \end{array}\right].$$

Since this must be true for all time, the coefficients of the sine and cosine terms in each equation must be equal. Thus, the coefficients are determined by the following four equations:

$$\begin{aligned}
\text{sine term, first equation} \quad &\Longrightarrow \quad -4a_1 = 2b_1 + b_2 \\
\text{cosine term, first equation} \quad &\Longrightarrow \quad 4b_1 = 2a_1 + a_2 \\
\text{sine term, second equation} \quad &\Longrightarrow \quad -4a_2 = 3b_2 \\
\text{sine term, first equation} \quad &\Longrightarrow \quad 4b_2 = 3a_2 + 1.
\end{aligned}$$

Solving these gives

$$\begin{aligned}
a_1 &= -\frac{1}{50} \\
a_2 &= -\frac{3}{25} \\
b_1 &= -\frac{1}{25} \\
b_2 &= \frac{4}{25}.
\end{aligned}$$

Thus the particular solution is

$$\xi_p(t) = \begin{bmatrix} -\frac{1}{50} \\ -\frac{3}{25} \end{bmatrix} \cos 4t + \begin{bmatrix} -\frac{1}{25} \\ -\frac{4}{25} \end{bmatrix} \sin 4t.$$

To compute the general solution, the homogeneous solution, *i.e.,* the solution to

$$\dot{\xi} = A\xi$$

is needed. A simple computation shows that the eigenvalues and eigenvectors of A are

$$\lambda_1 = 3, \quad \hat{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 2, \quad \hat{\xi}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, the general solution is

$$\xi(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -\frac{1}{50} \\ -\frac{3}{25} \end{bmatrix} \cos 4t + \begin{bmatrix} -\frac{1}{25} \\ -\frac{4}{25} \end{bmatrix} \sin 4t. \quad \blacksquare$$

In the previous example, note that the sine and cosine terms appear in *both* components of the solution even though the nonhomogeneous term contains $\cos 4t$ only in the second term. This is due to the fact that the equations are *coupled*, and the effect of the nonhomogeneity is not limited to the line in which it appears.

Equivalent Homogeneous Solution and Nonhomogeneous Term

The second complication is when the nonhomogeneous term is the exponential of an eigenvalue of the matrix A. When confronted with this problem in Chapter 2, the approach was to multiply the assumed form of the particular solution by the dependent variable. The approach for nonhomogeneous systems of first order equations with equivalent homogeneous solutions and nonhomogeneous terms is similar, but with a slight twist, as the following examples illustrate.

The first example is the second order scalar case, which is included to help you recall the procedure from Chapter 2.

3.10.4 Example (Review problem from Chapter 2) Determine the general solution to

$$\ddot{x} + 4x = \cos 2t. \tag{3.26}$$

Assuming a homogeneous solution of the form

$$x_h(t) = e^{\lambda t}$$

and substituting gives

$$\lambda^2 + 4 = 0 \qquad \Longrightarrow \qquad \lambda = \pm 2i.$$

For the particular solution, we are first inclined to assume a solution of the form

$$x_p(t) = a\cos 2t + b\sin 2t.$$

One that is observant and experienced in dealing with undetermined coefficients will immediately recognize that this will not work since it is actually a homogeneous solution. When $x_p(t)$ of this form is substituted into Equation 3.26 it will disappear leaving nothing to equate to the nonhomogeneous term since it is actually a solution of the homogeneous equation. In detail,

$$\ddot{x}_p(t) = -4a\cos 2t - 4b\sin 2t,$$

and substituting gives

$$\begin{aligned} -4a\cos 2t - 4b\sin 2t + 4\left(a\cos 2t + b\sin 2t\right) &= \cos 2t \\ 0 &= \cos 2t. \end{aligned}$$

The 0 on the left hand side of the previous equation is *guaranteed* to occur since $x_p(t)$ happens to satisfy

$$\ddot{x} + 4x = 0.$$

Recall, that the correct form to assume for the particular solution in this case would be

$$x_p(t) = t\left(a\cos 2t + b\sin 2t\right).$$

Then,

$$\ddot{x}_p(t) = -4\left((at - b)\cos 2t + (a + bt)\sin 2t\right)$$

and substituting and equating coefficients gives

$$\begin{aligned} -4\left((at-b)\cos 2t + (a+bt)\sin 2t\right) &\quad + \\ 4t\left(a\cos 2t + b\sin 2t\right) &= \cos 2t. \end{aligned}$$

Since this must be true for all t, the coefficients of $\sin 2t$, $t\sin 2t$, $\cos 2t$ and $t\cos 2t$ must be equal. Thus

$$\begin{aligned} -4a &= 0 \\ -4b+4b &= 0 \\ 4b &= 1 \\ -4a+4a &= 0 \end{aligned}$$

respectively. From this we obtain

$$\begin{aligned} a &= 0 \\ b &= \frac{1}{4}, \end{aligned}$$

and hence

$$x_p(t) = \frac{1}{4} t\sin 2t.$$

■

The analogous situation for a system of first order equations is when the nonhomogeneous term includes the exponential of one of the eigenvalues of the matrix A.

3.10.5 Example (Wrong approach number 1) Determine the general solution to

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

An easy computation shows that the eigenvalues and corresponding eigenvectors of A are

$$\lambda_1 = 2 \quad \hat{\xi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \lambda_2 = 3 \quad \hat{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since the nonhomogeneous term contains e^{3t} which is precisely the exponential of an eigenvalue of A, we should expect to run into trouble equating coefficients. Trying it anyway gives

$$\xi_p(t) = ae^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t}.$$

Thus

$$\dot{\xi}_p(t) = 3ae^{3t},$$

and substituting into the differential equation gives

$$3\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

Equating coefficients of e^{3t} in each equation gives

$$\begin{aligned} 3a_1 &= 2a_1 + a_2 \\ 3a_2 &= 3a_2 + 1. \end{aligned}$$

Since there is no value for a_2 that can satisfy the second equation, there is no solution, and hence, no way to determine the undetermined coefficients. It is left as a homework problem to see that exactly the same thing happens if the eigenvalue is purely imaginary (complex) and the nonhomogeneous term contains a sine or cosine at the same frequency. ■

Since the correct approach in Chapter 2 was to simply multiply the assumed form of the solution by the independent variable, t, one may assume that the same approach works in this case as well. Unfortunately, as the following example illustrates, it does not work.

3.10.6 Example (Wrong approach number 2) Again consider

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

Since the nonhomogeneous term contains e^{3t} which is precisely the exponential of an eigenvalue of A, we should expect to run into trouble equating coefficients. Thus assume

$$\xi_p(t) = ate^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{3t}.$$

Thus

$$\dot{\xi}_p(t) = 3ate^{3t} + ae^{3t}$$

and substituting into the differential equation gives

$$\left(3t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right)e^{3t} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{3t} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

Equating coefficients of e^{3t} and te^{3t} in each equation gives

$$\begin{aligned} a_1 &= 0 \\ 3a_1 &= 2a_1 + a_2 \\ a_2 &= 1 \\ 3a_2 &= 3a_2. \end{aligned}$$

Again, there is no solution. ■

The following example elaborates upon the reason why the simple approach of only multiplying by the independent variable t does not work.

3.10.7 Example Determine the general solution to

$$\frac{d}{dt}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}e^{2t}\\ e^{2t}\end{bmatrix}.$$

These equations are decoupled, so we can immediately see (or compute)

$$\begin{aligned}\xi_{1_h} &= e^{3t}\\ \xi_{2_h} &= e^{2t}.\end{aligned}$$

Since the homogeneous solution for ξ_2 is the same as the nonhomogeneous term, clearly assuming e^{2t} will be problematic. Thus, we need a term of the form te^{2t} in the assumed form of the particular solution for ξ_3. However, a term of the form e^{2t} in the particular solution is exactly what is needed for the first line since the homogeneous solution for ξ_1 contains e^{3t}, not e^{2t}. Thus, assuming

$$\xi_p(t) = ae^{2t}$$

will not work because of the ξ_2 component, and

$$\xi_p(t) = ate^{2t}$$

will not work because of the ξ_1 component. A solution containing *both* terms is necessary. ■

Unfortunately, there is still one final twist to this whole affair. Since it is necessary to assume a particular solution that is the sum of the independent variable, t times the homogeneous solution and the homogeneous solution itself, there will not be a unique particular solution. This is because of the the term in the homogeneous solution that is not multiplied by the independent variable in the assumed form of the particular solution can be combined with the homogeneous solution in an arbitrary manner. This (along with the correct approach) is illustrated by the following example.

3.10.8 Example (Right approach) Again consider

$$\frac{d}{dt}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}0\\ e^{3t}\end{bmatrix}.$$

Observing that $\lambda = 3$ is an eigenvalue of A we assume

$$\xi_p(t) = ate^{3t} + be^{3t} = \begin{bmatrix}a_1\\ a_2\end{bmatrix}te^{3t} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}e^{3t}.$$

Thus

$$\dot{\xi}_p(t) = 3ate^{3t} + ae^{3t} + 3be^{3t}$$

and substituting into the differential equation gives

$$\begin{aligned}&\left(3t\begin{bmatrix}a_1\\ a_2\end{bmatrix} + \begin{bmatrix}a_1\\ a_2\end{bmatrix} + 3\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right)e^{3t} =\\ &\qquad\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix}t + \begin{bmatrix}b_1\\ b_2\end{bmatrix}\right)e^{3t} + \begin{bmatrix}0\\ e^{3t}\end{bmatrix}.\end{aligned}$$

Equating coefficients of e^{3t} and te^{3t} in each equation gives

$$\begin{aligned} a_1 + 3b_1 &= 2b_1 + b_2 \\ 3a_1 &= 2a_1 + a_2 \\ a_2 + 3b_2 &= 3b_2 + 1 \\ 3a_2 &= 3a_2. \end{aligned}$$

Simplifying these equations gives only three independent equations

$$\begin{aligned} a_1 + b_1 &= b_2 \\ a_1 &= a_2 \\ a_1 &= 1. \end{aligned}$$

The reason there are less than four equations, and hence no unique solution, is because the vector b in the assumed form of the solution must be an eigenvector of A and hence can be combined in any linear way with one of the homogeneous solutions. One solution to the above three equations is

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ b_1 &= 0 \\ b_2 &= 1, \end{aligned}$$

and hence

$$\xi_p(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}. \tag{3.27}$$

This particular solution is *not* unique. Indeed,

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ b_1 &= -1 \\ b_2 &= 0, \end{aligned}$$

also work giving

$$\xi_p(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t}. \tag{3.28}$$

The reason both particular solutions work is that when they are combined with the homogeneous solution, they yield the same solution. In particular, from Example 3.10.5 we can write the homogeneous solution as

$$\xi_h(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}.$$

Then the general solution using the particular solution from Equation 3.27 gives

$$\xi(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t},$$

and the general solution using the particular solution from Equation 3.28 gives

$$\xi(t) = \hat{c}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \hat{c}_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t}.$$

For c_2 from the first equation and $\hat{c}_2$ from the second equation, if $\hat{c}_2 = c_2 + 1$ the equations are identical. ■

3.10.3 Undetermined Coefficients of Systems of Second Order Equations

3.10.4 Variation of Parameters

With all the complications involved in the method of undetermined coefficients, one may be hesitant to even venture into the realm of variation of parameters since, at least in Chapter 2 the derivation was rather complicated. Thankfully, in the case of nonhomogeneous systems of first order equations, variation of parameters is even more straightforward than in the scalar second order case.

Given

$$\dot{\xi} = A\xi + g(t) \tag{3.29}$$

where

i really want to reference equation 3.29.

$$\begin{aligned} A &\in \mathbb{R}^{n\times n} \\ \xi &\in \mathbb{R}^n \\ g(t) &\in \mathbb{R}^n \end{aligned}$$

assume that $\xi_{1_h}, \xi_{2_h}, \ldots, \xi_{n_h}$ are n linearly independent homogeneous solutions to Equation 3.29, *i.e.*, they satisfy

$$\dot{\xi}_{i_h} = A\xi_{i_h}.$$

Because it is useful subsequently, we first construct and define a matrix, $\Xi(t)$ where the columns of $\Xi(t)$ are the homogeneous solutions, $\xi_{i_h}(t)$.

