University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
Third Exam

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ID Number:______________________________

NAME:__________________________________

- Do not start or turn the page until instructed to do so.
- You have 50 minutes to complete this exam.
- This is an open book exam. You may consult the course text and anything you have written in it, but nothing else.
- You may **not** use a calculator or other electronic device.
- There are three problems. Problems 1 and 2 are worth 35 points and Problem 3 is worth 30 points.
- The second part of Problem 1 requires some thought. It may be best to save that for last (so that you have plenty of time to savor the experience).
- Your grade on this exam will constitute 15% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in the space provided on each page. If you need more space, use the back of the pages or use additional sheets of paper as necessary.
- If you do not have a stapler, do not take the pages apart.

Great minds discuss ideas; Average minds discuss events; Small minds discuss people.
–Eleanor Roosevelt

1. Consider a vibrating string described by the one dimensional wave equation

$$\rho \frac{\partial^2 u}{\partial t^2}(x,t) = \hat{T}\frac{\partial^2 u}{\partial x^2}(x,t),$$

where $\rho = 2$, $\hat{T} = 4$, $L = 3$, $u(0,t) = 0$, $u(L,t) = 0$ and

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 1, & a < x < b \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < a < b < L$. This models an impact on the string along the length of the string between a and b.

(a) Determine the solution. (25 points)

(b) Figure 1 illustrates $\sin(n\pi x/L)$ (left) and $\cos(n\pi x/L)$ (right), for $n = 1$ and $n = 10$. Consider the two cases

i. $a = 0.45$ and $b = 0.55$

ii. $a = 0.1$ and $b = 0.2$.

In which case will mode 1 be larger? In which case will mode 10 be larger? Explain your answer by specifically referring to features from the plots in Figure 1. (10 points)

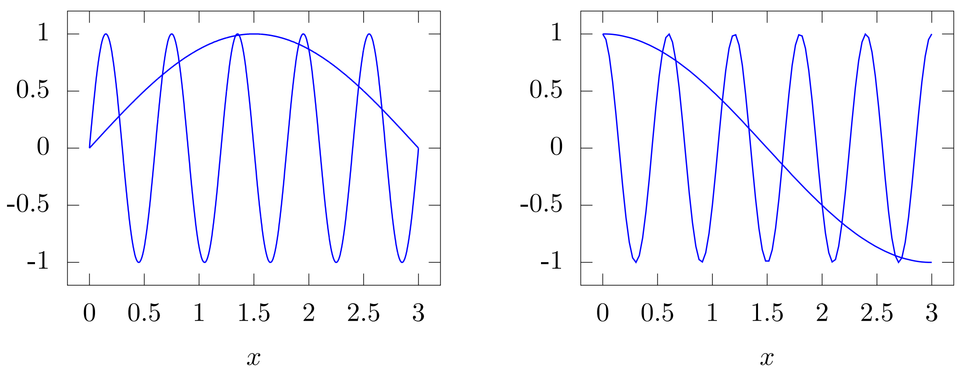

Figure 1. Plots of $\sin(n\pi x/L)$ (left) and $\cos(n\pi x/L)$ (right) for Problem 1 with $n = 1$ and $n = 10$.

2. Consider the one-dimensional wave equation with damping

$$\frac{\partial^2 u}{\partial^2 x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial u}{\partial t}(x,t), \tag{1}$$

where $u(0,t) = u(L,t) = 0$ and

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

(a) Assume a solution of the form $u(x,t) = X(x)T(t)$ and determine the ordinary differential equations that $X(x)$ and $T(t)$ satisfy.

(b) Determine the solutions for $X(x)$ $T(t)$.

(c) Write the solution $u(x,t)$ as an infinite series, which satisfies Equation 1 and the boundary conditions.

(d) Determine an expression for any constants that appear in your solution from the previous part.

(e) Indicate in your solution the feature that corresponds to adding damping and why it would have an effect that would be expected from damping.

3. Figure 2 illustrates solutions in the phase plane of

$$\ddot{x} + 0.3\dot{x} - 4x + x^2 = 0 \tag{2}$$

for various initial conditions.

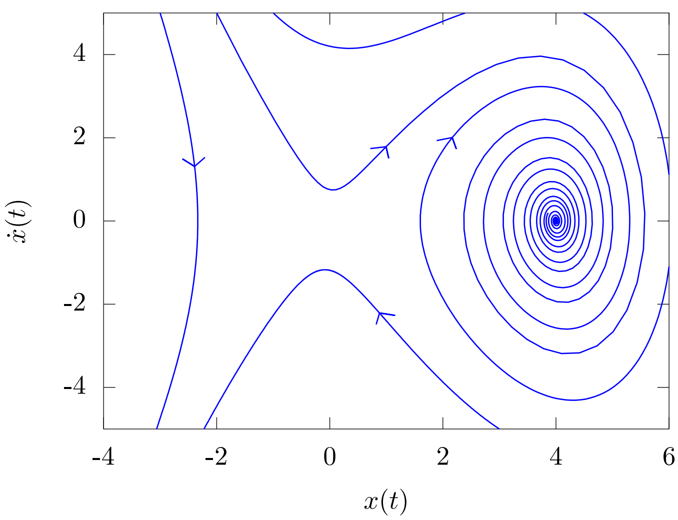

Figure 2. Phase plane solutions to Equation 2.

(a) Determine the equilibrium points. (10 points)

(b) Determine a linear ordinary differential equation that approximates Equation 2 near the point $(4, 0)$. (10 points)

(c) Solve the differential equation you determined in the previous part and indicate the manner to which it corresponds to the solutions in Figure 2. (10 points)