University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
Third Exam

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December 1, 2015

- Do not start or turn the page until instructed to do so. You have 75 minutes to complete this exam.
- This is an open book exam. You may consult the course text and your own course notes, but nothing else.
- There are five problems, each worth 20 points.
- You may **not** use a calculator or other computational aid.
- Your grade on this exam will constitute 20% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in a Blue Book.
- You may choose whatever method you like to solve the problems unless the problem specifies which method to use. Merely substituting into an equation from the book is totally fine as long as it answers the problem.

Whenever you find yourself on the side of the majority, it is time to pause and reflect.
— Mark Twain

1. Determine the solution to
$$\ddot{x} + t^2\dot{x} + x = 0$$
where $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

2. Determine a the solution to
$$\ddot{x} - 2t\dot{x} + 8x = 0$$
where $x(0) = x_0$ and $\dot{x}(0) = 0$.

   Note carefully the initial conditions. You should use a series solution method, but the answer is actually finite. Find the answer, not just the recursion relation.

3. Determine the Fourier series for
$$f(x) = \begin{cases} 30, & 0 \le x < 2 \\ 20, & 2 \le x < 4 \end{cases}$$
where $f(x+4) = f(x)$ for all x.

4. Determine the solution to the one-dimensional wave equation where $\alpha = 3$, $L = 10$, $u(x,0) = 0$ and
$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} 0, & 0 \le x < 3, \\ 1, & 3 \le x < 4 \\ 0, & 4 \le x < 10. \end{cases}$$

5. The one dimensional heat equation with an insulated end is given by
$$\alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial u}{\partial t}(x,t),$$
where $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$ and $u(x,0) = f(x)$.

   (a) Explain why the second boundary condition represents an insulated end.

   (b) By assuming $u(x,t) = X(x)T(t)$ and going through the usual process, show (derive) that the solution is given by Equations 11.39 and 11.40 in the course text which are
$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) \exp\left(\frac{-\alpha^2(2n+1)^2\pi^2 t}{4L^2}\right)$$
   where
$$c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{(2n+1)\pi x}{2L} dx.$$