Definition 3.10.9 (Fundamental Matrix Solution) *Let* $\xi_{1_h}, \xi_{2_h}, \ldots, \xi_{n_h}$ *satisfy*

$$\dot{\xi}_{i_h} = A\xi_{i_h}.$$

The fundamental matrix solution is the matrix

$$\Xi(t) = \begin{bmatrix} \xi_{1_h}(t) & \xi_{2_h}(t) & \cdots & \xi_{n_h}(t) \end{bmatrix},$$

i.e., the columns of $\Xi(t)$ *are the homogeneous solutions.*

3.10.10 Example Consider again the system from Example 3.7.15. In that example we computed the general solution to $\dot{\xi} = A\xi$ where

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Skipping the details that were presented in Example 3.7.15, the general solution was

$$\begin{aligned} \xi(t) &= c_1\hat{\xi}_1 e^{\lambda_1 t} + c_2\hat{\xi}_2 e^{\lambda_1 t} + c_3\hat{\xi}_3^1 e^{\lambda_3 t} + c_4\left(\hat{\xi}_3^2 + t\hat{\xi}_3^1\right)e^{\lambda_3 t} \\ &= c_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} + \\ & \quad c_4\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) e^{3t}. \end{aligned}$$

Since each term that is multiplied by a constant, c_i is a homogeneous solution simply construct a matrix with each one as a column to construct the fundamental matrix solution

$$\begin{aligned} \Xi(t) &= \begin{bmatrix} \hat{\xi}_1 e^{\lambda_1 t} & \hat{\xi}_2 e^{\lambda_2 t} & \hat{\xi}_3^1 e^{\lambda_3 t} & \left(\hat{\xi}_3^2 + t\hat{\xi}_3^1\right)e^{\lambda_3 t} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{2t} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} & \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}. \end{aligned}$$ ■

The fundamental matrix solution has one important property that will be used in the derivation of the variation of parameters solution; namely, the whole matrix satisfies the homogeneous equation. In other words, if $\Xi(t)$ is the fundamental matrix solution to

$$\dot{\xi} = A\xi$$

then

$$\dot{\Xi} = A\Xi.$$

This is true since each column of $\Xi(t)$ is a homogeneous solution and is illustrated by the following example.

3.10.11 Example From Example 3.10.10 we have

$$\Xi(t) = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}$$

so

$$\begin{aligned} \dot{\Xi}(t) &= \begin{bmatrix} 2e^{2t} & 0 & 0 & 0 \\ 0 & 2e^{2t} & 0 & 0 \\ 0 & 0 & 3e^{3t} & 3te^{3t} + e^{3t} \\ 0 & 0 & 0 & 3e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{3t} & te^{3t} \\ 0 & 0 & 0 & e^{3t} \end{bmatrix}. \end{aligned}$$

Thus $\dot{\Xi} = A\Xi$. ■

Similar to the approach for second order equations, the approach to find the particular solution for a nonhomogeneous system of first order equations is to assume that the particular solution is of the form of

$$\xi_p(t) = \Xi(t)u(t)$$

where $u(t)$ is a vector of unknown functions. To determine $u(t)$, simply substitute into Equation 3.29. First note that (dropping the explicit dependence on t)

$$\dot{\xi}_p = \dot{\Xi}u + \Xi\dot{u}.$$

Substituting into Equation 3.29 gives

$$\dot{\Xi}u + \Xi\dot{u} = A\Xi u + g.$$

Since

$$\dot{\Xi} = A\Xi \qquad \Longrightarrow \dot{\Xi}u = A\Xi u$$

so

$$\Xi\dot{u} = g.$$

Since Ξ contains n linearly independent solutions, it is invertible and hence

$$\dot{u} = \Xi^{-1}g \qquad \Longrightarrow \qquad u(t) = \int_{t_0}^{t} \Xi^{-1}(\tau)g(\tau)d\tau.$$

Substituting into the assumed form of the particular solution gives a complete expression for the particular solution as

$$\xi_p(t) = \Xi \int_{t_0}^{t} \Xi^{-1}(\tau)g(\tau)d\tau.$$

Note that to even compute the particular solution we need the fundamental matrix which contains a full set of homogeneous solutions. Since any linear combination of the homogeneous solutions can be expressed as

$$c_1\xi_{1_h} + c_2\xi_{2_h} + \cdots + c_n\xi_{n_h} = \Xi(t)c$$

where

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

the general solution to Equation 3.29 is

$$\xi(t) = \Xi(t)c + \Xi(t)\int_{t_0}^{t} \Xi^{-1}(\tau)g(\tau)d\tau. \tag{3.30}$$

Finally, if the initial conditions, $\xi(t_0)$ are specified, then

$$\xi(t_0) = \Xi(t_0)c$$

since the integral with the same upper and lower limits is zero. Hence

$$c = \Xi^{-1}(0)\xi(t_0)$$

and substituting into the general solution gives the entire answer as

$$\xi(t) = \Xi(t)\Xi^{-1}(t_0)\xi(t_0) + \Xi(t)\int_{t_0}^{t} \Xi^{-1}(\tau)g(\tau)d\tau. \tag{3.31}$$

An example illustrates the straightforward application of this method.

3.10.12 Example Solve

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} e^{-4t} \\ 0 \end{bmatrix}.$$

A simple computation determines the eigenvalues and eigenvectors for the matrix as

$$\lambda_1 = -4 \quad \hat{\xi}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = -2 \quad \hat{\xi}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

thus

$$\Xi(t) = \begin{bmatrix} -e^{-4t} & e^{-2t} \\ e^{-4t} & e^{-2t} \end{bmatrix}.$$

A simple computation determines that

$$\Xi^{-1}(t) = \frac{1}{2}\begin{bmatrix} -e^{4t} & e^{4t} \\ e^{2t} & e^{2t} \end{bmatrix},$$

and

$$\Xi^{-1}(t)g(t) = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2}e^{-2t} \end{bmatrix}.$$

Assuming that $t_0 = 0$,

$$\begin{aligned} \int_0^t \Xi^{-1}(\tau)g(\tau)d\tau &= \int_0^t \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2}e^{-2\tau} \end{bmatrix} d\tau \\ &= \begin{bmatrix} -\frac{1}{2}\tau \\ \frac{1}{4}\left(1-e^{-2t}\right) \end{bmatrix}. \end{aligned}$$

Then

$$\Xi(t)\int_0^t \Xi^{-1}(\tau)g(\tau)d\tau = \begin{bmatrix} \frac{1}{4}\left(e^{-2t}+2te^{-4t}-e^{-4t}\right) \\ \frac{1}{4}\left(e^{-2t}-2te^{-4t}-e^{-4t}\right) \end{bmatrix}.$$

So finally we have

$$\begin{aligned} \xi(t) &= \Xi(t)c + \Xi(t)\int_{t_0}^t \Xi^{-1}(\tau)g(\tau)d\tau \\ &= c_1 \begin{bmatrix} -e^{-4t} \\ e^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\left(e^{-2t}+2te^{-4t}-e^{-4t}\right) \\ \frac{1}{4}\left(e^{-2t}-2te^{-4t}-e^{-4t}\right) \end{bmatrix}. \quad \blacksquare \end{aligned}$$

3.11 Applications of Nonhomogeneous Systems of Equations

Chapter 4

Lagrangian Dynamics

4.1 Introduction

This chapter deals with the subject of Lagrangian dynamics, and in the overall context of this text deals with the problem of *determining* the differential equations that describe a particular system, as opposed to solving or analyzing differential equations. Thus, in a sense, it is perhaps more appropriately the starting point of the subject matter of this text. However, it is included at this point, later in the text, because the differential equations describing a mechanical system that are most conveniently obtained using Lagrange's equations are generally nonlinear and second order and hence naturally are considered with the specific tools for nonlinear systems and higher order systems.

The essential feature of Lagrange's equations, in contrast to a Newtonian ($F = ma$) approach, is that it is allows great flexibility in specifying which coordinates are used to describe the system.

4.2 Motivational Example

Consider the planar double pendulum illustrated in Figure 4.1. In this section the equations of motion for the system are derived using a Newtonian approach. After the derivation of Lagrange's equations in Section **??**, the problem will be revisited, with the hopefully obvious feature that the Lagrangian approach is *much* less work.

4.3 Derivation of Lagrange's Equations

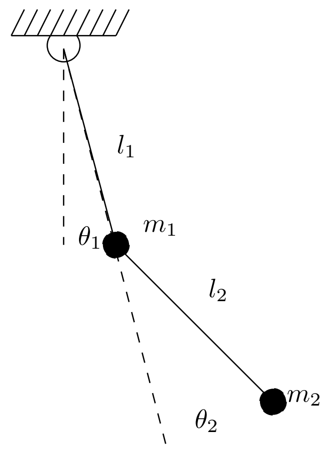

Figure 4.1. Double pendulum system (no gravity).

Chapter 5

Partial Differential Equations

Chapter 6

Transform Methods

This chapter deals primarily with the use of Laplace tranforms for solving linear ordinary differential equations and the application thereof to the analysis and design of feedback control systems. First a few details from complex variable theory are reviewed.

6.1 Review of Complex Variable Theory

Recall that a complex number may be represented in either Cartesian or polar form, *i.e.,*

$$\begin{aligned} z &= x + iy \\ &= (x, y) \\ &= (r, \theta) \end{aligned}$$

where

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan 2\,(y, x) \\ x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Note that in Cartesian form

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i\,(y_1 + y_2) \end{aligned}$$

and

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)\,(x_2 + iy_2) \\ &= x_1x_2 + x_1iy_2 + iy_1x_2 + iy_1iy_2 \\ &= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_1x_2) \end{aligned}$$

and in polar form

$$\begin{aligned} z_1 + z_2 &= r_1 \left(\cos\theta_1 + i\sin\theta_1\right) + r_2 \left(\cos\theta_2 + i\sin\theta_2\right) \\ &= \left(r_1\cos\theta_1 + r_2\cos\theta_2\right) + i\left(r_1\sin\theta_1 + r_2\sin\theta_2\right) \end{aligned}$$

and

$$\begin{aligned} z_1 z_2 &= r_1 \left(\cos\theta_1 + i\sin\theta_1\right) r_2 \left(\cos\theta_2 + i\sin\theta_2\right) \\ &= r_1 r_2 \left[\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)\right] \\ &= r_1 r_2 \left(\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right). \end{aligned}$$

So, that in general, it is more convenient to add and subtract complex numbers in Cartesian form and to multiply or divide them in polar form. Note in particular, in polar form when multiplying complex numbers, it is necessary only to multiply the magnitudes and add the arguments.

6.2 The Laplace Transform

Consider the following definition.

Definition 6.2.1 (Laplace Transform) *Let $f(t)$ be a function and $s \in \mathbb{C}$. Define the Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$ by*

$$\mathcal{L}\{f(t)\} = \lim_{a\to\infty} \int_0^a f(t)e^{st}dt = \int_0^\infty f(t)e^{st}dt.$$

Remark 6.2.2 *It is somewhat beyond the scope of this text, but obvserve that since the integral in the definition of the Laplace transform is an indefinite integral, convergence is not at all guaranteed for all values of s.*

6.3 The Fourier Transform

6.4 Using Laplace Transforms to Solve Linear Ordinary Differential Equations

6.5 The Transfer Function

This section introduces the transfer function, which is of great engineering importance due to the manner in which it concisely represents the relationship between the input and output of a system.

> **6.5.1 Example** Consider, again, the example from Section 3.2, which is reproduced in Figure 6.1 for convenience. In this problem consider the *input* to the system to be the force, $f(t)$. In controls problems considered subsequently, the task will be to pick $f(t)$ so that the response of the output

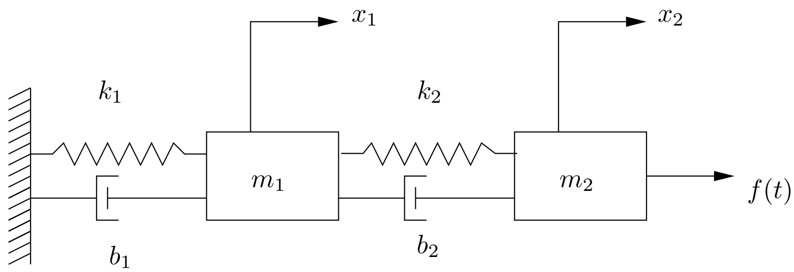

Figure 6.1. Two degree of freedom mass-spring-damper system.

of the system has some desirbale characteristic. Consider the *output* to be the position of mass 1, $x_1(t)$. The purpose of this example is to illustrate that using Laplace transforms provides a more convenient representation of the relationship between $f(t)$ and $x_1(t)$ than does the system of ordinary differential equations that we obtained and solved before.

