University of Notre Dame
Aerospace and Mechanical Engineering

AME 30315: Differential Equations, Vibrations and Controls II
Third Exam

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ID Number:______________________________

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- Do not start or turn the page until instructed to do so.
- You have 50 minutes to complete this exam.
- This is an open book exam. You may consult the course text and anything you have written in it, but nothing else.
- You may **not** use a calculator or other electronic device.
- There are four problems. The first three problems are worth 30 points and the fourth problem is worth 10 points. While not absolutely necessary, the problems sort of build on each other, so initially you should probably try to do them in order.
- Your grade on this exam will constitute 20% of your total grade for the course. *Show your work* if you want to receive partial credit for any problem.
- Answer each question in the space provided on each page. If you need more space, use the back of the pages or use additional sheets of paper as necessary.
- If you do not have a stapler, do not take the pages apart.

Are you sure you want to test your limits? 'Tis much more popular to limit your tests.
–Lazarus Lake

1. Consider

$$G(s) = \frac{4}{(s+2)(s^2+2s+2)}.$$

- On the axes on the following page, neatly sketch the root locus plot for $G(s)$. Be sure to include computations, if applicable, for each step in the root locus plotting process.
- From your root locus plot, determine the range of values for k for which the system placed in the feedback loop illustrated in Figure 9.44 of the course text is stable.
  - You do not need to do any elaborate trigonometry by hand – just make a neat plot and visually estimate some of the distances.
  - Indicate on the plot what the distances you used and what you used for the values.
  - You do not need to compute exactly where the root locus crosses the imaginary axis.

The root locus is illustrated in Figure 1. For each step in the root locus plotting procedure from Table 9.1 in the course text:

(a) There are no zeros. There is one pole at $s = -2$ and a complex conjugate pair of poles at

$$s = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

(b) The portion of the root locus on the real axis is to the left of the pole at $s = -2$.

(c) The asymptote angles are

$$\begin{aligned}
\theta_0 &= \frac{180}{-3} = -60^\circ \\
\theta_1 &= \frac{180+360}{-3} = -180^\circ = 180^\circ \\
\theta_0 &= \frac{180+720}{-3} = -300^\circ = 60^\circ.
\end{aligned}$$

(d) If we use the notation

$$\begin{aligned}
p_1 &= -2 \\
p_2 &= -1+i \\
p_3 &= -1-i
\end{aligned}$$

then the asymptote intersection point is at

$$s = \frac{0 - (-2 + (-1+i) + (-1-i))}{-3} = -\frac{4}{3}.$$

(e) Considering a point very close to p_1 and using the fact that points on the root locus must satisfy $\angle G(s) = \pm 180°$ gives

$$\angle G(s) = 0 - [\theta_{dep} + 45 + 90] = \pm 180 \quad \Longrightarrow \quad \theta_{dep} = 180 - 45 - 90 = 45°.$$

(f) Checking for break in or break away points on the real axis requires finding the solutions to

$$\begin{aligned}\frac{d}{ds}k = \frac{d}{ds}\frac{1}{G(s)} = \frac{d}{ds}\left[(s+2)\left(s^2+2s+2\right)\right] &= \left(s^2+2s+2\right) + (s+2)\left(2s+2\right) \\ &= s^2+2s+2+2s^2+6s+4 = 3s^2+8s+6 = 0,\end{aligned}$$

which has solutions

$$s = \frac{-8 \pm \sqrt{64-72}}{6}$$

which are not on the portion of the root locus on the real axis. Hence there are no break in or break out points.

Looking at the branch that goes along the $+60°$ asymptote, it appears the branch crosses the imaginar axis close to $s = 2i$. Hence

$$\begin{aligned}|2i - p_1| &\approx \sqrt{2^2+2^2} = 2\sqrt{2} \\ |2i - p_2| &\approx \sqrt{2}\,|2i - p1| \qquad \approx \sqrt{1^2+3^2} = \sqrt{10}.\end{aligned}$$

Hence,

$$|k| = \frac{1}{G\,(2i)} = \frac{(2)\sqrt{2}(2)\sqrt{10}}{4} \approx 4.5.$$

Since all the poles of $G(s)$ are in the left half plane, the lower limit

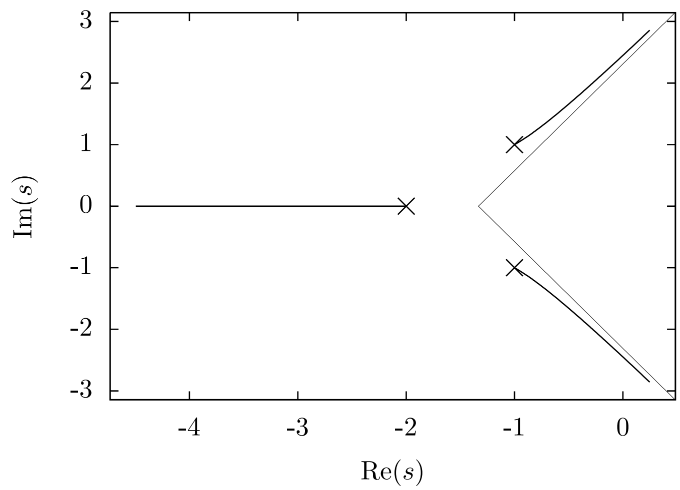

Figure 1. Root locus plot for Problem 1.

