University of Notre Dame

Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
Fall 2015
Homework 5, due September 30$^{\text{th}}$, 2015

1. Prove the following theorem regarding the principle of superposition for ordinary, linear, nth-order, homogeneous differential equations.

   Theorem. *Let the functions $x_1(t), ..., x_n(t)$ each satisfy the ordinary, nth-order, linear, homogeneous differential equation*

   $$f_n(t)\frac{d^n x}{dt^n} + f_{n1}(t)\frac{d^{n-1}x}{dt^{n-1}} + ... + f_1(t)\frac{dx}{dt} + f_0(t)x = 0. \tag{1}$$

   Then any linear combination of $x_1(t), ..., x_n(t)$, that is, $x(t) = c_1x_1(t) + ... + c_nx_n(t)$, also satisfies Equation (1).

2. In the case of distinct real roots where the solution is a linear combination of two exponential functions, that is, $x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$, it may seem initially that it is only possible for the solutions to decay or grow. However, due to the linear combination it is possible for the slope of the solution to change once. Let $\lambda_1 = -3$ and $\lambda_2 = -2$ and plot $x(t)$ for the following combinations of c_1 and c_2.

   a. $c_1 = -1$ and $c_2 = -1$.
   b. $c_1 = 1$ and $c_2 = 1$.
   c. $c_1 = -1$ and $c_2 = 1$.
   d. $c_1 = 1$ and $c_2 = -1$.

   Explain in words why the characteristics of each solution make sense.

3. For

   $$a\ddot{x} + b\dot{x} + cx = 0$$

   prove that if $b/a < 0$ or $c/a < 0$ then one or both of the homogeneous solutions is unstable; that is, as t gets large, the magnitude of the solution gets large (i.e. diverges). If one of the two homogeneous solutions blows up, is it mathematically possible for a linear combination to remain bounded? Is it practically possible if the equation represents a real system for the solution to remain bounded?

4. Use the method of undetermined coefficients to determine the general solution to the following equations.

   (a) $\ddot{x} + 4\dot{x} + 3x = t^3$.

   (b) $\ddot{x} + 9x = \cos(2t)$.

   (c) $\ddot{x} + 4\dot{x} = \cos(2t) + e^{2t} + t^2$.

5. Find the solution to the initial value problem $\ddot{x} + x = \sec(t)$ where $x(0) = 1$ and $\dot{x}(0) = 0$ for $0 \leq t < \pi/2$ (this restriction on t is only to keep $\sec(t)$ defined).

6. For each of the following differential equations, which of the following methods may be used to find the solution?

   a. Undetermined coefficients

   b. Variation of parameters

   c. A numerical approach (Euler's method or ODE45)

   d. None of the above

   Which would be the best method to use and why?

   (1) $t\ddot{x} + te^{2\pi}\dot{x} + t\sinh(5)x = \sin(t)$.

   (2) $\ddot{x} + e^{1/2}\dot{x} + \sinh(5)x = \sin(t)/t$.

   (3) $t^2\ddot{x} + t\dot{x} - x = t\sin(t)$, for $t > 0$,
   where you know that $x_1(t) = t$ and $x_2(t) = 1/t$ are solutions to $t^2\ddot{x} + t\dot{x} - x = 0$.

   (4) $\ddot{x} + x\dot{x} + x = 0$.

   (5) $\dfrac{\partial^2 u}{\partial x^2} = c^2 \dfrac{\partial^2 u}{\partial t^2}$.

   (6) $\ddot{x} - \dot{x} + x = 2e^{2t}\sin(t)$.

   (7) $\ddot{x} + e^t\dot{x} + \sin(5)x = 0$.

   (8) $t\ddot{x} + te^t\dot{x} + t\sin(5)x = \sin(t)$.

   (9) $\ddot{x} + \dot{x} + e^{\pi}x = (e^t)t^4 + t^2\sin(t)$.

   (10) $\ddot{x} + \dot{x} + e^{\pi}x = (e^t)/t^4 + t^2\sin(t)$.

7. Solve **Problem 3.29** in the textbook[1]

[1] Engineering Differential Equations: Theory and Applications, by Bill Goodwine