From Section 3.2

$$\begin{aligned} m_1\ddot{x}_1 + (b_1+b_2)\,\dot{x}_1 - b_2\dot{x}_2 + (k_1+k_2)\,x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - b_2\dot{x}_1 + b_2\dot{x}_2 - k_2x_1 + k_2x_2 &= f(t), \end{aligned} \tag{6.1}$$

and assume for simplicity that

$$\begin{aligned} x_1(0) &= 0 \\ \dot{x}_1(0) &= 0 \\ x_2(0) &= 0 \\ \dot{x}_2(0) &= 0. \end{aligned}$$

Previously, at this point our goal was to solve these equations. The controls problem considered in subsequent sections of this chapter is not simply to solve the equations, but to pick $f(t)$ so that the solution behaves in a particular manner.

Without any motivation other than to observe that this example is in the "transform methods" chapter Laplace transforming both equations in Equation 6.1, gives

$$\begin{aligned} m_1s^2X_1(s) + (b_1+b_2)\,sX_1(s) - b_2sX_2(s) + (k_1+k_2)\,X_1(s) - k_2X_2(s) &= 0 \\ m_2s^2X_2(s) - b_2sX_1(s) + b_2sX_2(s) - k_2X_1(s) + k_2X_2(s) &= F(s) \end{aligned}$$

or collecting terms

$$\begin{aligned} X_1(s)\left(m_1s^2 + (b_1+b_2)\,s + (k_1+k_2)\right) - X_2(s)\,(b_2s+k_2) &= 0 \\ -X_1(s)\,(b_2s+k_2) + X_2(s)\left(m_2s^2 + b_2s + k_2\right) &= F(s). \end{aligned}$$

Solving the first equation for $X_2(s)$ gives

$$X_2(s) = \frac{m_1 s^2 + (b_1 + b_2) + (k_1 + k_2)}{b_2 s + k_2} X_1(s)$$

and substituting into the second equation gives

$$-X_1(s)\,(b_2 s + k_2) + \frac{m_1 s^2 + (b_1 + b_2) + (k_1 + k_2)}{b_2 s + k_2} X_1(s)\left(m_2 s^2 + b_2 s + k_2\right) = F(s).$$

Solving for $X_1(s)$ and not making much attempt to simplify things gives

$$X_1(s) = \frac{(b_2 s + k_2)}{\left(m_1 s^2 + (b_1 + b_2)\,s + (k_1 + k_2)\right)\left(m_2 s^2 + b_2 s + k_2\right) - (b_2 s + k_2)^2} F(s). \tag{6.2}$$

Let

$$G(s) = \frac{(b_2 s + k_2)}{\left(m_1 s^2 + (b_1 + b_2)\,s + (k_1 + k_2)\right)\left(m_2 s^2 + b_2 s + k_2\right) - (b_2 s + k_2)^2}$$

so that

$$X_1(s) = G(s)F(s).$$

As will be defined subsequently, the term that relates the input, $F(s)$ to the output, $X_1(s)$ is called the transfer function from $f(t)$ to $x_1(t)$. ■

Remark 6.5.2

1. *Note that given the force, $f(t)$ we can first find $F(s)$, then $X_1(s)$ and finally $x_1(t)$ directly from Equation 6.2 without having to screw around with $x_2(t)$ or $X_2(s)$ since $X_2(s)$ does not explicity appear in the equation. It is still represented in the equation, though, in the polynomials in s.*

2. *If we did not Laplace transform the equations in Equation 6.1, there is no way to reduce the system to one equation like we were able to do by Laplace transforming.*

3. *Note that the transfer function, $H(s)$ is a function only of the system itself, i.e., the m's, b's and k's, so regardless of the nature of the input, $f(t)$, the transfer function is a complete representation of the relationship between the input and output.*

Definition 6.5.3 (Transfer Function) *The transfer function of a linear, time invariant system of differential equations is the ratio of the Laplace transform of the output of the system, $Y(s)$ to the input of the system, $R(s)$, i.e., $G(s) = \frac{Y(s)}{R(s)}$.*

An often unspecified bit of information that accompanies the definition of a transfer function is that if the initial conditions are not explicitly specified, *they are assumed to be zero.*

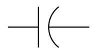	capacitor	$i(t) = C\frac{dv(t)}{dt}$	$V(s) = RI(s)$
	inductor	$v(t) = L\frac{di(t)}{dt}$	$I(s) = CsV(s)$
	resistor	$v(t) - i(t)R$	$V(s) = LsI(s)$

Figure 6.2. Basic electrical components.

6.5.1 Review of Kirchoff's Laws and DC Motor Laws

Since electic motors are often used to apply torques and, hence, indirectly, forces, to mechanical systems, we must review simple means to model electic motors as well as the electic circuits attached to them. First we will consider Kirchoff's laws and then motor modeling.

Definition 6.5.4 (Kirchoff's Laws)

1. *Kirchoff's current law (KCL): the sum of currents leaving a node in an electic circuit is equal to the sum of currents entering the node.*
2. *Kirchoff's voltage law (KVL): the sum of all volatages around any closed path in an electic circuit is equal to zero.*

The basic elements and their governing equations of a circuit we consider are the resistor, capacitor and inductor. The schematic graphic as well as the equations describing them are illustrated in Figure **??**.

A simple application of Kirchoff's laws is illustrated by the following example.

6.5.5 Example Determine the transfer function relating the input voltage, $v_i(t)$ to the output voltage, $v_o(t)$ in the circuit illustrated in Figure 6.3.

Referring to Figure 6.4, clearly, $V_o(s) = R_2 I(s)$, if we denote the current through R_2 by $i(t)$ and its Laplace transform by $I(s)$. This is either by simple inspection, or by using KVL around the loop through R_2 and across the output voltage, $V_o(s)$ indicated by the arrow and labeled by A. Also, considering the closed loop with R_1 and the capacitor, if $v_c(t)$ denotes the voltage across the capacitor and v_{R_1} denotes the voltage across R_1, then KVL gives

$$v_c(t) = v_{R_1}(t). \tag{6.3}$$

Also, considering the current, $i(t)$, indicated by the arrow and using KCL at the node labeled by B

$$i(t) = C\frac{dv_c(t)}{dt} + \frac{v_{R_1}(t)}{R_1} \tag{6.4}$$

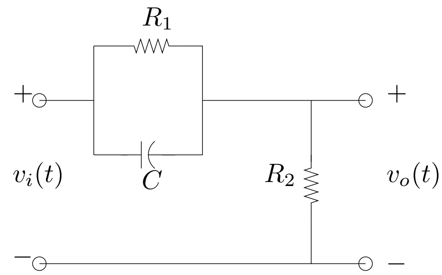

Figure 6.3. Electrical circuit for Example 6.5.5.

using the governing equations for a capacitor and resistor, respectively. Laplace transforming Equations 6.3 and 6.4 gives

$$V_o(s) = R_2 I(s) \tag{6.5}$$

$$I(s) = CsV_c(s) + \frac{V_{R_1}(s)}{R_1} = CsV_c(s) + \frac{V_c(s)}{R_1} \tag{6.6}$$

Finally, using KVL around the loop indiacted by E,

$$V_i(s) = V_c(s) + V_o(s)$$

or

$$V_c(s) = V_i(s) - V_o(s). \tag{6.7}$$

Solving equation 6.5 for $I(s)$ and substituting into equation 6.6, and also substituting the expression for $V_c(s)$ in equation 6.7 into equation 6.6 gives

$$\frac{V_o(s)}{R_2} = Cs\left(V_i(s) - V_o(s)\right) + \frac{V_i(s) - V_o(s)}{R_1}.$$

Solving for the transfer function gives

$$\frac{V_o(s)}{V_i(s)} = \frac{R_1 R_2 Cs + R_2}{R_1 R_2 Cs + R_1 + R_2}.$$

Obvserve that the approach to determining transfer functions is pretty straightfoward: Laplace transform all the governing equations and there should be enough equations and unknown variables to solve for the ratio of the output to the input.

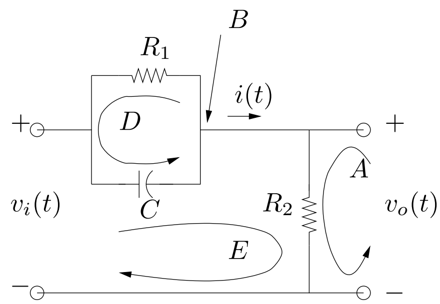

Figure 6.4. Electrical circuit for Example 6.5.5. ∎

6.6 Basic Stability Analysis

6.7 Second Order System Response

6.8 The Root Locus Design Method

6.9 Frequency Response Analysis and Design

Chapter 7

Classical Control Theory

With the exception of time delay, all the transfer functions consider thus far have been ratios of polynomials in the variable s. Henceforth, we will assume that any transfer function is a rational function, *i.e.,* it is a ratio of polynomials in s and that the order of the polynomial in the denominator is greater than the numerator. This assumption regarding the relative degree of the denominator and numerator is generally true for all physical engineering systems and will be a basic assumption throughout the rest of this chapter.

what about pole zero cancelation?

elaborate on relative degree

Since a transfer function is a ratio of polynomials in s, it is convenient to consider it in a factored form, *i.e.,*

$$\frac{Y(s)}{R(s)} = a\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-p_j)}.$$

The values $s = z_i$ are called the *zeros* of the transfer function and the values $s = p_j$ are called the poles. Since s is the independent variable in a transfer function, clearly values z_i and p_j completely determine the transfer function.

7.0.1 Example Consider a mass-spring-damper system with a force of $r(t)$ applied to the mass. The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = r(t)$$

and the transfer function is

$$\frac{X(s)}{R(s)} = \frac{1}{ms^2 + bs + k},$$

or in factored form

$$\frac{X(s)}{R(s)} = \frac{1}{m}\frac{1}{\left(s + \frac{b}{2m} + \frac{i}{2m}\sqrt{4km - b^2}\right)\left(s + \frac{b}{2m} - \frac{i}{2m}\sqrt{4km - b^2}\right)}.$$

So for this system, there are no zeros, one pole is located at $s = -\frac{b}{2m} - \frac{i}{2m}\sqrt{b^2 - 4km}$ and the other pole is located at $s = -\frac{b}{m} + \frac{i}{2m}\sqrt{b^2 - 4km}$.

Note that if $b^2 > 4km$ then the two poles are real, if $b^2 < 4km$ the poles are a complex conjugate pair and if $b^2 = 4km$ they are the same, *i.e.,* the are repeated.

Note that if the two poles are real, then the step response of the system will be of the form of two exponentials. If the poles are a complex conjugate pair, then they will contain sinusoidal terms.

$$\frac{X(s)}{R(s)} = \frac{1}{m}\frac{1}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}}.$$

■

The basis for the so-called "root locus" design technique in classical control is an attempt to manipulate the location of poles of a transfer function to affect the desired response of the system. In order to have a basis for this approach, however, the basic response of a system as a function of pole location must be thoroughly understood.

7.1 Step Response *vs.* Pole Location

The most basic controller design methodologies, particularly the root locus method, are based upon knowledge of a system's response to various inputs based upon the location of the transfer function poles and zeros. This section describes the qualitative behavior of the step response of transfer functions as a function of the location of the poles of the transfer function. The location of the zeros will be treated as a modification of the response versus pole location and will be outlined in Section 7.3. The approach taken is fundamentally *geometric*, which, when the complete context of the problem is understood, is indeed a natural approach to the problem. The following sections merely provide the details to develop a catalog of responses for various transfer function pole locations.

7.1.1 One real pole

Consider a transfer function of the form

$$\frac{Y(s)}{R(s)} = \frac{-p}{s-p}. \tag{7.1}$$

Note that this transfer function has one pole located at $s = p$. Consider the case when the input is a step input, *i.e.,* $R(s) = \frac{1}{s}$. A partial fraction expansion gives

$$Y(s) = \frac{1}{s} - \frac{1}{s-p}.$$

The reason for the $-p$ numerator in the transfer function is to make the partial fraction numerators both equal to 1. Even if the numerator were not $-p$, the following discussion would still be qualitatively correct. From Table **??** the solution is

$$y(t) = 1 - e^{pt}.$$

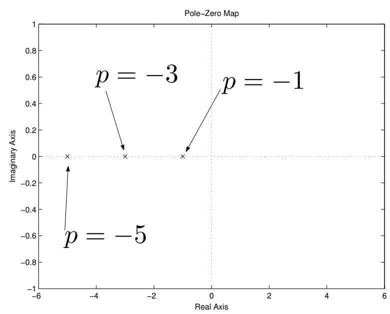

Figure 7.1. Pole locations for Equation 7.1 for $p = -1, -3$ and -5.

