University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
First Exam

B. Goodwine
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- Do not start or turn the page until instructed to do so. You have 75 minutes to complete this exam.
- This is an open book exam. You may consult the course text and your own course notes, but nothing else.
- There are five problems, each worth 20 points.
- You may **not** use a calculator or other computational aid.
- Your grade on this exam will constitute 20% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in a Blue Book.
- You may choose whatever method you like to solve the problems unless the problem specifies which method to use. Merely substituting into an equation from the book is totally fine as long as it answers the problem.

Consider the subtleness of the sea; how its most dreaded creatures glide under water, unapparent for the most part, and treacherously hidden beneath the loveliest tints of azure. Consider also the devilish brilliance and beauty of many of its most remorseless tribes, as the dainty embellished shape of many species of sharks. Consider, once more, the universal cannibalism of the sea; all whose creatures prey upon each other, carrying on eternal war since the world began.
Consider all this; and then turn to the green, gentle, and most docile earth; consider them both, the sea and the land; and do you not find a strange analogy to something in yourself? For as this appalling ocean surrounds the verdant land, so in the soul of man there lies one insular Tahiti, full of peace and joy, but encompassed by all the horrors of the half-known life. God keep thee! Push not off from that isle, thou canst never return!

— Herman Melville, *Moby Dick*

1. This is like Exercise 2.31 in the book.

   For relatively high velocities the drag due to the motion of the body through air is proportional to the square of the velocity of the body. Hence, if the direction of positive velocity is down, Newton's law on the body can be represented as

$$m\frac{dv}{dt}(t) = mg - kv^2(t). \tag{1}$$

   (a) If a body falls from a sufficiently high altitude, it will reach its *terminal velocity* which is the velocity at which it will stop accelerating. Determine an expression for the terminal velocity, v_{term} from Equation (1).

   (b) What method would you use to solve this equation? Explain why it would be the best approach.

   (c) Do not solve the equation, but sketch what you expect the solution would look like.

   (d) *Extra credit:* solve the equation. You may find the attached table of integrals useful.

2. Consider the "curved funnel" illustrated in Figure 1.

   - The funnel has a circular cross section as does the outlet pipe. The shape of the funnel is given by $y = x^2$.
   - The radius of the outlet pipe is r.
   - Let v_{out} denote the flow rate of the fluid out of the outlet.
   - Let $h(t)$ denote the height of the water in the funnel. Then $\dot{h}$ would be the velocity of the fluid at the top in the downward direction, $v_{top} = \dot{h}$.
   - Use Bernoulli's equation

$$\frac{1}{2}\rho v_{out}^2(t) = \frac{1}{2}\rho v_{top}^2(t) + \rho g h(t)$$

     and equate the mass flow rates at the outlet and through a cross section of the funnel at the top of the fluid to determine a differential equation describing how the height in the funnel changes with time.
   - What method would you use to solve this equation?
   - Solving it in the present form may be hard. Assume that the radius at the top of the fluid is much greater than the radius of the outlet and simplify your equation based on that assumption. Then solve the simplified equation.
   - *Extra credit:* would chaning the shape from $y = x^2$ to $y = x^4$ make it drain faster or slower? Explain your reasoning.

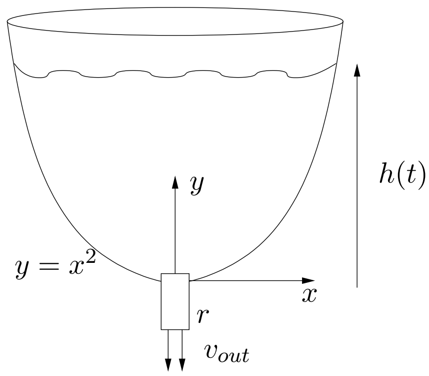

Figure 1: Funnel for Problem 2.

3. Make up or find 5 exact differential equations that differ by more than coefficient values. In other words, if $\dot{x} + 3x = 0$ is exact, then you can not use $\dot{x} + 4x = 0$ or $2\dot{x} + 4x = 0$ as other ones.

4. The most basic epidemiological model we have for the spread of disease is that "the rate at which people are infected is proportional to the number of people infected" which leads to the differential equation $\dot{x} = kx$ where x is the number of people infected with the solution
$$x(t) = ce^{kt}. \tag{2}$$
However, this has two very obvious shortcomings when it comes to accuracy, not the least of which is that no infectious disease has yet killed all human life on earth.

   (a) People change their behavior over time in response to knowledge of the disease to avoid being infected. Consider
   $$\dot{x} = kx - \alpha tx. \tag{3}$$
      i. Explain why the $(-\alpha tx)$ term may approximate this added feature.
      ii. Find the general solution to Equation 3.
      iii. On the same plot, make a qualitative sketch of the solutions to Equation 3 and Equation 2. At very large time, which one will be larger?

   (b) If a disease is very contagious, it may "saturate" an area. If everyone around me is already infected, I'm not going to infect any new people. So, for example, if all of Indiana has been infected by the Zombie plague, then only the people at the boundaries of the state will infect new people. If people are pretty much evenly spread out over an area, then the number of people at the boundary are proportional to the square root of the number. Hence, we have

$$\dot{x} = k\sqrt{x}. \tag{4}$$

i. Find the general solution to Equation 4.
ii. On the same plot, make a qualitative sketch of the solutions to Equations 4, 3 and 2.

5. Determine the general solutions to each of the following differential equations.

   (a) $\dot{x} + x = \cos t$
   (b) $\dot{x} + 2x = e^{-2t}$
   (c) $\dot{x}\left(1 - x^2\right) = t$
   (d) $\dot{x} + tx = 0$