University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
First Exam

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- Do not start or turn the page until instructed to do so. You have 75 minutes to complete this exam.
- This is an open book exam. You may consult the course text and your own course notes, but nothing else.
- There are five problems. Problems 1 and 2 are worth 20 points each, Problem 3 is worth 40 points and Problems 4 and 5 are worth 10 points each.
- You may **not** use a calculator or other electronic device.
- Your grade on this exam will constitute 20% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in a Blue Book.
- You may choose whatever method you like to solve the problems unless the problem specifies which method to use. Merely substituting into an equation from the book is totally fine as long as it answers the problem.

Long term consistency trumps short term intensity.

— Bruce Lee

1. Find the solution to
$$\dot{x} + 3x = \cos t + \sin 2t$$
where $x(0) = 0$.

2. Find the general solution to
$$\left(t^2 + 2x \sin t\right) \dot{x} + 2tx + x^2 \cos t = 0.$$

3. The most basic epidemiological model we have for the spread of disease is that "the rate at which people are infected is proportional to the number of people infected" which leads to the differential equation $\dot{x} = kx$ where x is the number of people infected with the solution
$$x(t) = ce^{kt}. \tag{1}$$
However, this has two very obvious shortcomings when it comes to accuracy, not the least of which is that no infectious disease has yet killed all human life on earth.

   (a) People change their behavior over time in response to knowledge of the disease to avoid being infected. Consider
   $$\dot{x} = kx - \alpha t x. \tag{2}$$
      i. Explain why the $(-\alpha t x)$ term may approximate this added feature.
      ii. Find the general solution to Equation 2.
      iii. On the same plot, make a qualitative sketch of the solutions to Equation 2 and Equation 1. At very large time, which one will be larger?

   (b) If a disease is very contagious, it may "saturate" an area. If everyone around me is already infected, I'm not going to infect any new people. So, for example, if all of Indiana has been infected by the Zombie plague, then only the people at the boundaries of the state will infect new people. If people are pretty much evenly spread out over an area, then the number of people at the boundary are proportional to the square root of the number. Hence, we have
   $$\dot{x} = k\sqrt{x}. \tag{3}$$
      i. Find the general solution to Equation 3.
      ii. On the same plot, make a qualitative sketch of the solutions to Equations 3, 2 and 1.

4. Prove that all separable equations are exact.

5. Find the general solution to
$$\dot{x} + x = 0.$$