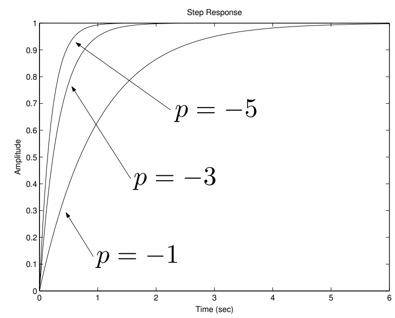

Figure 7.2. Step response for Equation 7.1 for $p = -1, -3$ and -5.

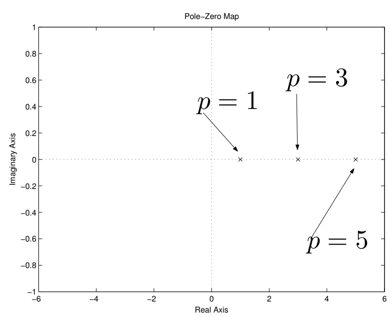

Figure 7.3. Pole locations for Equation 7.1 for $p = 1, 3$ and 5.

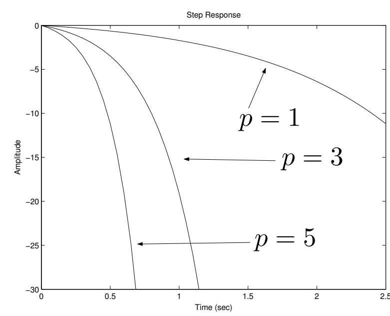

Figure 7.4. Step response for Equation 7.1 for $p = 1, 3$ and 5.

Figures 7.1 and 7.3 illustrate the pole locations for the pole values $p = -5, -3, -1, 1, 3$ and 5. Figures 7.2 and 7.4 illustrate the corresponding solution, $y(t)$ for the same pole values $p = -5, -3, -1, 1, 3$ and 5. Observe that

- If the pole is to the left of the imaginary axis, the step response is stable. If the pole is to the right of the imaginary axis, the response is unstable.
- The farther away from the imaginary axis the pole is, the faster the response is.

7.1.2 Purely imaginary complex conjugate pair

Now consider

$$\frac{Y(s)}{R(s)} = \frac{a^2}{s^2 + a^2}. \tag{7.2}$$

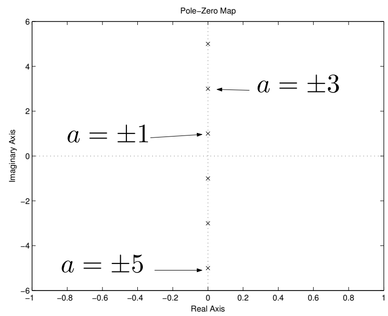

Figure 7.5. Pole locations for Equation 7.2 for $a = \pm 1, \pm 3$ and ± 5.

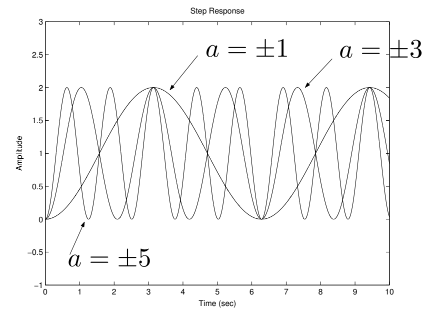

Figure 7.6. Step response for Equation 7.1 for $a = \pm 1, \pm 3$ and ± 5.

Note that this transfer function has two poles located at $s = \pm ai$, and again consider the step response, *i.e.,* $R(s) = \frac{1}{s}$. Partial fractions gives

$$Y(s) = \frac{1}{s} + \frac{1}{s^2 + a^2},$$

where again the numerator in the transfer function was picked solely for computational convenience. Again, the following discussion would still be qualitatively true with a different numerator. From Table **??** the solution is

$$y(t) = 1 + \sin at.$$

Figures **??** and **??** illustrate the pole locations and step response for the pole values $p = \pm 1, \pm 3$ and ± 5. Observe that all the solutions are non-decaying sinusoidal solutions. As the imaginary pair of poles moves farther away from the real axis, the frequency of the response increases.

Figures **??** and **??** illustrate the pole locations for the pole values $p = \pm 1, \pm 3$ and ± 5. Observe that all the solutions are non-decaying sinusoidal solutions. As the imaginary pair of poles moves farther away from the real axis, the frequency of the response increases.

7.1.3 Complex conjugate poles

Now consider a transfer function of the form

$$\frac{Y(s)}{R(s)} = \frac{a^2 + b^2}{(s-a)^2 + b^2}, \tag{7.3}$$

which has two poles located at $s = a \pm ib$. Again the numerator was picked solely for convenience and the following discussion is qualitatively true for other numerator values as well. The response is

$$y(t) = 1 - e^{at} \cos bt + \frac{a}{b} e^{at} \sin bt,$$

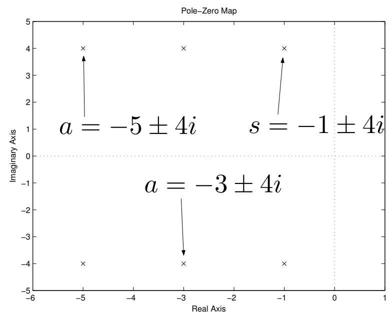

Figure 7.7. Pole locations for Equation 7.3 for $a = -1, -3$ and -5 and $b = \pm 1$.

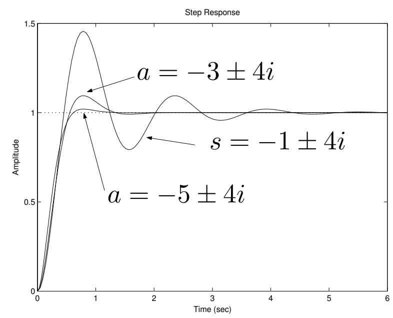

Figure 7.8. Step response for Equation **??** for $a = -1, -3$ and -5 and $b = \pm 1$.

and the poles and step response are plotted for $a = -1, -3$ and -5 and $b = \pm 4$ in Figures 7.7 and 7.8. The poles and step response are plotted for $a = 1$ and 3 and $b = \pm 4$ in Figures **??** and **??**. The poles and step response are plotted for $b = \pm 1, \pm 3$ and ± 5 and $a = -1$ in Figures **??** and **??**. The poles and step response are plotted for $b = \pm 1, \pm 3$ and ± 5 and $a = 1$ in Figures **??** and **??**.

7.1.4 Combination of poles

7.2 Time Domain Response of a Second Order System

As has already been studied extensively in Chapter **??** second order, constant coefficient, linear homogeneous systems have a solution that is relatively easy to characterize. The most basic approach to feedback control design is to attempt to design a controller so that the step response of a system with feedback control has desirable features similar to that of a second order system. First, we need to characterize the response of a second order system.

7.2.1 Response *vs.* pole location for a complex conjugate poles

Consider the differential equation

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = r(t)$$

where $b^2 < 4km$. The transfer function from $r(t)$ to $x(t)$ is

$$\frac{X(s)}{R(s)} = \frac{1}{ms^2 + bs + k} = \left(\frac{1}{m}\right)\left(\frac{1}{\left(s + \frac{b}{2m}\right)^2 + \frac{4km - b^2}{4m^2}}\right)$$

and the poles are at $s = -\frac{b}{2m} \pm \frac{i}{2m}\sqrt{4km - b^2}$. Assuming that $r(t)$ is a unit step then $R(s) = \frac{1}{s}$, which gives

$$X(s) = \left(\frac{1}{m}\right)\left(\frac{1}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}}\right)\left(\frac{1}{s}\right).$$

Partial fractions gives

$$X(s) = \frac{A}{s} + \frac{Bs + C}{ms^2 + bs + k} = \left(\frac{1}{s}\right)\left(\frac{1}{ms^2 + bs + k}\right)$$

which gives

$$\begin{aligned} A &= \frac{1}{k} \\ B &= -\frac{m}{k} \\ C &= -\frac{b}{k} \end{aligned}$$

so

$$\begin{aligned} X(s) &= \frac{\frac{1}{k}}{s} - \frac{\frac{m}{k}s + \frac{b}{k}}{ms^2 + bs + k} \\ &= \frac{\frac{1}{k}}{s} - \frac{\frac{1}{k}s + \frac{b}{mk}}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}} \\ &= \frac{1}{k}\left(\frac{1}{s} - \frac{s + \frac{b}{m}}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}}\right) \\ &= \frac{1}{k}\left(\frac{1}{s} - \frac{s + \frac{b}{2m}}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}} - \frac{\frac{b}{2m}}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}}\right) \\ &= \frac{1}{k}\left(\frac{1}{s} - \frac{s + \frac{b}{2m}}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}} - \left(\sqrt{\frac{b^2}{4km-b^2}}\right)\frac{\sqrt{\frac{4km-b^2}{4m^2}}}{\left(s+\frac{b}{2m}\right)^2 + \frac{4km-b^2}{4m^2}}\right). \end{aligned}$$

Thus, from the Table **??**

$$x(t) = \frac{1}{k}\left[1 - e^{-\frac{b}{2m}t}\left(\cos\left(\frac{\sqrt{4km-b^2}}{2m}t\right) + \sqrt{\frac{b^2}{4km-b^2}}\sin\left(\frac{\sqrt{4km-b^2}}{2m}t\right)\right)\right].$$

If we denote

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}} \\ \zeta &= \frac{b}{2m\omega_n} \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} \end{aligned}$$

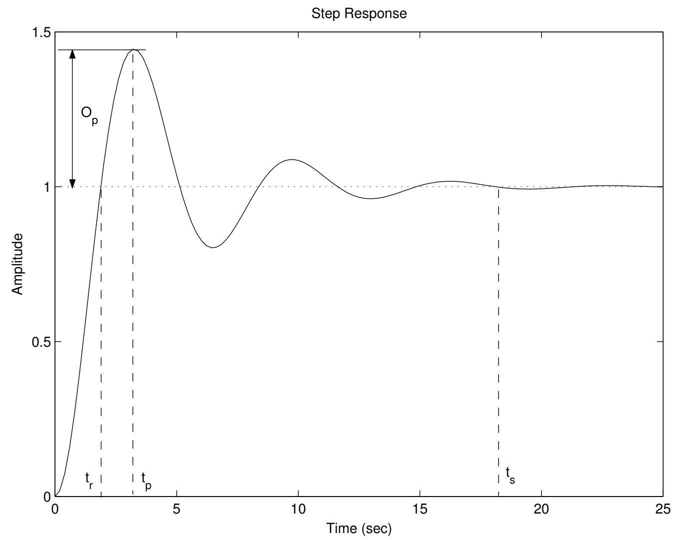

Figure 7.9. Second order step response.

then

$$x(t) = \frac{1}{k}\left[1 + e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right].$$

7.2.2 Time Domain Specifications

Consider the typical second order step response illustrated in Figure 7.9. We will consider four *time domain specifications*: percent overshoot, rise time, peak time and settling time.

Define

- he *maximum overshoot,* O_x is the maximum value that $x(t)$ obtains minus the steady state value, $x_{ss} = \lim_{t\to\infty} x(t)$ and the *percentage overshoot* is given by
$$M_p = \frac{O_x - x_{ss}}{x_{ss}};$$
- the *peak time* is the (first) time at which $x(t) = O_x$;
- the *rise time* is the time at which $x(t)$ first is equal to x_{ss}; and,
- the *S % settling time* is the first time for which $|x(t) - x_{ss}| < \frac{S}{100}x_{ss}$.

