It is possible to complete all of these exercises by hand.

1. Each of the matrices in this problem has a full set of linearly independent eigenvectors. For each one, find the general solution to $\dot{\xi} = A\xi$ and indicate whether Theorem 6.5.1 or 6.5.6 applies:

$$A_1 = \begin{bmatrix} 6 & -4 \\ 0 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -3 & 0 & 0 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -8 & 7 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_5 = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 1 \end{bmatrix} \qquad A_6 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 9 \\ 0 & 2 & 1 & 4 \\ 0 & -4 & 0 & 14 \end{bmatrix} \qquad A_8 = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & -4 \end{bmatrix} \qquad A_9 = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -5 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

2. For A_2, A_3, A_4, A_8 and A_9 in Exercise 1, determine the solution if

$$\xi(0) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

3. Each of the matrices in this problem have some complex eigenvalues. Determine the general solution to $\dot{\xi} = A\xi$ for:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -\frac{7}{2} & \frac{15}{2} & -3 \\ -\frac{3}{2} & -\frac{1}{2} & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} -1 & -4 \\ 4 & -1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 11 & 0 & 17 \\ 0 & -6 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad A_5 = \begin{bmatrix} -5 & 1 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 2 & -5 \end{bmatrix} \qquad A_6 = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & -1 & -7 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -5 & 6 \\ 0 & -3 & 1 \end{bmatrix} \qquad A_8 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & -5 & 0 \\ 3 & 0 & 2 \end{bmatrix} \qquad A_9 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & -1 \end{bmatrix}.$$

4. For A_2, A_4, A_7 and A_8 in Exercise 3, determine the solution if

$$\xi(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

5. Each of the matrices in this problem has some repeated eigenvalues. Determine the general solution to $\dot{\xi} = A\xi$ for:

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad A_3 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 5 & 1 \\ 1 & -1 & 7 \end{bmatrix} \quad A_5 = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad A_6 = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

6. Prove Theorem 6.7.13 by substituting Equation 6.29 into $\dot{\xi} = A\xi$ and making use of the properties of generalized eigenvectors.

7. Find the general solution to

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}.$$

8. Consider

$$\frac{d}{dt}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}.$$

- What happens when you assume

$$\xi_p(t) = ae^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t}.$$

  Explain why it does not work.

- What happens when you assume

$$\xi_p(t) = ate^{3t} = t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t}.$$

  Explain why it does not work.

9. Determine the solution to $\dot{\xi} = A\xi + g(t)$ where

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & -4 \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

- using the method of undetermined coefficients;
- by determining a coordinate transformation that diagonalizes A; and,
- using the method of variation of parameters.

10. Determine the solution to $\dot{\xi} = A\xi + g(t)$ where

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}, \quad g(t) = \begin{bmatrix} e^{-4t} \\ 0 \\ 0 \end{bmatrix}$$

- using the method of undetermined coefficients;
- by determining a coordinate transformation that diagonalizes A; and,
- using the method of variation of parameters.

Since $A = A^T$, make use of the fact that $T^{-1} = T^T$. Verify this fact by showing that $T^T T = I$.

11. Compute the matrix exponential for A_1, A_2 and A_9 from Exercise 1. For the initial condition given in Expercise 2, A_3 and A_9 verify that

$$\xi(t) = e^{At}\xi(0)$$

is the same solution as was computed in Exercise 2.