Method of Variation of Parameters for Constant or Variable Coefficients for 2nd Order Linear Differential Equations

AME 30314 Engineering Differential Equations September 12, 2008 Michelle M. Michalenko



Overview

- Method of Variation of Parameters
- Review of 1st Order Differential Equations
 - Constant Coefficients
 - Variable Coefficients
- 2nd Order Diff EQs Variation of Parameters
 - Constant Coefficients
 - Variable Coefficients



Variation of Parameters

- 1. Find the homogeneous solution $x_h(t)$
- 2. Assume the particular solution has the form,

 $\mathbf{x}_{\mathrm{p}}(t) = \mu(t) \mathbf{x}_{\mathrm{h}}(t)$

- 3. Differentiate $x_p(t)$ as needed and substitute into differential equation.
- 4. Determine equation for $\mu(t)$. If possible solve for $\mu(t)$.
- 5. Then the differential equation has solution of the form,

1st Order: $x(t) = \mu(t) x_h(t)$ 2nd Order: $x(t) = \mu_1(t) x_1(t) + \mu_2(t) x_2(t)$



Variation of Parameters: 1st Order Differential Equations

Consider the ordinary, 1st Order, Linear, inhomogeneous Differential Equations:

Constant Coefficients Variable Coefficients* $\dot{x}(t) + \alpha x(t) = f(t)$ $\dot{x}(t) + h(t)x(t) = f(t)$

*For variable coefficients finding the homogeneous solution is going to change (no longer just $x(t) = re^{rt}$) Variation of Parameters: 1st Order Differential Equations To Solve:

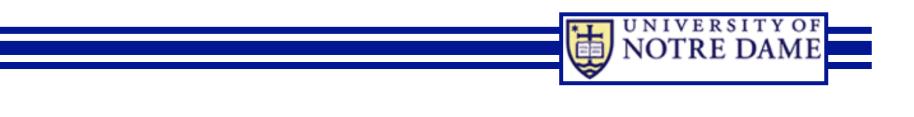
1. Solve for the homogeneous solution.

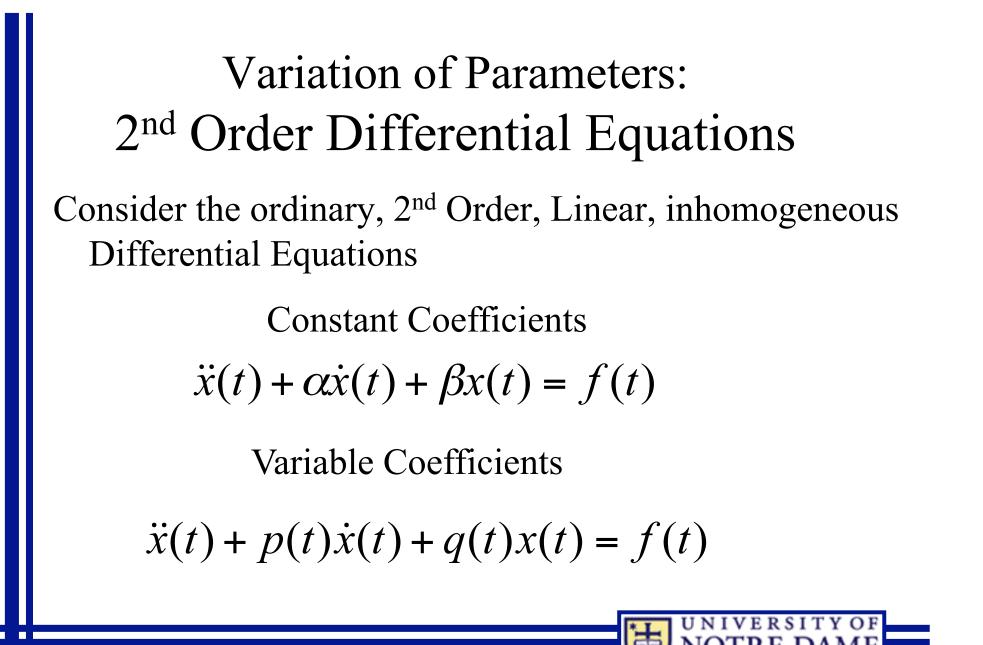
 $\dot{x}(t) + h(t)x(t) = 0$

2. Assume a particular solution of the form,

 $x(t) = \mu(t) x_h(t)$

- 3. Differentiate and substitute x(t) into original equation.
- 4. Solve for $\mu(t)$.