7.2.3 Time domain specifications and pole location geometry

Consider the complex conjugate pairs of poles illustrated in Figure 7.10 and recall that the poles are located at $s = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{4km - b^2}$. Note that from some simple algebra and geometry

- $\left|s^2\right| = \frac{b^2}{4m^2} + \frac{4km-b^2}{4m^2} = \frac{k}{m} = \omega_n^2$;
- $\sin\theta = \frac{\frac{-b}{2m}}{\omega_n} = \zeta$;
- $\mathrm{Re}(s) = \omega_n \sin\theta = \zeta\omega_n$; and,
- $\mathrm{Im}(s) = \cos\theta\omega_n = \sqrt{1 - \sin^2\theta}\omega_n = \sqrt{1-\zeta^2}\omega_n = \omega_d$.

Given these relationships, it is relatively straight-forward to determine the relationship among the time domain specifications, M_p, t_r, t_p and t_s and the geometry of the pole locations. In particular, solving $\frac{dx(t)}{dt} = 0$ (homework problem **??**) for t gives

$$t_p = \frac{\pi}{\omega_d}. \tag{7.4}$$

Substituting this into the solution for $x(t)$ gives

$$x(t_p) = \frac{1}{k}\left(1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}t}\right),$$

and since $\lim_{t\to\infty} x(t) = \frac{1}{k}$,

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}. \tag{7.5}$$

Note that the $S\%$ settling time occurs when

$$e^{-\zeta\omega_n t_s} = \frac{S}{100}$$

or

$$t_s = -\frac{1}{\omega_n\zeta}\ln\left(\frac{S}{100}\right). \tag{7.6}$$

Finally, for the rise time, t_r the expression would be rather complicated, but simply by inspecting Figure 7.9 gives the approximate relationship

$$t_r \approx \frac{1.8}{\omega_n}.$$

So, finally, considering how a pole may "move" as some parameter is varied, we have the results in Figure **??**.

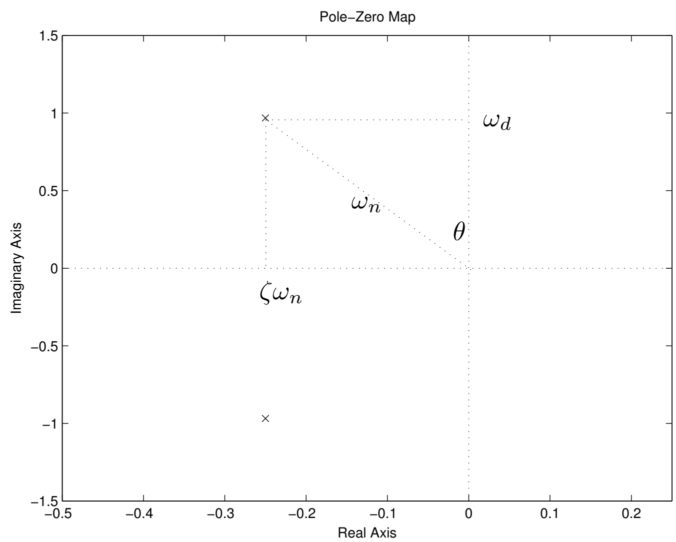

Figure 7.10. Complex conjugate pole geometry.

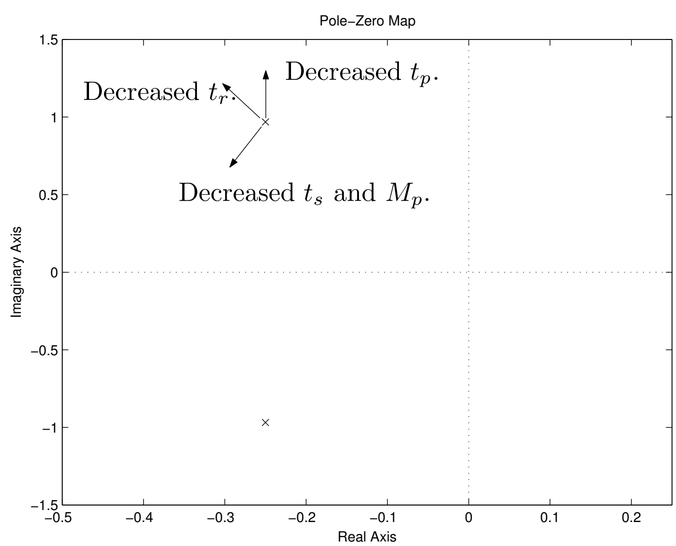

Figure 7.11. How pole displacement affects time domain specifications.

7.3 Effect of Additional Poles and Zeros

7.4 The Root Locus Design Method

The root locus design method is a technique to generate a plot, called the *root locus plot* that graphically depicts how the poles of a transfer function vary as some parameter in the system, typically a controller gain, is varied. The procedure is rather straight forward, as is the derivation of the rules. Recall, that the nature of the response of the system is primarily dictated by the location of the poles of the transfer function; hence, a graphical depiction of how the poles move as a parameter is varied is extremely informative.

7.4.1 Quick Review of Complex Variables

To understand the root locus plotting rules, two facts regarding multiplication of complex numbers must be at the forefront and are worth repeating.

1. If $a, b \in \mathbb{C}$, then $\angle(ab) = \angle a + \angle b$, that is, the phase of the product of two complex numbers is the sum of the phases of the two numbers.
2. If $a, b \in \mathbb{C}$, then $|ab| = |a||b|$, that is, the magnitude of the product of two complex numbers it the product of the magnitudes of the two numbers.

For division, the rules are similar.

1. If $a, b \in \mathbb{C}$, then $\angle\left(\frac{a}{b}\right) = \angle a - \angle b$, that is, the phase of the product of two complex numbers is the difference of the phases of the two numbers.
2. If $a, b \in \mathbb{C}$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, that is, the magnitude of the product of two complex numbers it the quotient of the magnitudes of the two numbers.

Also, since we are going to deal graphically with the difference between complex numbers, it is worth pointing out that if s and a are complex numbers, then the complex number $s - a$ graphically is the vector pointing from a to s. This is illustrated in Figure 7.12.

7.4.2 Graphical Interpretation of a Transfer Function

Now, consider a transfer function of the form

$$G(s) = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} = \frac{s^m + a_1 s^{m-1} + a_2 s^{m-2} + \cdots + a_{m-1}s + a_m}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + a_{n-1}s + a_n},$$

where the middle term is simply expressed in factored form and the right hand term is simply expressed as the individual terms of the polynomial in s. In the next section, graphically determining the magnitude and phase of $G(s)$ for a given value of s based upon its pole and zero values will be necessary. From the

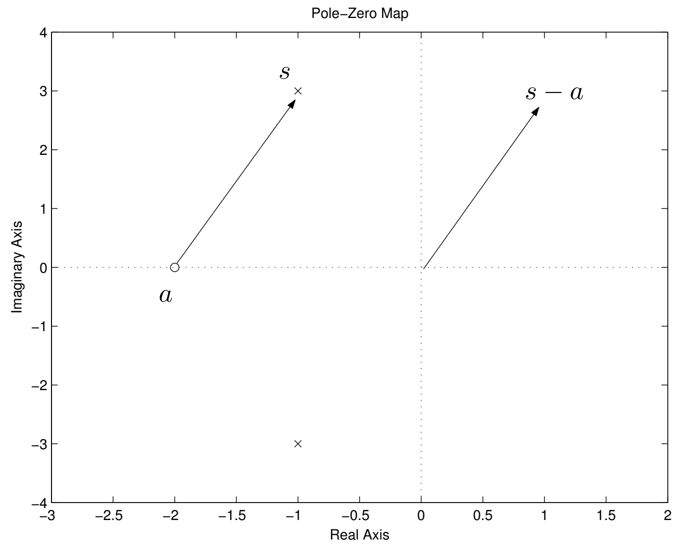

Figure 7.12. Vector interpretation of the difference between two complex numbers.

above facts regarding multiplication and division of complex numbers it follows that for any given value of s

$$\angle G(s) = \sum_{i=1}^{m} \angle (s - z_i) - \sum_{j=1}^{n} (s - p_j)$$

and

$$|G(s)| = \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{j=1}^{n} |s - p_j|}.$$

The significance of these two equations is critical. Simply by plotting the location of the poles and zeros of $G(s)$, it is easy to determine both the phase and magnitude (and hence the value) of $G(s)$ graphically. This is illustrated by the following example.

7.4.1 Example Let

$$G(s) = \frac{s+2}{(s+4)^2 + 9}. \tag{7.7}$$

Determine the magnitude and phase of $G(-4+0i) = G(-4)$.

Referring to Figure 7.13 and letting $s = -4$

$$\begin{aligned} \angle (s - z_1) &= 180^\circ \\ \angle (s - p_1) &= -90^\circ \\ \angle (s - p_2) &= 90^\circ. \end{aligned}$$

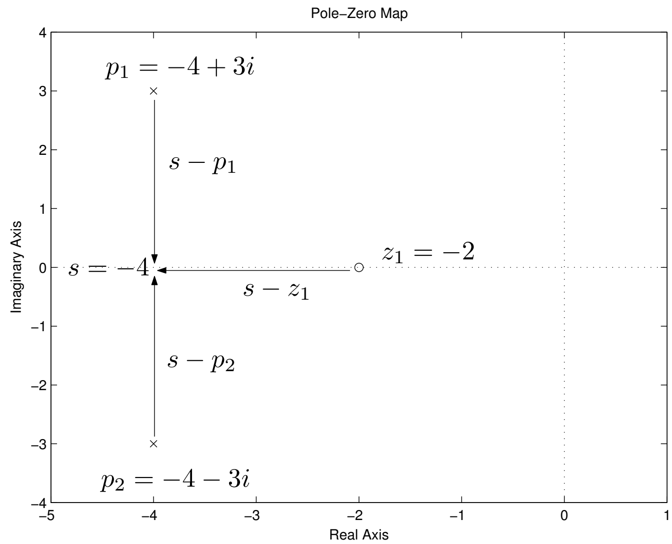

Figure 7.13. Graphically evaluating $G(-4)$ for Equation 7.7.

Hence

$$\begin{aligned}\angle G(s) &= \angle G(-4)\\ &= \sum_{i=1}^{m} \angle (s - z_i) - \sum_{j=1}^{n} (s - p_j)\\ &= 180^\circ - (-90^\circ + 90^\circ)\\ &= 180^\circ.\end{aligned}$$

Also,

$$\begin{aligned}\angle |s - z_1| &= 2\\ \angle |s - p_1| &= 3\\ \angle |s - p_2| &= 3.\end{aligned}$$

Hence

$$\begin{aligned}|G(s)| &= |G(-4)|\\ &= \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{j=1}^{n} |s - p_j|}\\ &= \frac{2}{3 \cdot 3}\\ &= \frac{2}{9}.\end{aligned}$$

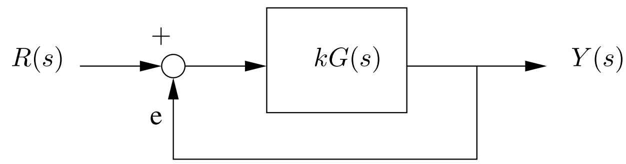

Figure 7.14. Block diagram for Example 7.4.2.

This can be verified by direct substitution as well since

$$G(-4) = \frac{-4+2}{(-4+4)^2+9} = \frac{-2}{9},$$

so

$$\begin{aligned} \angle G(-4) &= 180^\circ \\ |G(-4)| &= \frac{2}{9}. \end{aligned}$$ ■

7.4.3 Root Locus Plotting Rules

Consider a transfer function with a characteristic equation of the form

$$1 + kG(s) = 0. \tag{7.8}$$

This form of a characteristic equation is quite common, as is illustrated by the following examples.

7.4.2 Example Consider the block diagram in Figure 7.14. Clearly

$$\frac{Y(s)}{R(s)} = \frac{kG(s)}{1+kG(s)},$$

so the characteristic equation of this transfer function is of the form of Equation 7.8. ■

7.4.3 Example Consider the block diagram in Figure 7.15. Clearly

$$\frac{Y(s)}{R(s)} = \frac{kG_1(s)}{1+kG_1(s)G_2(s)},$$

so the characteristic equation of this transfer function is of the form of Equation 7.8 if $G(s) = G_1(s)G_2(s)$. ■

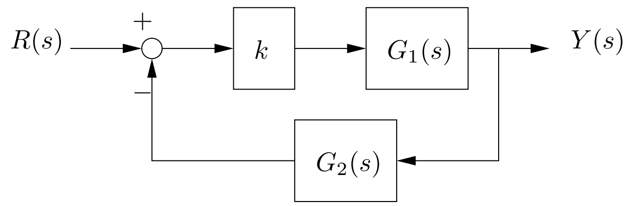

Figure 7.15. Block diagram for Example 7.4.3.