2. Use a Routh array to verify the range of k values in Problem 1 for stability.

The closed loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{kG(s)}{1+kG(s)} = \frac{4k}{(s+2)\,(s^2+2s+2)+4k} = \frac{4k}{s^3+4s^2+6s+4+4k}.$$

For stability, we need to find the values of k for which there are no right half plane solutions to

$$s^3+4s^2+6s+4+4k=0.$$

The Routh array is

$$\begin{array}{c|cccc} s^3 & 1 & 6 & 0 & 0 \\ s^2 & 4 & 4+4k & 0 & 0 \\ \hline s^1 & -\frac{4+4k-24}{4} & 0 & 0 & \\ s^0 & 4+4k & 0 & & \end{array}$$

For the first term in the s^1 row to be positive we need

$$4+4k-24>0 \quad \Longrightarrow \quad k<5.$$

The last row requires $s>-1$, which does not have to be computed since the problem was restricted to positive k values only.

Hence, $k<5$, which is verifies our approximate answer from the root locus plot.

3. Sketch the Bode plot for $G(s)$ in Problem 1.

   - For clarity, split the transfer functions into terms and call them something like "term 1" and "term 2" and label each of the curves on the plot on this page as corresponding to "term 1," "term 2," *etc.*
   - Be sure to indicate the values of all the slopes in the magnitude plot when they are not zero.
   - Indicate on the Bode plot the feature that corresponds to the maximum k value for stability that you determined in the first two problems, *i.e.,* how you could have determined the maximum k from the Bode plot. On the blank plots on this page, sketch the individual terms of the plot, and on the next page plot the combined plot.

If we split the transfer function into two terms as

$$G(s) = \left(\frac{4}{s+2}\right)\left(\frac{1}{s^2+2s+2}\right)$$

then the plot for the first term is shown in Figure 2 and for the second in Figure 3. The combined plot and final answer is illustrated in Figure 4.

Note that it would probably be easier to split the 4 in the numerator into a 2 for each term so that the low frequency magnitude for each term would be 0 dB. There is no right and wrong way to split it. Since calculators are not allowed, having the exact low-frequency values is not required.

The important features for Figure 2 are

(a) For $\omega \ll 2$ the slope of the magintude plot is zero, and for $\omega \gg 2$ the slope is -20 db/decade.

(b) For $\omega \ll 2$ the phase is $0°$ and for $\omega \gg 2$ the phase is $-90°$.

(c) There should be an appxoximately linear relationship between phase and frequency between $\omega = .2$ and $\omega = 20$.

The important features of Figure 3 are

(a) Computing that $\omega_n = \sqrt{2}$ and $\zeta = 1/\sqrt{2}$.

(b) For $\omega \ll \sqrt{2}$ the slope of the magintude plot is zero, and for $\omega \gg \sqrt{2}$ the slope is -40 db/decade.

(c) Since $\zeta \approx 0.7$ there is basically no resonance peak.

(d) For $\omega \ll \sqrt{2}$ the phase is $0°$, and for $\omega \gg \sqrt{2}$ the phase is $-180°$.

(e) The manner in which the phase transistions from $0°$ to $-180°$ depends on the damping ratio. The plot shows it exactly, but much latitude should be given when grading this part.

The important features of Figure 4 are:

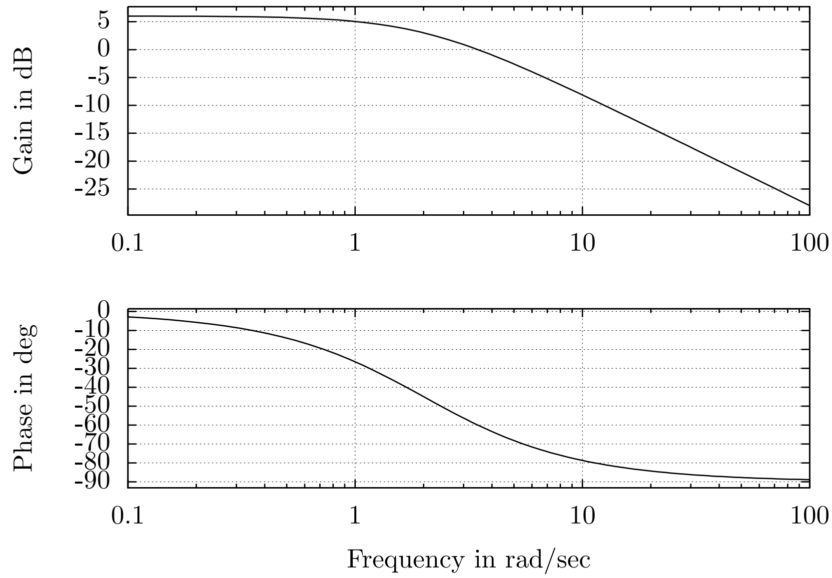

Figure 2. Bode plot for $G(s) = 4/(s+2)$.