Variation of Parameters: 2nd Order Differential Equations To Solve:

1. Solve for the homogeneous solution. $\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0$

2. Assume a particular solution of the form,

$$x(t) = \mu_1(t)x_1(t) + \mu_2(t)x_2(t)$$
(2)

Variation of Parameters: 2nd Order Differential Equations

3. Differentiate once:

$$\dot{x}_p = \dot{\mu}_1 x_1 + \mu_1 \dot{x}_1 + \dot{\mu}_2 x_2 + \mu_2 \dot{x}_2$$

4. Make the requirement that

$$\dot{\mu}_1 x_1 + \dot{\mu}_2 x_2 = 0 \qquad (3)$$

Variation of Parameters: 2nd Order Differential Equations 5. Because of (4) we can simplify x_p

$$\dot{x} = \dot{\mu}_{1}x_{1} + \mu_{1}\dot{x}_{1} + \dot{\mu}_{2}x_{2} + \mu_{2}\dot{x}_{2}$$

$$\dot{x} = \mu_1 \dot{x}_1 + \mu_2 \dot{x}_2$$

6. So when you differentiate again you obtain,

$$\ddot{x} = \dot{\mu}_1 \dot{x}_1 + \mu_1 \ddot{x}_1 + \dot{\mu}_2 \dot{x}_2 + \mu_2 \ddot{x}_2$$

Variation of Parameters: 2nd Order Differential Equations

7. Substitute x (t) into original equation. Rearranging you obtain,

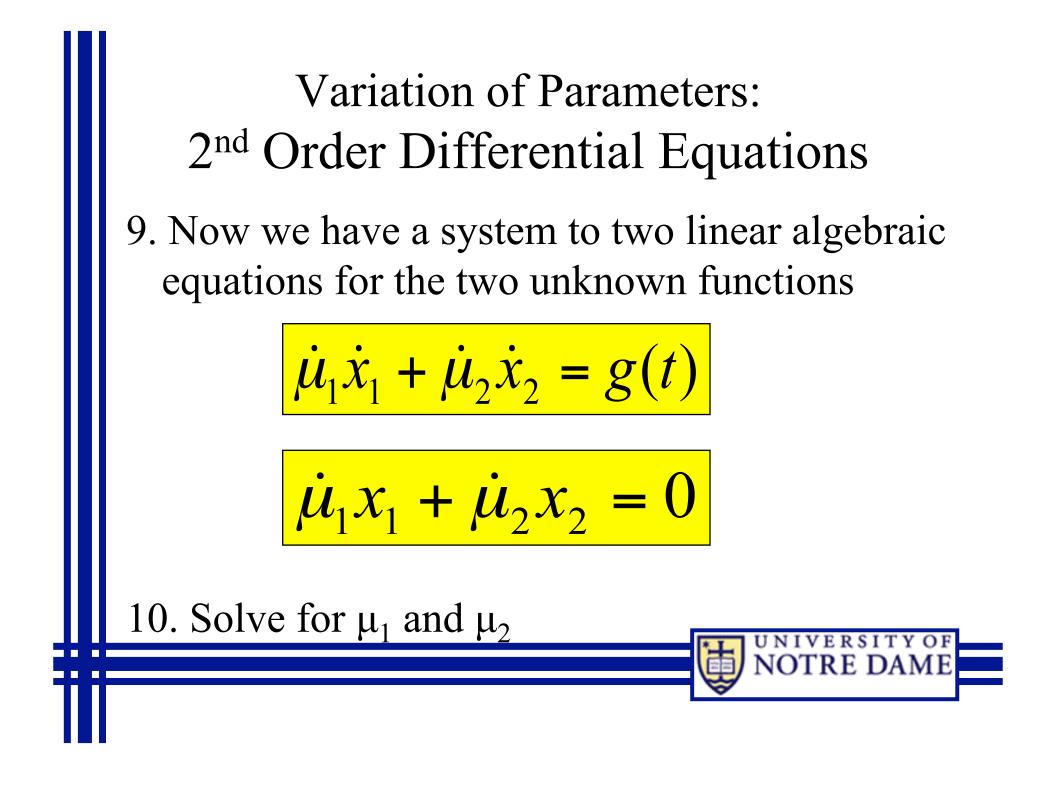
$$\ddot{x} + p\dot{x} + qx = \mu_1(\ddot{x}_1 + p\dot{x}_1 + qx_1) + \mu_2(\ddot{x}_2 + p\dot{x}_2 + qx_2)$$
(4)
$$+ \dot{\mu}_1\dot{x}_1 + \dot{\mu}_2\dot{x}_2 = g(t)$$