If the denominator of a transfer function is of the form of Equation 7.8, then the values of s which satisfy it also satisfy

$$G(s) = -\frac{1}{k}.$$

If we assume that $0 < k < +\infty$, then

$$\angle\left(-\frac{1}{k}\right) = \pm 180°.$$

Thus, the s values that satisfy it are such that

$$\angle G(s) = \pm 180°.$$

Thus, on the complex plane, any s value that satisfies Equation 7.8 must be such that

$$\sum_{i=1}^{m} \angle\,(s - z_i) - \sum_{j=1}^{n} (s - p_j) = \pm 180° \tag{7.9}$$

where the p_j and z_i are the poles and zeros of $G(s)$ respectively. Similarly

$$\frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{j=1}^{n} |s - p_j|} = \left|\frac{1}{k}\right|. \tag{7.10}$$

Equations 7.9 and 7.10 are the basis for developing the rules to plot a root locus plot.

7.4.4 Root Locus Plotting Rules

The following list is the steps to plot a root locus plot for a system with a characteristic equation of the form of Equation 7.8.

1. **Plot the poles and zeros of $G(s)$ on the complex plane.**

   Note that regardless of the value of k, Equation 7.8 will always have n solutions because if both sides of Equation 7.8 is multiplied by the denominator of $G(s)$, which is a nth order polynomial in s, the result is an nth order polynomial in s. Thus, the root locus has n "branches" of solutions. Also note that $G(s)$ has n poles. Since

$$\lim_{k\to 0}\left|\frac{1}{k}\right| = \infty$$

for k values near 0, the s values that satisfy Equation 7.8 must be such that

$$\lim_{k\to 0}|G(s)| = \infty,$$

which are precisely the poles of $G(s)$. Thus for k values near 0, the root locus is near the poles of $G(s)$.

Also since

$$\lim_{k\to\infty}\left|\frac{1}{k}\right| = 0$$

the values of s that satisfy Equation 7.8 must be such that

$$\lim_{k\to\infty}|G(s)| = 0,$$

which are precisely the zeros of $G(s)$. Note, however, that Equation 7.8 has n solutions, but $G(s)$ has only m zeros and we have assumed that $m < n$. Hence $n - m$ of the solutions mustgo elsewhere. Since

$$G(s) = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{j=1}^{n}(s - p_j)} = \frac{s^m + a_1 s^{m-1} + a_2 s^{m-2} + \cdots + a_{m-1}s + a_m}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + a_{n-1}s + a_n}$$

and $n > m$, then

$$\lim_{|s|\to\infty}|G(s)| = 0.$$

Hence, the other $n - m$ values of s which are such that

$$\lim_{k\to\infty}|G(s)| = 0,$$

are complex numbers s such that $|s| \to \infty$.

Hence, if we think of the root locus starting when $k = 0$ and ending as $k \to \infty$, then **the n branches of the root locus start at the n poles of $G(s)$ and end either at the m zeros of $G(s)$ or ∞.**

2. **On the real axis root locus is to the left of an odd number of poles plus zeros.**

   This rule focuses only on determining solutions to Equation 7.8 that are real. Consider Figure 7.16 which has the poles and zeros of the transfer function

$$G(s) = \frac{(s+2)(s-1)\left(s^2+2s+12\right)}{(s^2+8s+25)(s+6)(s-2)(s+8)} \tag{7.11}$$

   plotted in accordance with step 1.

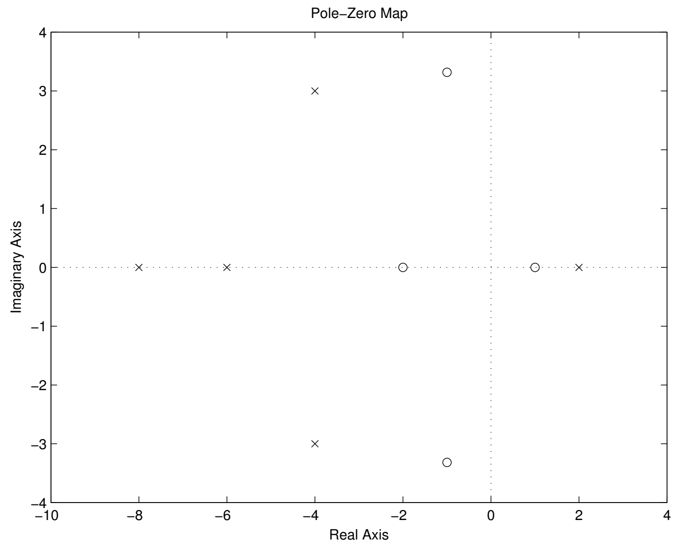

Figure 7.16. The poles and zeros for $G(s)$ in Equation 7.11.

Recall that the phase of $G(s)$ is the sum of all the angles from the zeros to s minus the sum of all the angles from the poles to s. Note that for an s value anywhere on the real axis, the contribution to the phase of $G(s)$ due to either the complex poles or zeros will be zero since the angle from each one in a complex conjugate pair exactly cancel each other as is illustrated in Figure 7.17.

Since the net contribution to $\angle G(s)$ for s values on the real axis by the complex conjugate poles and zeros is zero, we only need to consider the contribution to $\angle G(s)$ by the real poles and zeros. This is especially simple since the angles from all the poles and zeros on the real axis to s values on the real axis can only be $0°$ or $\pm 180°$.

For any real s values to the right of all the poles and zeros of $G(s)$ on the real axis $\angle G(s)$ must be zero since all the angles from the poles and zeros on the real axis are zero, as is illustrated in Figure 7.18. Now, proceeding along the real axis to the left, once s passes through the first pole, there will be a net change in $\angle G(s)$ of $-180°$. Hence, for any s value on the real axis between the pole located at $s = 2$ and the zero located at $s = 1$, $\angle G(s) = -180°$, as is illustrated in Figure 7.19.

Happily proceeding farther to the left on the real axis between the zero located at $s = 1$ and the zero located at $s = -2$, $\angle G(s) = 0°$ since $\angle G(s) = 180°$ (from the zero at $s = 1$) $+ -180°$ (from the pole at $s = 2$) $= 0°$.

Continuing in this manner yields obviously yields rule for Step 2. Fig-

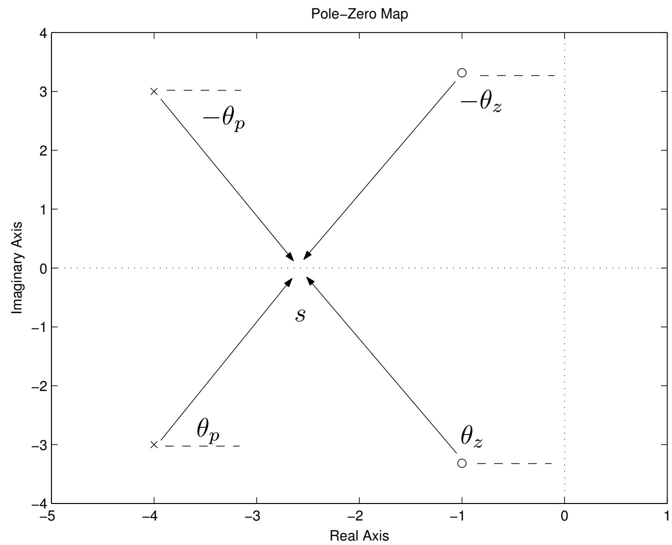

Figure 7.17. Phase contribution due to complex conjugate poles and zeros to $G(s)$ for $s \in \mathbb{R}$ is zero.

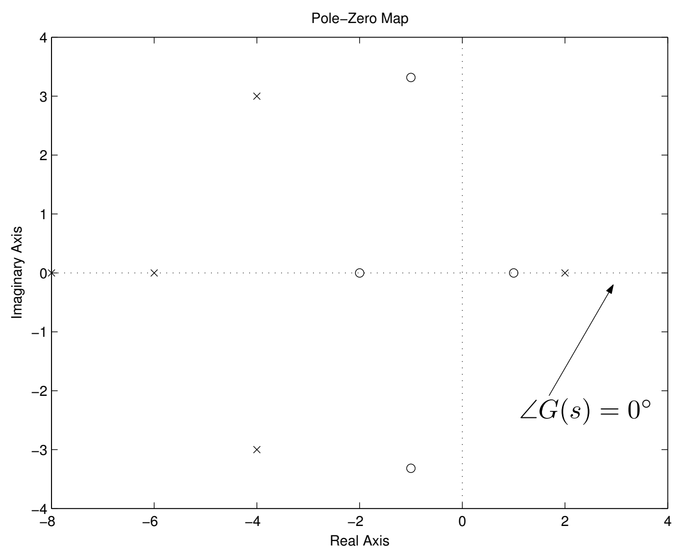

Figure 7.18. On the real axis to the right of all real poles and zeros of $G(s)$, $\angle G(s) = 0°$.

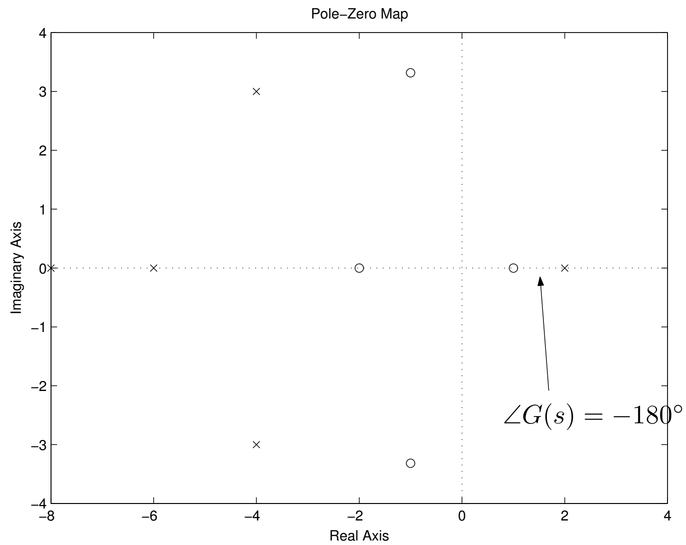

Figure 7.19. On the real axis to the left of an odd number of poles plus zeros of $G(s)$, $\angle G(s) = \pm 180°$.

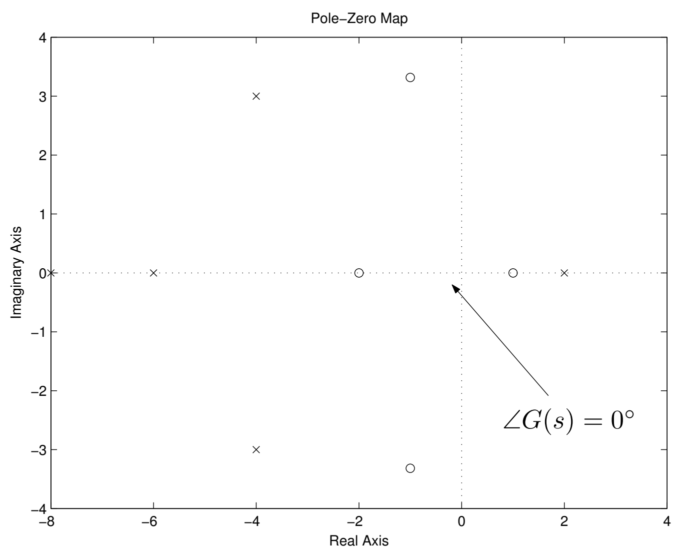

Figure 7.20. On the real axis to the left of an even number of poles plus zeros of $G(s)$, $\angle G(s) = 0°$.