(a) A magnitude of zero for $\omega \ll \sqrt{2}$.

(b) A slope of -60 dB/decade for $\omega \gg 2$.

(c) A phase of zero for $\omega \ll \sqrt{2}$.

(d) A phase of $-270°$ for $\omega \gg 2$.

(e) Indicating that the magnitude is less than zero where the phase crosses $-180°$, which is the gain margin. There is no need for the student to compute the logarithms, but from the plot the gain margin is approximately 14 dB, and

$$10^{\frac{14}{20}} \approx 5.$$

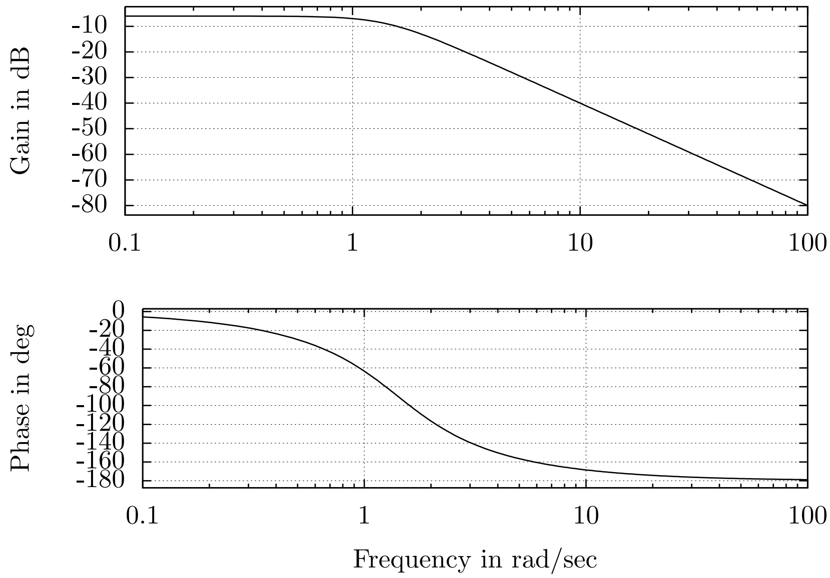

Figure 3. Bode plot for $G(s) = 1/(s^2 + 2s + 2)$.

4. If a lead compensator of the form

$$C(s) = \frac{s+z}{s+p}$$

where $0 < z < p < \infty$ is added to the feedback system in series with $G(s)$, *i.e.*, in the usual way, will it be possible to determine values for the zero and pole such that the closed-loop system is stable for all positive values of k? Explain your answer.

A lead compensator adds one pole and one zero, which will *not* change the asymptotes since it does not change the difference in the number of poles and zeros. Since there are asymptotes at $\pm 60°$ the branches of the root locus must cross the imaginary axis and hence a lead compensator can *not* make this system stable for all positive k.

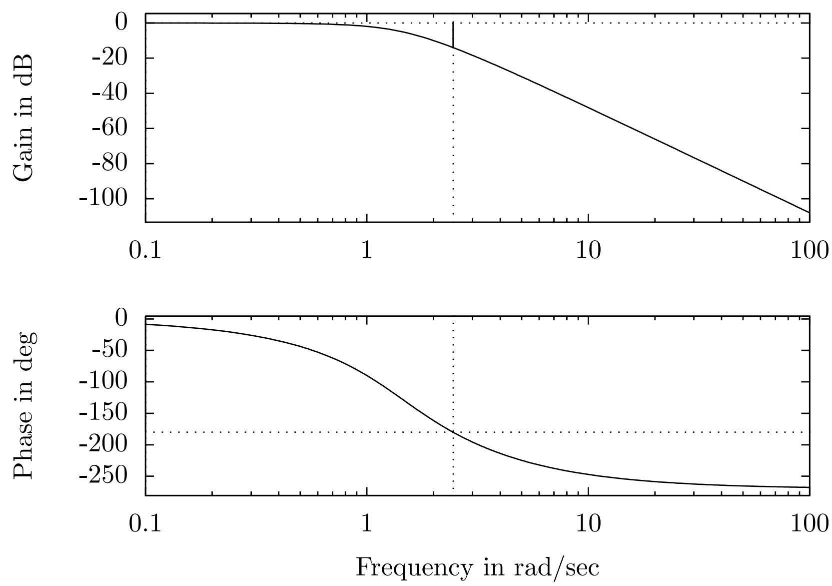

Figure 4. Bode plot for $G(s) = 4/[(s+2)(s^2+2s+2)]$.