University of Notre Dame
Aerospace and Mechanical Engineering

AME 301: Differential Equations, Vibrations and Controls I
Exam 1

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October 6, 2004

NAME: ______________________________

- Do not start or turn the page until instructed to do so.
- You have 50 minutes to complete this exam.
- This is an open book exam. You may consult the course texts, any other text book, your class notes, homework solutions and your own homework sets. You may **not** use a calculator.
- There are four problems, each is worth 25 points.
- Your grade on this exam will constitute 25% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in the space provided on each page. If you need more space, use the back of the pages or use additional sheets of paper as necessary.

"It is ever so with the things that Men begin: there is a frost in Spring, or a blight in Summer, and they fail of their promise," said Gimli.

Yet seldom do they fail of their seed," said Legolas. "And that will lie in the dust and rot to spring up again in times and places unlooked-for. The deeds of Men will outlast us, Gimli."

"And yet come to naught in the end but might-have-beens, I guess," said the Dwarf.

"To that, the Elves do not know the answer," said Legolas.

— J.R.R. Tolkien, *The Lord of the Rings*

1. Find $x(t)$ such that

$$\begin{aligned} \dot{x} - 2x &= 5 \\ x(0) &= -\frac{3}{2}. \end{aligned}$$

(25 points)

You can do this one of three ways.

(a) Using an integrating factor we have

$$\mu(t) = e^{\int -2dt} = e^{-2t}$$

so

$$x(t) = e^{2t}\left(\int 5e^{-2t}dt + c\right) = -\frac{5}{2} + ce^{2t}.$$

Evaluating the initial condition gives $c = 1$ so

$$x(t) = e^{2t} - \frac{5}{2}.$$

(b) You can also use undetermined coefficients. Computing the homogeneous solution gives a characteristic equation of

$$\lambda - 2 = 0 \qquad \Longrightarrow x_h(t) = e^{2t}.$$

Assuming a particular solution that is a constant $x_p = A$ and substituting gives

$$x_p = -\frac{5}{2}$$

so the complete solution is

$$x(t) = x_h(t) + x_p(t) = e^{2t} - \frac{5}{2}.$$

(c) Finally, you could recognize that this is exactly Equation 2 on page 32 of the course text with $a = -2$ and $b = 5$ and substitute into Equation 5 directly gives

$$x(t) = -\frac{5}{2} + ce^{2t},$$

and just like above the initial condition gives $c = 1$ so

$$x(t) = -\frac{5}{2} + e^{2t}.$$

2. Find $x(t)$ such that

$$\begin{aligned}\ddot{x} - 4\dot{x} + 4x &= 2e^{2t} \\ x(0) &= 0 \\ \dot{x}(0) &= 0.\end{aligned}$$

(25 points)

You can use either undetermined coefficients or variation of parameters. In either case, you need two linearly independent homogeneous solutions. Assuming $x_h(t) = e^{\lambda t}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \qquad \Longrightarrow \qquad \lambda = 2$$

a repeated root. Thus

$$x_h(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

(a) Using undetermined coefficients, we first are inclined to assume a solution of the form $x_p(t) = Ae^{2t}$. However, this is one of the homogeneous solutions. Naturally, we next would try $x_p(t) = Ate^{2t}$ which is also a homogeneous solution. Thus we must assume

$$x_p(t) = At^2 e^{2t}.$$

Thus $\dot{x}_p(t) = 2Ate^{2t} + 2At^2e^{2t}$ and $\ddot{x}_p(t) = 2Ae^{2t} + 8Ate^{2t} + 4At^2e^{2t}$. Substituting into the equation and simplifying gives

$$2Ae^{2t} = 2e^{2t} \qquad \Longrightarrow \qquad A = 1.$$

Thus

$$x(t) = x_h(t) + x_p(t) = c_1 e^{2t} + c_2 t e^{2t} + 2t^2 e^{2t}.$$

Evaluating the initial conditions gives

$$x(0) = c_1 = 0 \qquad \Longrightarrow \qquad c_1 = 0$$

and

$$\dot{x}(0) = c_2 = 0 \qquad \Longrightarrow \qquad c_2 = 0.$$

Thus

$$x(t) = t^2 e^{2t}.$$

(b) Using variation of parameters we first need the Wronskian

$$W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix} = e^{4t}.$$

Substituting into Equation 28 on page 189 gives

$$\begin{aligned} x_p(t) &= -e^{2t} \int_{t_0}^{t} \frac{2se^{2s}e^{2s}}{e^{4s}} ds + te^{2t} \int_{t_0}^{t} \frac{2e^{2t}e^{2s}}{e^{4s}} ds \\ &= -e^{2t} \left(s^2\right)\Big|_{t_0}^{t} + te^{2t} \left(2s\right)\Big|_{t_0}^{t} \\ &= t^2 e^{2t} \end{aligned}$$

