# Method of Variation of Parameters for Constant or Variable Coefficients for 2nd Order Linear Differential Equations

AME 30314 Engineering Differential Equations

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# Overview

- Method of Variation of Parameters
- Review of 1st Order Differential Equations
  - Constant Coefficients
  - Variable Coefficients
- 2nd Order Diff EQs Variation of Parameters
  - Constant Coefficients
  - Variable Coefficients

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# Variation of Parameters

1. Find the homogeneous solution $x_h(t)$
2. Assume the particular solution has the form,

   $$x_p(t) = \mu(t)\, x_h(t)$$

3. Differentiate $x_p(t)$ as needed and substitute into differential equation.
4. Determine equation for $\mu(t)$. If possible solve for $\mu(t)$.
5. Then the differential equation has solution of the form,

   1st Order: $x(t) = \mu(t)\, x_h(t)$

   2nd Order: $x(t) = \mu_1(t)\, x_1(t) + \mu_2(t)\, x_2(t)$

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# Variation of Parameters: $1^{st}$ Order Differential Equations

Consider the ordinary, $1^{st}$ Order, Linear, inhomogeneous Differential Equations:

| Constant Coefficients | Variable Coefficients* |
| --- | --- |
| $\dot{x}(t) + \alpha x(t) = f(t)$ | $\dot{x}(t) + h(t)x(t) = f(t)$ |

*For variable coefficients finding the homogeneous solution is going to change
(no longer just $x(t) = re^{rt}$ )

# Variation of Parameters: $1^{st}$ Order Differential Equations

To Solve:

1. Solve for the homogeneous solution.

$$\dot{x}(t) + h(t)x(t) = 0$$

2. Assume a particular solution of the form,

$$x(t) = \mu(t)x_h(t)$$

3. Differentiate and substitute x(t) into original equation.

4. Solve for μ(t).

# Variation of Parameters: 2nd Order Differential Equations

Consider the ordinary, 2nd Order, Linear, inhomogeneous Differential Equations

Constant Coefficients

$$\ddot{x}(t) + \alpha\dot{x}(t) + \beta x(t) = f(t)$$

Variable Coefficients

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = f(t)$$

# Variation of Parameters: 2nd Order Differential Equations

To Solve:

1. Solve for the homogeneous solution.

$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0$$

2. Assume a particular solution of the form,

$$\boxed{x(t) = \mu_1(t)x_1(t) + \mu_2(t)x_2(t)} \qquad (2)$$

# Variation of Parameters:
# 2nd Order Differential Equations

3. Differentiate once:

$$\dot{x}_p = \dot{\mu}_1 x_1 + \mu_1 \dot{x}_1 + \dot{\mu}_2 x_2 + \mu_2 \dot{x}_2$$

4. Make the requirement that

$$\boxed{\dot{\mu}_1 x_1 + \dot{\mu}_2 x_2 = 0} \qquad (3)$$

# Variation of Parameters: 2nd Order Differential Equations

5. Because of (4) we can simplify $x_p$

$$\dot{x} = \cancel{\dot{\mu}_1 x_1} + \mu_1 \dot{x}_1 + \cancel{\dot{\mu}_2 x_2} + \mu_2 \dot{x}_2$$

$$\dot{x} = \mu_1 \dot{x}_1 + \mu_2 \dot{x}_2$$

6. So when you differentiate again you obtain,

$$\ddot{x} = \dot{\mu}_1 \dot{x}_1 + \mu_1 \ddot{x}_1 + \dot{\mu}_2 \dot{x}_2 + \mu_2 \ddot{x}_2$$

# Variation of Parameters: 2nd Order Differential Equations

7. Substitute x (t) into original equation. Rearranging you obtain,

$$
\begin{aligned}
\ddot{x} + p\dot{x} + qx = {} & \mu_1(\ddot{x}_1 + p\dot{x}_1 + qx_1) \\
& + \mu_2(\ddot{x}_2 + p\dot{x}_2 + qx_2) \qquad (4) \\
& + \dot{\mu}_1\dot{x}_1 + \dot{\mu}_2\dot{x}_2 \\
= {} & g(t)
\end{aligned}
$$

# Variation of Parameters: $2^{nd}$ Order Differential Equations

8. Each expression in parentheses the Equation (4) is zero because both $x_1$ and $x_2$ are solutions of the homogeneous equation.

