University of Notre Dame
Aerospace and Mechanical Engineering

AME 301: Differential Equations, Vibrations and Controls I
Exam 2

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NAME: ______________________________

- Do not start or turn the page until instructed to do so.
- You have 50 minutes to complete this exam.
- This is an open book exam. You may consult the course texts, any other text book, your class notes, homework solutions and your own homework sets.
- You may **not** use a calculator.
- There are three problems. The first problem is worth 25 points. The second problem is worth 40 points and the third problem is worth 35 points.
- Your grade on this exam will constitute 25% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in the space provided on each page. If you need more space, use the back of the pages or use additional sheets of paper as necessary.
- A table of integrals is attached at the back of the exam for your reference.

"Shut up! Shut up, you American. You always talk, you Americans. You talk and you talk and say 'let me tell you something' and 'I just wanna say this'. Well, you're dead now, so shut up!"

— The Grim Reaper, *The Meaning of Life*

1. Consider the function

$$f(x) = \begin{cases} +1 & -2 < x \leq -1 \\ 0 & -1 < x \leq +1 \\ +1 & +1 < x \leq +2 \end{cases} \tag{1}$$

   (a) Circle one: this function is (5 points)

      i. EVEN

      ii. ODD.

   (b) Determine the Fourier series for $f(x)$ in equation 1. (20 points)

2. Consider the heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

which, as you know, describes the temperature, $u(x,t)$ at time t at location x in a uniform bar of length L.

Let

$$\begin{aligned} \alpha &= 1 \\ L &= 10 \\ u(0,t) &= 0 \\ u(10,t) &= 10 \\ u(x,0) &= 5 - \frac{x}{2}. \end{aligned}$$

(a) Determine $u(x,t)$. (30 points)

(Intentionally left blank.)

(b) Sketch a plot of $u(x, 0)$. (5 points)

(c) Sketch a plot of $\lim_{t\to\infty} u(x, t)$. (5 points)

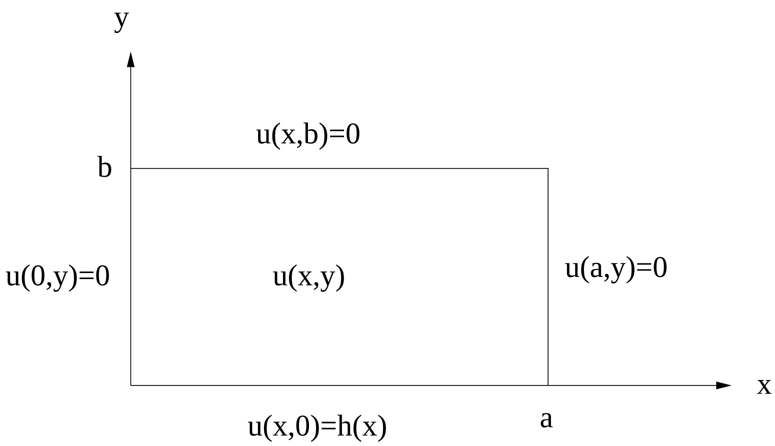

Figure 1. Domain for Laplace's equation in Problem 3.

3. Consider Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

in the domain illustrated in Figure 1.

As illustrated,

$$\begin{aligned} u(a, y) &= 0 \\ u(0, y) &= 0 \\ u(x, b) &= 0 \\ u(x, 0) &= h(x). \end{aligned} \tag{3}$$

Following the method of separation of variables outlined in the book where we assume $u(x, t) = X(x)Y(y)$ and substitute into we obtain two ordinary differential equations:

$$\begin{aligned} X'' - \lambda X &= 0 \\ Y'' + \lambda Y &= 0. \end{aligned}$$

(a) (10 points) Using the boundary conditions in Equation 3, you can determine three of the following. Complete each of these and write "cannot be determined" in the appropriate one:

$$\begin{aligned} X(0) &= \underline{\hspace{3cm}} \\ X(a) &= \underline{\hspace{3cm}} \\ Y(0) &= \underline{\hspace{3cm}} \\ Y(b) &= \underline{\hspace{3cm}}. \end{aligned}$$

(b) (5 points) You may assume that λ must be real. Must it be (circle one)

i. positive

ii. negative

iii. zero.

Prove your answer (20 points).