Variation of Parameters: 2nd Order Differential Equations

- 8. Each expression in parentheses the Equation
 (4) is zero because both x₁ and x₂ are solutions of the homogeneous equation.
 - Therefore Equation (4) reduces to

$$\dot{\mu}_1 \dot{x}_1 + \dot{\mu}_2 \dot{x}_2 = g(t)$$





Example 1

Find the solution of

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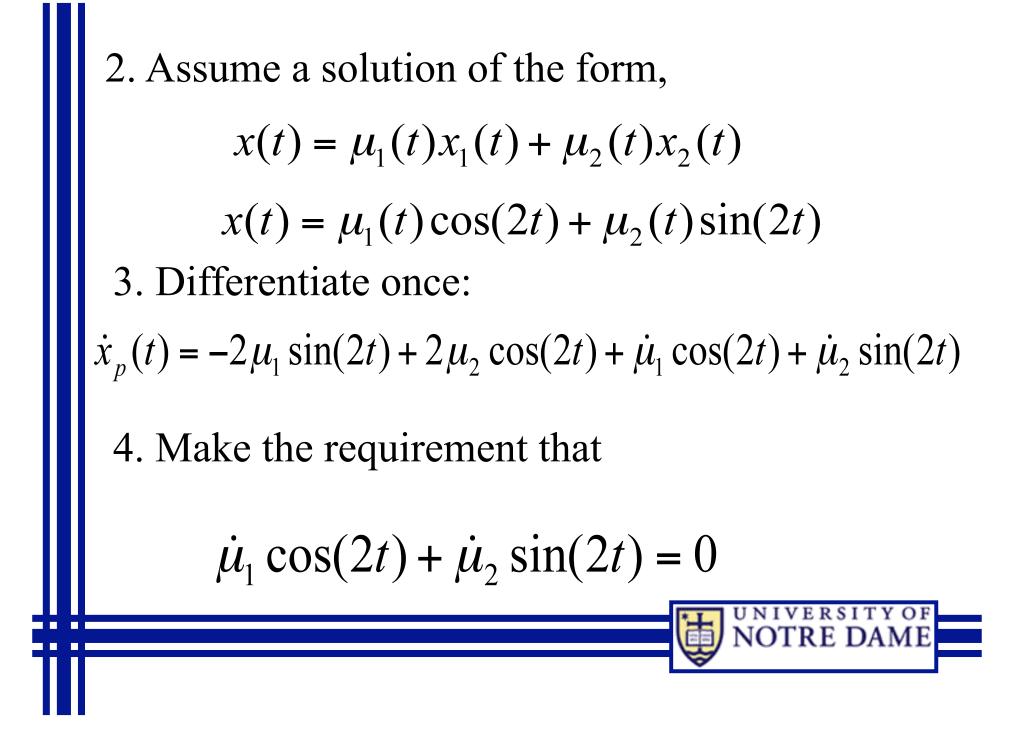
$$\ddot{x} + 4x = 3\csc(t)$$
1. Solve for homogeneous equation