(picking $t_0 = 0$). The initial conditions evaluate exactly as above, so

$$x(t) = t^2 e^{2t}.$$

3. Consider

$$\begin{aligned} \ddot{x} - 4\dot{x} + 4x &= 2e^{2t} \\ x(0) &= 0 \\ \dot{x}(0) &= 0. \end{aligned} \tag{1}$$

(a) Write this second order equation as a system of two first order equations. (5 points)

If we let $x_1 = x$ and $x_2 = \dot{x}$ then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 2e^{2t} - 4x_1 + 4x_2 \end{bmatrix}.$$

(b) The following code is used to determine an approximate numerical solution to Equation 1 using Euler's method. Fill in the blanks. (10 points)

```
#include<stdio.h>
#include<math.h>

main() {
  double t,dt,x1,x2;
  double t_start=0.0,t_finish=25.0;
  FILE *fp;

  dt = 0.01;
  x1 = 0.0;
  x2 = 0.0;

  fp = fopen("data.d","w");

  for(t=t_start;t<=t_finish;t+=dt) {
    fprintf(fp,"%.3f\t%.3f\t%.3f\n",t,x1,x2);
    x1 += x2*dt;
    x2 += (2.0*exp(2.0*t) - 4.0*x1 + 4.0*x2)*dt;
  }

  fclose(fp);
}
```

(c) The following code is used to determine an approximate numerical solution to Equation 1 using the 4th order Runge-Kutta method. Fill in the blanks. (10 points)

```
#include<stdio.h>
#include<math.h>

main() {
```

```
  double t,dt,x1,x2;
  double k1,k2,k3,k4,l1,l2,l3,l4;
  double t_start=0.0,t_finish=25.0;
  FILE *fp;

  dt = 0.1;
  x1 = 0.0;
  x2 = 0.0;

  fp = fopen("data.d","w");

  for(t=t_start;t<=t_finish;t+=dt) {
    fprintf(fp,"%.3f\t%.3f\t%.3f\n",t,x1,x2);

    k1 = x2*dt;
    l1 = (2.0*exp(2.0*t) - 4.0*x1 + 4.0*x2)*dt;

    k2 = (x2+l1/2.0)*dt;
    l2 = (2.0*exp(2.0*(t+dt/2.0)) - 4.0*(x1+k1/2.0) + 4.0*(x2+l1/2.0))*dt;

    k3 = (x2+l2/2.0)*dt;
    l3 = (2.0*exp(2.0*(t+dt/2.0)) - 4.0*(x1+k2/2.0) + 4.0*(x2+l2/2.0))*dt;

    k4 = (x2+l3)*dt;
    l4 = (2.0*exp(2.0*(t+dt)) - 4.0*(x1+k3) + 4.0*(x2+l3))*dt;

    x1 += (k1 + 2.0*k2 + 2.0*k3 + k4)/6.0;
    x2 += (l1 + 2.0*l2 + 2.0*l3 + l4)/6.0;
  }

  fclose(fp);
}
```

Note, there is also the alternative, but equivalent way, that the book presents that places the dt terms in other places.

4. Figure 1 is a plot of the solution to

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= F_0 \cos\omega t \\ x(0) &= 0 \\ \dot{x}(0) &= 0 \end{aligned} \tag{2}$$

where $m = 1$, $b = 0.5$, $k = 1$, $\omega = 1$ and $F_0 = 1$.

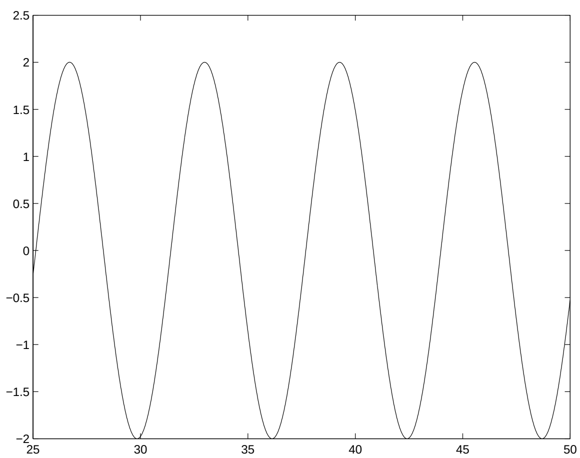

Figure 1. Response of system in equation 2.

Match each of the following four cases with the plots illustrated in Figures 2 through 5 and explain your reasoning. (6 points each — 1 free point!)

