University of Notre Dame
Aerospace and Mechanical Engineering

AME 30314: Differential Equations, Vibrations and Controls I
First Exam

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September 22, 2016

- Do not start or turn the page until instructed to do so. You have 75 minutes to complete this exam.
- This is an open book exam. You may consult the course text and your own course notes, but nothing else.
- There are five problems, each worth 20 points. For most students, they are probably not ordered from the easiest to the hardest.
- You may **not** use a calculator or other computational aid.
- Your grade on this exam will constitute 15% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in a Blue Book.
- You may choose whatever method you like to solve the problems unless the problem specifies which method to use. Merely substituting into an equation from the book will receive full credit as it answers the problem as long as you refer to the equation you used, *e.g.*, "substituting into equation 1.23 gives..."

Poets say science takes away from the beauty of the stars - mere globs of gas atoms. I, too, can see the stars on a desert night, and feel them. But do I see less or more?

— Richard P. Feynman

1. Determine the general solution to

$$\dot{x}(t) + 100x(t) = M \sin \omega t.$$

   (a) Whatever the value it has at $t = t_0$, how much smaller is the homogeneous solution one second later at $t = t_0 + 1$?

   (b) Plot the magnitude of the particular solution versus the frequency of the right hand side, ω. Plot it for positive ω values from small values, $\omega \approx 1/10$ to large values, $\omega \approx 1000$.

   (c) If your answer to the first part essentially justifies ignoring the homogeneous solution, then why might things with dynamics described by this equation be called *low pass filters*?

2. The basic population model is

$$\dot{x}(t) = kx(t)$$

which represents the fact that the more people, plants, bacteria, whatever, there are, the more offspring they will produce.

However, in an environment with limited resources, competition among members of the population will reduce the rate of reproduction. A common model for that is

$$\begin{aligned}\dot{x}(t) &= kx(t) - \hat{\alpha}x^2(t) \\ &= kx(t)\left(\alpha - x(t)\right).\end{aligned}$$

where k and α are positive, so the $x^2(t)$ term takes away some of the population growth.

   (a) Find the general solution. Make sure that your work clearly communicates the method you are using to solve this, even if you do not make it to the end. *Hint:* start with $\alpha\dot{x} = x\left(\alpha - x\right)(k\alpha)$ and note that

$$\frac{\alpha}{x(t)\left(\alpha - x(t)\right)} = \frac{1}{x(t)} + \frac{1}{\alpha - x(t)}.$$

   (b) Sketch the solution. Even if you do no successfully solve it above, you can probably do this directly from the differential equation.

3. Determine the general solutions to each of the following differential equations.

   (a) $\dot{x} + 2x = \cos 2t$

   (b) $\ddot{x} + 2x = \cos 2t$

(c) $\dot{x}\left(1-x^3\right)=e^t$

(d) $\dot{x}+\sin(t)x=0$

4. Find $x(t)$ that satisfies

$$\begin{aligned}\ddot{x}(t)+\dot{x}(t)+x(t)&=t\\ x(0)&=2\\ \dot{x}(0)&=0.\end{aligned}$$

5. Solve and sketch the solution to

$$\begin{aligned}\ddot{x}+4x&=0\\ x(0)&=1\\ \dot{x}(0)&=0.\end{aligned}$$

On the same plot, sketch the solution if the 4 is increased slightly. Sketch the solution if the 4 is decreased slightly. Explain your reasoning.