$$\ddot{x} + 4x = 0$$

$$x_h = c_1 \cos(2t) + c_2 \sin(2t)$$

$$x_1 = c_1 \cos(2t) \qquad x_2 = c_2 \sin(2t)$$

$$with the equation of the equation o$$



5. Because of (4) we can simplify x $\dot{x}(t) = -2\mu_1 \sin(2t) + 2\mu_2 \cos(2t)$

6. So when you differentiate again you obtain, $\ddot{x}(t) = -4\mu_1 \cos(2t) - 4\mu_2 \sin(2t) - 2\dot{\mu}_1 \sin(2t) + 2\dot{\mu}_2 \cos(2t)$

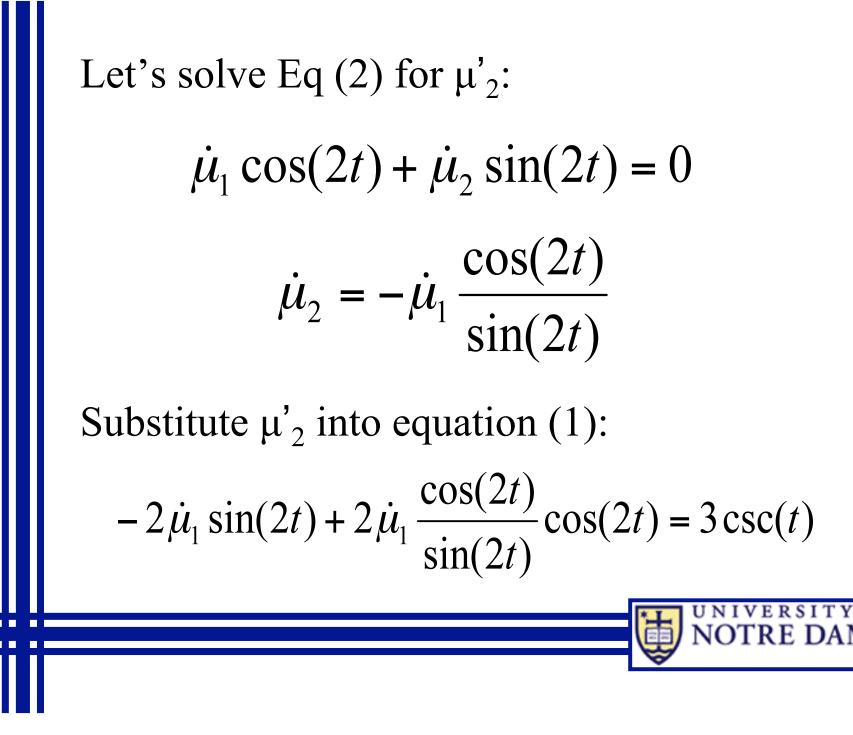
If we substitute x" and x into the original equation and simplify we find that μ_1 and μ_2 must satisfy