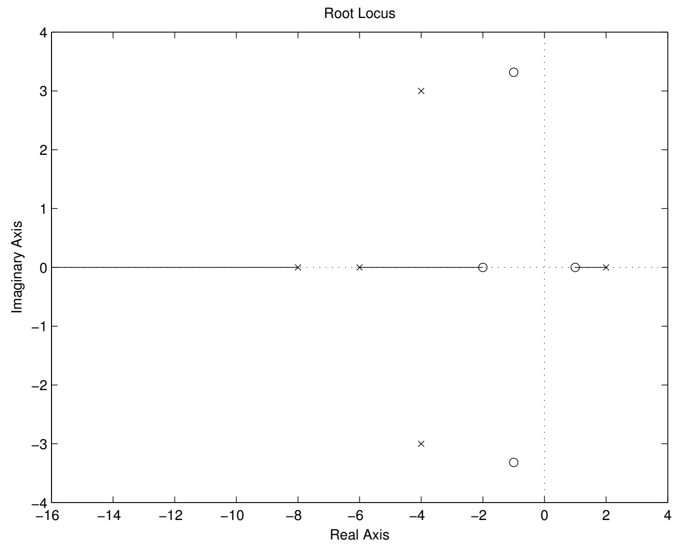

Figure 7.21. Segments of the root locus on the real axis for $G(s)$ given in Equation 7.11.

ure 7.21 illustrated the intervals of the real axis that correspond to this rule for the example consider thus far.

3. **As $k \to \infty$, $n-m$ branches of the root locus go to (complex) infinity along asymptotes with angles $\alpha = \frac{(2j-1)}{n-m}180^\circ$, where $j = 1, 2, \ldots, (n-m)$.**

Considering again the transfer function in Equation 7.11, Figure 7.22 illustrates a plot of the poles and zeros of $G(s)$ and an s value with a very large magnitude. Clearly, for such large values of s, the angles from all the poles and zeros of $G(s)$ are nearly identical. If this value is denoted by α, then

$$\begin{aligned}
\angle G(s) &= \sum_{i=1}^{m} \angle (s - z_i) - \sum_{j=1}^{n} (s - p_j) \\
&= m\alpha - n\alpha \\
&= (m - n)\,\alpha.
\end{aligned}$$

In order to satisfy Equation 7.8, $\angle G(s)$ must be equal to $\pm 180^\circ$ or $\pm 180^\circ +$

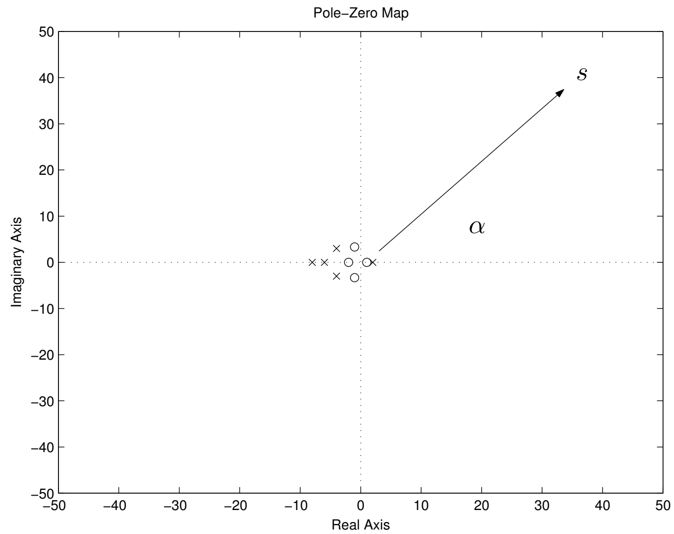

Figure 7.22. Determining $\angle G(s)$ for values of s with large magnitudes.

$j360°$ for $j = 0, 1, 2, \ldots$. Hence

$$\begin{aligned} \alpha &= \frac{\pm 180° + 2j180°}{m-n} \\ &= \frac{2j-1}{n-m}180° \end{aligned}$$

for $j = 1, 2, \ldots, (n-m)$.

4. **The asymptotes are centered on the real axis at the point given by $\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$.**

   There is not much to learn from the derivation of this.

5. **The point(s) where the locus either breaks away from or into the real axis is determined by solving $\frac{dK}{ds} = 0$.**

   Since the characteristic equation

   $$1 + kG(s) = 0$$

   may have either real or complex conjugate roots for various values of k, these "break away" or "break in" points are the points where the root switch from being a pair of real poles to a complex conjugate pair and conversely, respectively.

7.4.4 Example Consider

$$\begin{aligned} 1 + kG(s) &= 1 + k\frac{1}{(s+2)(s+4)} \\ &+ s^2 + 6s + (8+k) \end{aligned}$$

This has roots

$$s = -3 \pm \sqrt{1-k}$$

which are real for $k < 1$, complex for $k > 1$ and repeated for $k = 1$. These solutions are illustrated in Figure 7.23, which is nothing more than the root locus for $G(s)$.

Observe that solving the characteristic equation for k gives

$$k = -(s+2)(s+4).$$

Solving $\frac{dk}{ds} = 0$ gives

$$\frac{dk}{ds} = -(2s+3) = 0 \qquad \Longrightarrow \qquad s = -3,$$

which is exactly the value of s where the roots switch from two distinct real poles to a complex conjugate pair of poles. ■

While they are necessarily slightly more complicated to construct, as will be part of subsequent root locus examples, it is conversely possible for a complex conjugate pair of poles to switch to two real poles a k increases.

The basis for this rule is obvious upon a moment's reflection. If a distinct pair of real poles move closer together as k increases and then split into a complex conjugate pair, the value for k at which this occurs will be a local extremum along the real axis; hence, $\frac{dk}{ds} = 0$ at the point at which that occurs. Similarly, if a pair of complex conjugate poles move to the real axis and split into a distinct pair as k increases, the k value at which this occurs on the real axis is a local minimum. Note that not all the values of s for which $\frac{dk}{ds} = 0$ correspond to break in or break away points. This is because k as a function of s may have multiple extrema, but only some of them will happen to be real, and furthermore happen to be on the portions of the real axis where the root locus happens to exist.

7.4.5 Example Plot the root locus for the system illustrated in Figure 7.24. From block diagram algebra,

$$\frac{Y(s)}{R(s)} = \frac{k\left(\frac{1}{(s+2)(s+4)}\right)}{1 + k\left(\frac{1}{(s+2)(s+4)}\right)\left(\frac{1}{s+6}\right)},$$

so the characteristic equation is

$$1 + kG(s) = 0,$$

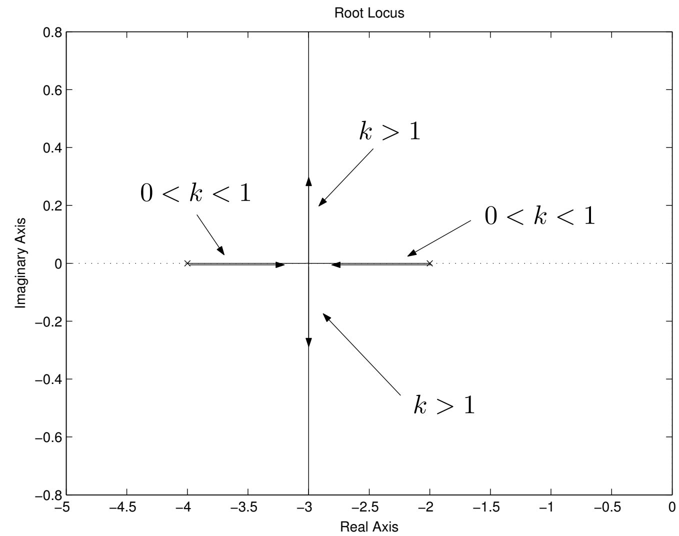

Figure 7.23. Roots of a characteristic equation may change from a real pair to a complex conjugate pair.

where

$$G(s) = \frac{1}{(s+2)\,(s+4)\,(s+6)}.$$

Applying the rules developed so far, constructs the root locus illustrated in Figure 7.25.

(a) The poles of $G(s)$ are located at $s = -2, -4$ and -6.

(b) On the real axis, the locus is between the two poles at $s = -2$ and $s = -4$ and to the left of the pole at $s = -6$.

(c) Since there are three poles and no zeros, $n = 3$ and $m = 0$. Hence there are three asymptotes. The angles are

$$\begin{aligned}\alpha_1 &= \frac{2(1)-1}{3}180^\circ = 60^\circ \\ \alpha_2 &= \frac{2(2)-1}{3}180^\circ = 180^\circ \\ \alpha_3 &= \frac{2(3)-1}{3}180^\circ = 300^\circ.\end{aligned}$$

(d) The asymptotes are centered at

$$\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{(-2-4-6)-(0)}{3} = -4.$$

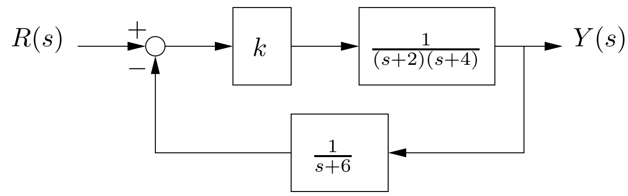

Figure 7.24. Block diagram for Example 7.4.5.

(e) Solving for k in the characteristic equation gives

$$k = -(s+2)(s+4)(s+6) = -\left(s^3 + 12s^2 + 44s + 48\right),$$

so

$$\frac{dk}{ds} = -\left(3s^2 + 24s + 44\right) = 0$$

which gives

$$\begin{aligned} s &= -4 \pm \frac{\sqrt{144 - 3 * 44}}{3} \\ &= -4 \pm \frac{2}{\sqrt{3}} \\ &= -5.1547, -2.8453. \end{aligned}$$

The first solution is not on the locus. The second is the breakaway point.

This is all the information necessary to construct the root locus. The final root locus plot is illustrated in Figure 7.25. ∎

The following is a slightly more complicated example.

7.4.6 Example Plot the solutions for

$$1 + kG(s) = 0$$

where

$$G(s) = \frac{s+8}{(s+2)(s+4)}.$$

Again, following the rules that have been developed thus far constructs the root locus plot.

(a) There is a pole at $s = -2$, another pole at $s = -4$ and a zero at $s = -8$.

(b) On the real axis, the root locus is between the two poles and then to the left of the zero.

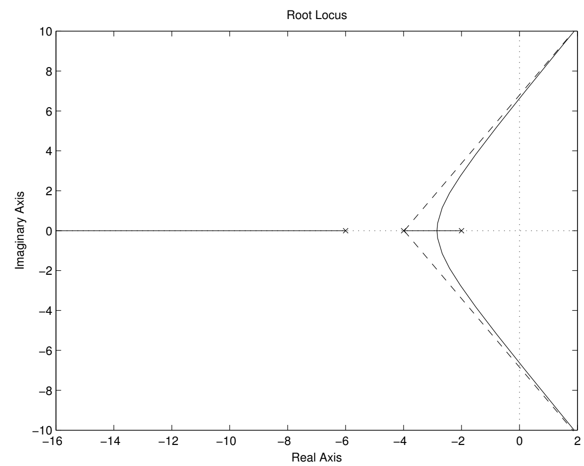

Figure 7.25. Root locus diagram for Example 7.4.5.

(c) Since $n = 2$ and $m = 1$, there is only one asymptote at $\alpha = 180°$, and, in fact, the root locus has already been completed there by the previous rule.

(d) Since the only asymptote is along the real axis, where it is centered on the real axis is meaningless.

(e) At $k = 0$ the locus starts at the poles. As $k \to \infty$ they approach either the zero or ∞ along the asymptote. Thus it must be the case that the locus breaks away between the two poles and breaks in to the left of the zero. Since there is no locus on the real axis between the pole at $s = -4$ and the zero at $s = -8$, the locus must do this. In particular, solving for k gives

$$k = -\frac{(s+2)(s+4)}{s+8}$$

and

$$\frac{dk}{ds} = -\frac{s^2 + 16s + 40}{(s+8)^2} = 0.$$

Hence,

$$s = -8 \pm 2\sqrt{6} = -12.899, -3.101.$$

The first value is the break in point to the left of the zero and the second value is the break away point between the two poles.

The final root locus plot is illustrated in Figure 7.26. Note that it is not possible to simply sketch the *exact path* that the locus takes

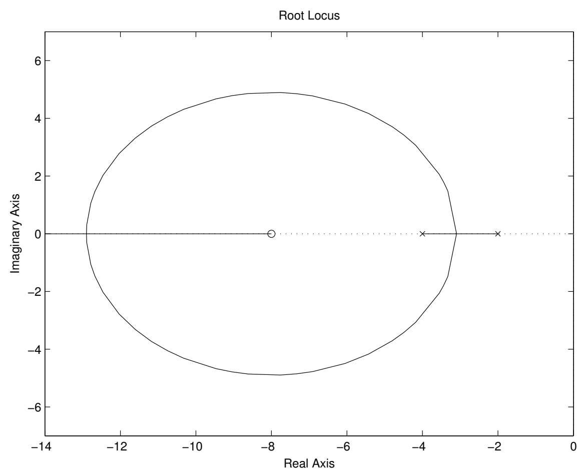

Figure 7.26. Root locus diagram for Example 7.4.6.

between the break away point and break in point. However, keep in mind that the root locus is a plot of the solutions of a relatively low order polynomial. Hence, it should be rather intuitive that it cannot follow a path with a large variation in curvature with many inflection points, *i.e.,* as long as the polynomial is not of a very large order, the roots cannot vary in some crazy manner as k is varied.