- In each case only one of the parameters is different from the system given above and is indicated in **bold** type.
- It is acceptable that your reasoning be based upon a comparison between the given figures and Figure 1. It is not necessary to use any equations in your explanation, but it may be helpful.
- The time axis starts at $t = 25$ in each figure; therefore, it is acceptable to only consider the particular solution. By the time $t = 25$ the effect of the homogeneous solution is negligible.
- Note that scale of the axis of the ordinate varies dramatically among the various figures.

With the give parameter values,

$$\omega_n = 1 \qquad \text{and} \qquad \zeta = \frac{1}{4}$$

so this is case 13 in the vibrations summary handout which gives

$$x_p(t) = \frac{F_0}{\sqrt{\left(k - \omega^2 m\right)^2 + \left(b\omega\right)^2}} \cos\left(\omega t - \phi\right). \tag{3}$$

(a) Figure 2 is where $m = 1$, $b = 0.5$, $k = 1$, $\omega = \mathbf{2}$ and $F_0 = 1$ because the forcing frequency is twice as large; hence, the particular solution will have a frequency twice as large by comparison to Figure 1.

(b) Figure 4 is where $m = 1$, $b = 0.5$, $k = 1$, $\omega = 1$ and $\mathbf{F_0 = 2}$. because clearly from Equation 3 doubling the forcing amplitude will double the magnitude of the response.

(c) Figure 5 is where $m = 1$, $b = 0.5$, $\mathbf{k = 2}$, $\omega = 1$ and $F_0 = 1$. because increasing k will increase the magnitude of the denominator in Equation 3 reducing the amplitude of the particular solution. By comparison with Figure 1, Figure 5 has a smaller amplitude.

(d) Figure$\tilde{3}$ is where $m = 1$, $\mathbf{b = 0.0}$, $k = 1$, $\omega = 1$ and $F_0 = 1$. because this is the only possible case for the response which corresponds to undamped resonance.

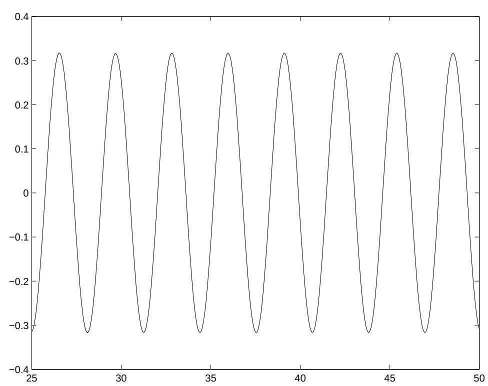

Figure 2.

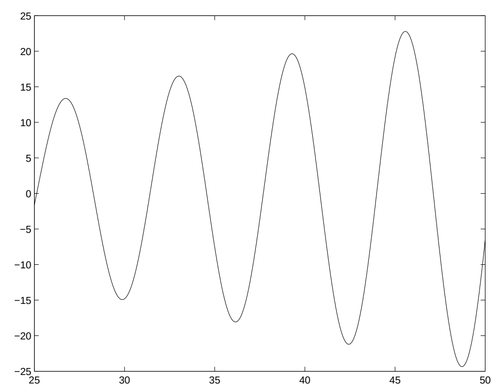

Figure 3.

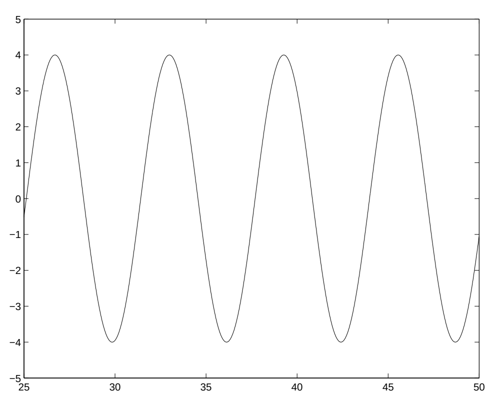

Figure 4.

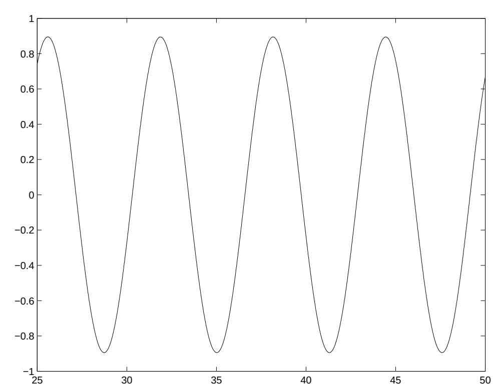

Figure 5.