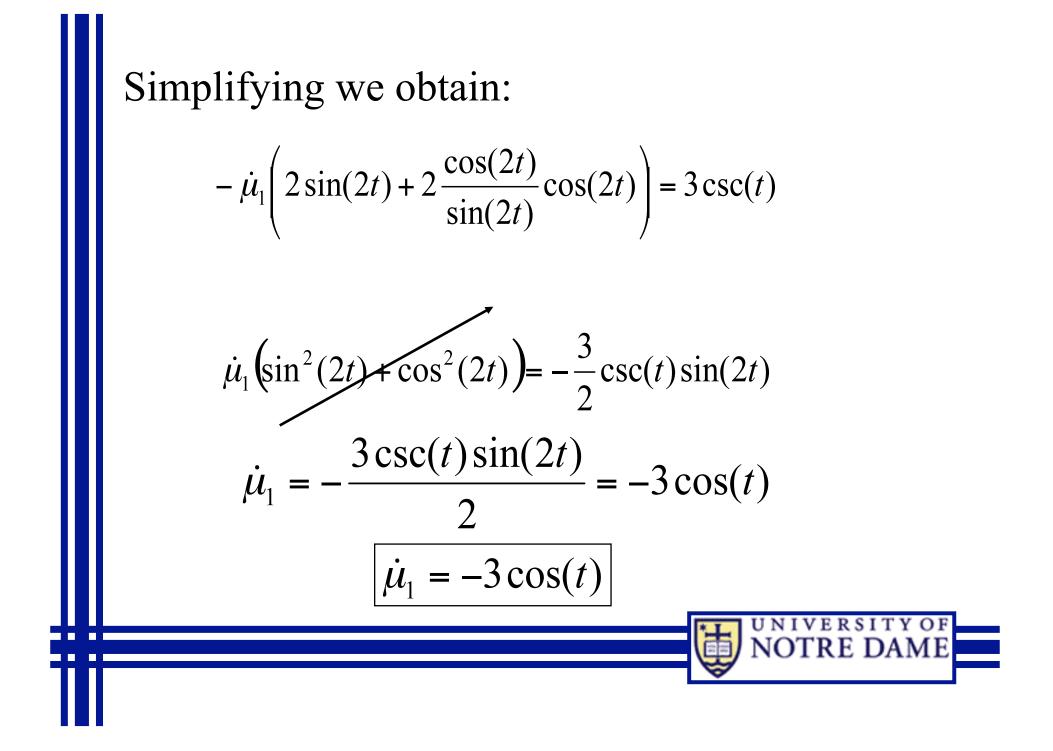
$$-2\dot{\mu}_1\sin(2t) + 2\dot{\mu}_2\cos(2t) = 3\csc(t)$$

Now we have 2 linear algebraic equations and 2 unknowns:

(1) $-2\dot{\mu}_1 \sin(2t) + 2\dot{\mu}_2 \cos(2t) = 3\csc(t)$ (2) $\dot{\mu}_1 \cos(2t) + \dot{\mu}_2 \sin(2t) = 0$







Further, now we can plug in and solve for μ_2 :

$$\dot{\mu}_2 = -\dot{\mu}_1 \frac{\cos(2t)}{\sin(2t)} \qquad \qquad \dot{\mu}_1 = -3\cos(t)$$

$$\dot{\mu}_2 = -(-3\cos(t))\frac{\cos(2t)}{\sin(2t)}$$

$$\dot{\mu}_2 = \frac{3}{2}\csc(t) - 3\sin(t)$$



Having obtained μ'_1 and μ'_2 we now can integrate to obtain μ_1 and μ_2

$$\mu_1(t) = -3\sin(t) + c_1$$

$$\mu_2(t) = \frac{3}{2} \ln \left| \csc(t) - \cot(t) \right| + 3\cos(t) + c_2$$

Finally substituting in for the answer

$$x(t) = \mu_1(t)\cos(2t) + \mu_2(t)\sin(2t)$$



Sections in Notes

- <u>Section 2.3.2/2.3.3</u> Variation of Parameters for 1st Order Differential Equations
- <u>Section 3.4.2</u> Variation of Parameters for 2nd
 Order Differential Equations



nth Order systems

- If you have the differential equation $P_n(x)y^{(n)} + P_{n-1}(x)y^{n-1} + \dots + P_1(x)y' + P_0(x)y = f(x)$
- We propose that the particular solution is of the form $y_p = \sum_{i=1}^n u_i(x) y_i(x)$
- Then the requirement is set that

$$\sum_{i=1}^{n} u'_{i} y_{i} = 0$$