Therefore Equation (4) reduces to

$$\dot{\mu}_1 \dot{x}_1 + \dot{\mu}_2 \dot{x}_2 = g(t)$$

# Variation of Parameters: 2nd Order Differential Equations

9. Now we have a system to two linear algebraic equations for the two unknown functions

$$\dot{\mu}_1 \dot{x}_1 + \dot{\mu}_2 \dot{x}_2 = g(t)$$

$$\dot{\mu}_1 x_1 + \dot{\mu}_2 x_2 = 0$$

10. Solve for $\mu_1$ and $\mu_2$

# Example 1

Find the solution of

$$\ddot{x} + 4x = 3\csc(t)$$

1. Solve for homogeneous equation

$$\ddot{x} + 4x = 0$$

$$x_h = c_1 \cos(2t) + c_2 \sin(2t)$$

$$x_1 = c_1 \cos(2t) \qquad x_2 = c_2 \sin(2t)$$

2. Assume a solution of the form,

$$x(t) = \mu_1(t)x_1(t) + \mu_2(t)x_2(t)$$

$$x(t) = \mu_1(t)\cos(2t) + \mu_2(t)\sin(2t)$$

3. Differentiate once:

$$\dot{x}_p(t) = -2\mu_1 \sin(2t) + 2\mu_2 \cos(2t) + \dot{\mu}_1 \cos(2t) + \dot{\mu}_2 \sin(2t)$$

4. Make the requirement that

$$\dot{\mu}_1 \cos(2t) + \dot{\mu}_2 \sin(2t) = 0$$

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5. Because of (4) we can simplify x

$$\dot{x}(t) = -2\mu_1 \sin(2t) + 2\mu_2 \cos(2t)$$

6. So when you differentiate again you obtain,

$$\ddot{x}(t) = -4\mu_1 \cos(2t) - 4\mu_2 \sin(2t) - 2\dot{\mu}_1 \sin(2t) + 2\dot{\mu}_2 \cos(2t)$$

If we substitute x" and x into the original equation and simplify we find that $\mu_1$ and $\mu_2$ must satisfy

$$-2\dot{\mu}_1 \sin(2t) + 2\dot{\mu}_2 \cos(2t) = 3\csc(t)$$

Now we have 2 linear algebraic equations and 2 unknowns:

$$(1)\quad -2\dot{\mu}_1 \sin(2t) + 2\dot{\mu}_2 \cos(2t) = 3\csc(t)$$

$$(2)\quad \dot{\mu}_1 \cos(2t) + \dot{\mu}_2 \sin(2t) = 0$$

Let’s solve Eq (2) for μ’$_2$:

$$\dot{\mu}_1 \cos(2t) + \dot{\mu}_2 \sin(2t) = 0$$

$$\dot{\mu}_2 = -\dot{\mu}_1 \frac{\cos(2t)}{\sin(2t)}$$

Substitute μ’$_2$ into equation (1):

$$-2\dot{\mu}_1 \sin(2t) + 2\dot{\mu}_1 \frac{\cos(2t)}{\sin(2t)} \cos(2t) = 3\csc(t)$$

Simplifying we obtain:

$$-\dot{\mu}_1\left(2\sin(2t)+2\frac{\cos(2t)}{\sin(2t)}\cos(2t)\right)=3\csc(t)$$

$$\dot{\mu}_1\left(\sin^2(2t)+\cos^2(2t)\right)=-\frac{3}{2}\csc(t)\sin(2t)$$

$$\dot{\mu}_1=-\frac{3\csc(t)\sin(2t)}{2}=-3\cos(t)$$

$$\boxed{\dot{\mu}_1=-3\cos(t)}$$

Further, now we can plug in and solve for $\mu_2$:

$$\dot{\mu}_2 = -\dot{\mu}_1 \frac{\cos(2t)}{\sin(2t)}$$

$$\dot{\mu}_1 = -3\cos(t)$$

$$\dot{\mu}_2 = -(-3\cos(t))\frac{\cos(2t)}{\sin(2t)}$$

$$\boxed{\dot{\mu}_2 = \frac{3}{2}\csc(t) - 3\sin(t)}$$

Having obtained $\mu'_1$ and $\mu'_2$ we now can integrate to obtain $\mu_1$ and $\mu_2$

$$\mu_1(t) = -3\sin(t) + c_1$$

$$\mu_2(t) = \frac{3}{2}\ln\left|\csc(t) - \cot(t)\right| + 3\cos(t) + c_2$$

Finally substituting in for the answer

$$x(t) = \mu_1(t)\cos(2t) + \mu_2(t)\sin(2t)$$

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# Sections in Notes

- **<u>Section 2.3.2/2.3.3</u>** - Variation of Parameters for 1st Order Differential Equations

- **<u>Section 3.4.2</u>** - Variation of Parameters for 2nd Order Differential Equations

# $\mathrm{n}^{\mathrm{th}}$ Order systems

- If you have the differential equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y' + P_0(x)y = f(x)$$

- We propose that the particular solution is of the form

$$y_p = \sum_{i=1}^{n} u_i(x)y_i(x)$$

- Then the requirement is set that

$$\sum_{i=1}^{n} u'_i y_i = 0$$