A good rule of thumb in such cases is to make the path somewhat of a semi-circle. If a more exact plot is necessary, then one must resort to tabulating various values for k and actually computing the roots of the characteristic equation for each value of k. ■

6. **Departure angles from the poles and arrival angles at the zeros that are not on the real axis are determined by selecting a point very near the pole or zero and applying the angle conditions.**

   Since
$$\angle G(s) = \sum_{i=1}^{m} \angle\,(s - z_i) - \sum_{i=1}^{n} \angle\,(s - p_i) = \pm 180^\circ,$$
   we can solve for any one of the angle terms, say the one corresponding to p_j, *i.e.,*
$$\angle\,(s - p_j) = \sum_{i=1}^{m} \angle\,(s - z_i) - \sum_{i=1, i\neq j}^{n} \angle\,(s - p_i) \pm 180^\circ. \qquad (7.12)$$

If a point is very close to p_j, *i.e.,* $s \approx p_j$, then Equation 7.12 can be written as

$$\angle(s - p_j) = \sum_{i=1}^{m} \angle(p_j - z_i) - \sum_{i=1, i \neq j}^{n} \angle(p_j - p_i) \pm 180^\circ. \tag{7.13}$$

Note that the right hand side of Equation 7.13 is composed of the angles from all the poles and zeros to the pole p_j.

A similar consideration gives the arrival angle for a complex zero as

$$\angle(s - z_j) = \pm 180^\circ + \sum_{i=1}^{n} \angle(z_j - p_i) - \sum_{i=1, i \neq j}^{m} \angle(z_j - z_i). \tag{7.14}$$

7.4.7 Remarks

(a) Equations 7.13 and 7.14 are only valid for s values very close to p_j and z_j respectively.

(b) For poles and zeros on the real axis, the departure and arrival angles are handled automatically by the "left of an odd number of poles plus zeros" rule; hence, applying this rule, while giving the correct answer, is redundant.

(c) Since complex roots of a polynomial equation always occur in complex conjugate pairs, it is only necessary to compute the departure or arrival angles for one of the two poles or zeros in a complex conjugate pair. The root locus must be symmetric about the real axis so the departure or arrival angle corresponding to the conjugate of the pole or zero just computed will simply be -1 times the angle computed. ■

A few examples are in order.

7.4.8 Example Plot the root locus for

$$G(s) = \frac{1}{(s+2)^2 + 9}.$$

(a) There are two poles located at $s = -2 \pm 3i$.

(b) There are no real poles or zeros, so there is no region on the real axis that is to the left of an odd number of poles plus zeros.

(c) There are two asymptotes. The asymptote angles are

$$\begin{aligned} \alpha_1 &= \frac{1}{2} 180^\circ = 90^\circ \\ \alpha_1 &= \frac{3}{2} 180^\circ = 270^\circ. \end{aligned}$$

Observe that the asymptote angles only depend on the number of poles and zeros of $G(s)$ and not the location of the poles and zeros. Thus, the angles are always 90° and 270° when there are two more poles than zeros.

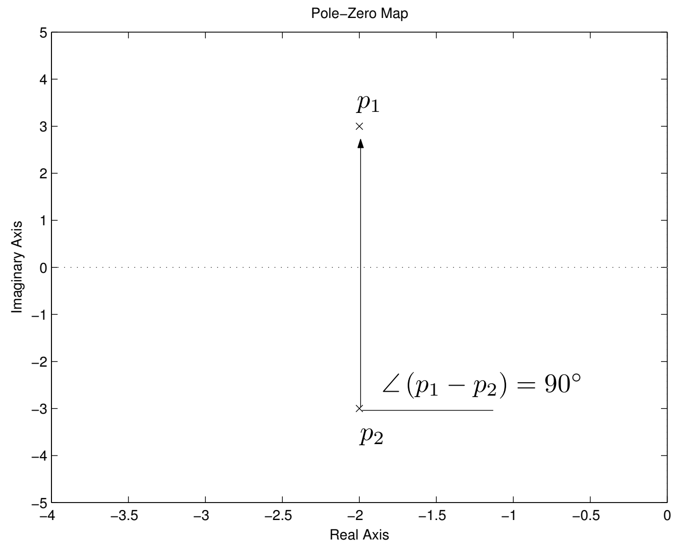

Figure 7.27. Departure angle computation for Example 7.4.8.

(d) The asymptotes are centered at

$$\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m} = \frac{-4}{2} = -2.$$

(e) The locus is never on the real axis, thus there are no break in or break away points.

(f) The departure angle from each pole is given by Equation 7.13. Let $p_1 = -2+3i$ and $p_2 = -2-3i$ as is illustrated in Figure 7.27. A simple computation gives for any $s \approx p_1$

$$\begin{aligned} \angle(s - p_1) &= \sum_{i=1}^{m} \angle(p_1 - z_i) - \sum_{i=1, i\neq 1}^{m} \angle(p_1 - p_i) \pm 180° \\ &= -90° \pm 180° \\ &= 90° \text{ or } -270°, \end{aligned}$$

which in either case is "straight up." One can compute the departure angle for p_2 or use the symmetry property to finally construct the root locus plot illustrated in Figure 7.28. ∎

The following is slightly more complicated.

7.4.9 Example Plot how the poles of the transfer function for the system illustrated in Figure 7.29 change as k varies from 0 to $+\infty$.

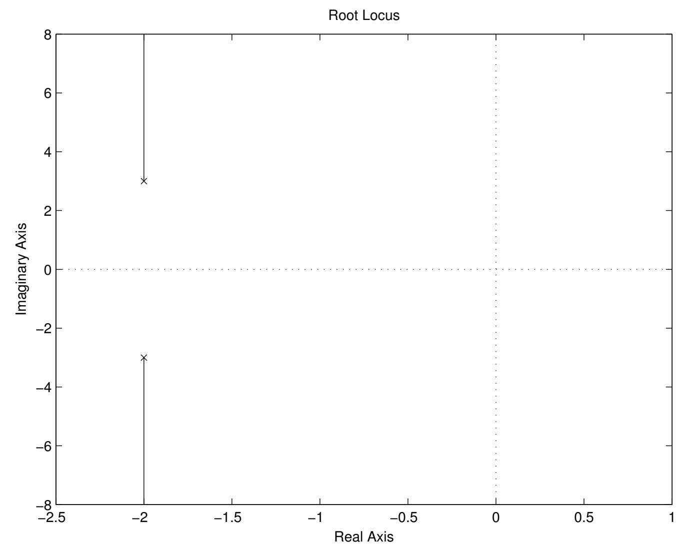

Figure 7.28. Departure angle computation for Example 7.4.8.

(a) There are two located at $s = -2 \pm 3i$ and one zero at $s = -8$.

(b) The root locus on the real axis is to the left of the zero.

(c) Since there are two poles and one zero, there is only one asymptote with angle $\alpha = \pm 180°$.

(d) Computing the center point of the asymptote is meaningless since it is along the real axis.

(e) Solving for k gives

$$k = -\frac{s^2 + 4s + 13}{s + 8}$$

and differentiating with respect to s gives

$$\frac{dk}{ds} = -\frac{s^2 + 16s + 19}{(s + 8)^2} = 0$$

which gives

$$\begin{aligned} s &= -8 \pm 3\sqrt{5} \\ &= -14, 71, -1.29. \end{aligned}$$

The former is the break in point to the left of the zero. The latter is not on the locus on the real axis, so is ignored.

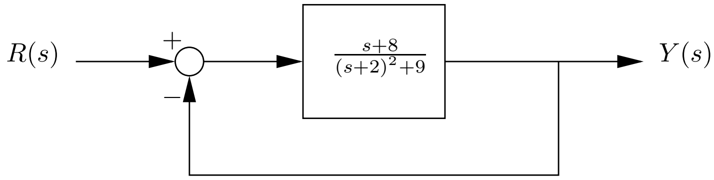

Figure 7.29. Block diagram for Example 7.4.9.

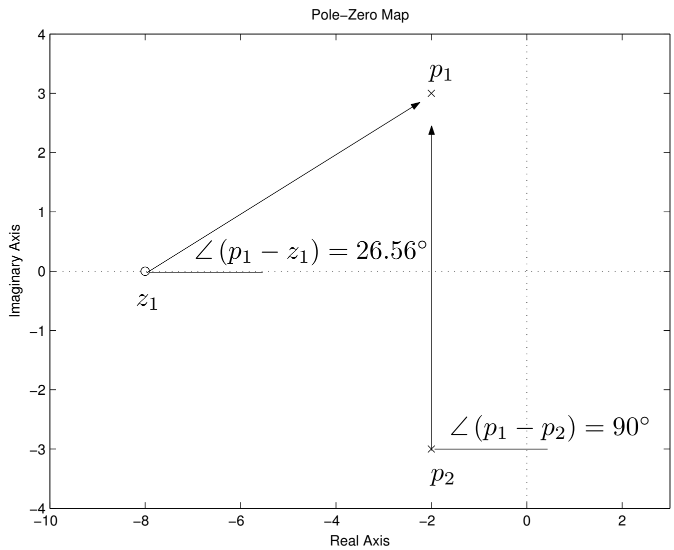

Figure 7.30. Departure angle computation for Example 7.4.9.

(f) Referring to Figure 7.30, the departure angle from p_1 is computed as the following for $s \approx p_1$

$$\begin{aligned}\angle(s - p_1) &= 26.56° - 90° \pm 180° \\ &= 116.57°.\end{aligned}$$

The complete root locus is illustrated in Figure 7.31. Note that the locus leaves p_1 at an angle of 116.57° and leaves p_2 at an angle of −116.57°. ∎

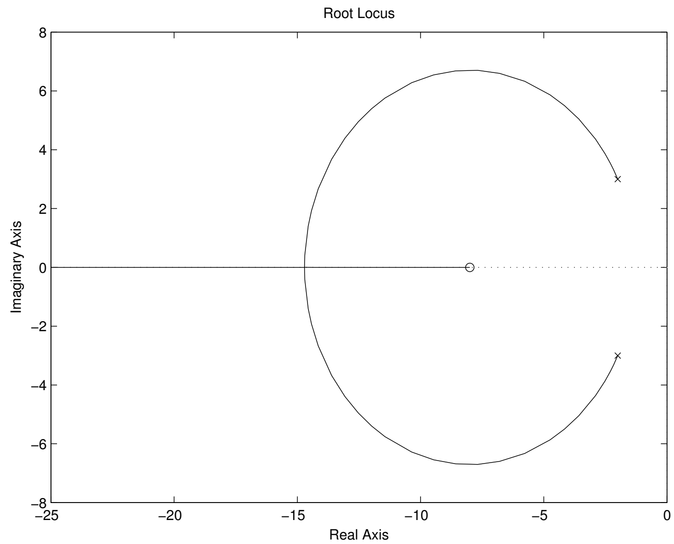

Figure 7.31. Complete root locus plot for Example 7.4.9.

7.5 Problems

(a) Plot the solutions to
$$1 + kG(s) = 0$$
for
$$G(s) = \frac{1}{(s+4)\left((s+2)^2+9\right)}.$$

(b) Plot the solutions to
$$1 + kG(s) = 0$$
for
$$G(s) = \frac{s+8}{(s+2)(s+4)(s+6)}$$

(c) Plot the solutions to
$$1 + kG(s) = 0$$
for
$$G(s) = \frac{s+8}{(s+2)(s+4)}.$$

(d) Plot the solutions to
$$1 + kG(s) = 0$$

for

$$G(s) = \frac{s+6}{(s+4)\left((s+2)^2+9\right)}.$$

Bibliography

[1] Joel N. Franklin. *Matrix Theory.* Prentice-Hall, Inc., 1968.

[2] Morris W. Hirsch and Stephen Smale. *Differential Equations, Dynamical Systems, and Linear Algebra.* Academic Press, Inc., 1974.