University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
Third Exam

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NAME: ______________________________

- Do not start or turn the page until instructed to do so.
- You have 50 minutes to complete this exam.
- This is an open book exam. You may consult the course text and four pages of written notes, but nothing else.
- You may **not** use a calculator or other electronic device.
- There are three problems. Problems 1 and 3 are worth 35 points and Problem 2 is worth 30 points.
- Your grade on this exam will constitute 20% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in the space provided on each page. If you need more space, use the back of the pages or use additional sheets of paper as necessary.

All right ramblers, let's get rambling! —*Joe*

1. Consider the wave equation with $L = 5$ and $\alpha = 1$ which describes the oscillations of a string fixed at both ends.

   (a) Determine the solution when the initial displacement of the string is zero and the initial velocity is given by

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 0, & 0 < x \leq 3 \\ 1, & 3 < x \leq 4 \\ 0, & 4 < x \leq 5 \end{cases}$$

   (b) Determine the solution when the initial displacement of the string is zero and the initial velocity is given by

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 0, & 0 < x \leq 1 \\ 1, & 1 < x \leq 2 \\ 0, & 2 < x \leq 5 \end{cases}$$

   (c) By specifically referring to your answers from the previous two parts, describe why the sound produced by the string will be the same or different for the two cases.

The first two parts are just plug and chug into the wave equation solution from the text.

(a) Using the notation from the book, he coefficients are $b_n = 0$ since $f(x) = 0$ and

$$\begin{aligned} a_n &= \frac{2}{\alpha n\pi}\int_0^5 g(x)\sin\frac{n\pi x}{5}dx \\ &= \frac{2}{n\pi}\int_3^4 g(x)\sin\frac{n\pi x}{5}dx \\ &= \frac{2}{n\pi}\frac{5}{n\pi}\left(\cos\frac{3n\pi}{5} - \cos\frac{4n\pi}{5}\right) \\ &= \frac{10}{n^2\pi^2}\left(\cos\frac{3n\pi}{5} - \cos\frac{4n\pi}{5}\right). \end{aligned}$$

Hence, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{10}{n^2\pi^2}\left(\cos\frac{3n\pi}{5} - \cos\frac{4n\pi}{5}\right)\sin\frac{n\pi x}{5}\cos\frac{n\pi t}{5}\right].$$

(b) The second part is just like the first, but with a 1 and 2 instead of a 3 and 4. Hence, the solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{10}{n^2\pi^2}\left(\cos\frac{n\pi}{5} - \cos\frac{n\pi}{5}\right)\sin\frac{n\pi x}{5}\cos\frac{n\pi t}{5}\right].$$

(c) The will sound the same. The reason is that while the coefficients are different, they have the same magnitude. This is clear from the fact that they differ by $\frac{1}{5}$ of a wavelength. It's also clear from physical grounds. Impacts at different locations will produce different sounds generally because they will excite a different set of harmonics. However, in this problem the locations of the impacts were symmetric.

2. Determine the solution to the heat equation

$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$$

on the domain $0 \le x \le 3$ where

$$\begin{aligned} u(0,t) &= 0 \\ u(3,t) &= 1 \end{aligned}$$

and

$$u(x,0) = 0.$$

This is just substituting from section 12.3.3 in the text.

The solution to the heat equation with inhomogeneous boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} + \frac{T_R - T_L}{L}x + T_L$$

where

$$c_n = \frac{2}{L}\int_0^L f(x) - \left(\frac{T_L - T_R}{L}x + T_L\right)\sin\frac{n\pi x}{L}dx.$$

For this problem,

$$\alpha = \frac{\sqrt{2}}{2},$$

and hence

$$\begin{aligned} c_n &= \frac{2}{3}\int_0^3 -\frac{1}{3}x\sin\frac{n\pi x}{3}dx \\ &= -\frac{2}{9}\int_0^3 x\sin\frac{n\pi x}{3}dx \\ &= -\frac{2}{n\pi}\cos n\pi. \end{aligned}$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{2}{n\pi}(\cos n\pi)\left(\sin\frac{n\pi x}{3}\right)e^{-\left(\frac{n\pi}{3\sqrt{2}}\right)^2 t} + \frac{1}{3}x.$$

3. In this problem you will solve the heat condition problem in a one dimensional bar where one end is held at a fixed temperature and the other end is *insulated*. Because the rate of heat transfer is proportional to the temperature gradient (energy is transferred from hot to cold), the fact that one end is insulated is mathematically represented by the fact that the slope of the temperature will be zero there.

Hence, we are going to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$$

with

$$\begin{aligned} u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \end{aligned}$$

and

$$u(x,0) = f(x).$$

(a) Assume that

$$u(x,t) = X(x)\,T(t)$$

and determine two ordinary differential equations that $X(x)$ and $T(t)$ must satisfy as well as the general solutions to them.

(b) Use the boundary conditions to determine a solution that is an infinite series that satisfies the differential equation for $X(x)$ and the boundary conditions.

(c) Determine the solution for $T(t)$.

(d) Write the general solution to the partial differential equation with the given boundary conditions.

(e) Write an expression to determine the unknown constants in the general solution so that the solution will satisfy the initial condition.

(a) Assuming $u(x,t) = X(x)T(t)$ and substituting gives

$$X''T = \frac{1}{\alpha}XT' \quad \Longrightarrow \quad \frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T}.$$

Since the left hand side only depends on x and the right hand side only depends on t, these ratios must be a constant. Hence,

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T} = -\lambda,$$

which gives

$$X'' + \lambda X = 0$$

and

$$T' + \alpha\lambda T = 0.$$

You could go through all the arguments why λ must be postive or simply recognize that this is similar. Either way

$$X(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$$

and

$$T(t) = ce^{-\alpha \lambda t}.$$

(b) Using the first boundary condition

$$X(0) = c_2 = 0.$$

The second boundary condition is where this problem is different. It has to be applied to the derivative, so

$$X'(L) = c_1 \sqrt{\lambda} \cos \sqrt{\lambda} L = 0.$$

Since $\cos \theta = 0$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots = \frac{(2n+1)\pi}{2}$ for $n = 0, 1, 2, \ldots$, we have

$$\sqrt{\lambda} L = \frac{(2n+1)\,\pi}{L} \quad \Longrightarrow \quad \lambda = \left(\frac{(2n+1)\,\pi}{2L} \right)^2.$$

Hence,

$$X(x) = \sum_{n=0}^{\infty} c_n \sin \frac{(2n+1)\,\pi x}{2L}.$$

(c) This is simply substituting λ into $T(t)$, which is

$$T(t) = ce^{-\alpha \left(\frac{(2n+1)\pi}{2L} \right)^2 t}.$$

(d) Putting theset together gives

$$u(x,t) = \sum_{n=0}^{\infty} c_n \left(\sin \frac{(2n+1)\,\pi x}{2L} \right) e^{-\alpha \left(\frac{(2n+1)\pi}{2L} \right)^2 t}.$$

(e) Substituting $t = 0$ gives

$$u(x,0) = \sum_{n=0}^{\infty} c_n \sin \frac{(2n+1)\,\pi x}{2L}.$$

Similar to before, multiply by $\sin \frac{(2m+1)\pi x}{L}$ and integrate from 0 to L and all the terms are zero except for when $n = m$.
Thus,

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n+1)\pi x}{2L} dx.$$