AME 90951: Geometric Nonlinear Control Theory
Homework 5[1]

Problem 1: Consider the system

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)),\end{aligned}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz with $f(0,0) = 0$ and $h\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is continuous with $h(0,0) = 0$. Assume there exists a continously differentiable positive semi-definite function $V(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ and positive constant $\delta > 0$ such that

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) + \delta y^T y,$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Prove that this system is finite-gain $\mathcal{L}_2$ stable.

Problem 2: Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2^5 - \gamma x_2 \\ \dot{x}_2 &= -x_1^3 - x_2^5 + \gamma(x_1 + \pi x_2).\end{aligned}$$

If we let $\gamma = 0$, show that the origina is globally asymptotically stable.
With $0 < \gamma < \frac{1}{2}$ show that the origin is unstable and that the solutions of this system are globally ultimately bounded. Determine the ultimate bound.

Problem 3: Show that the following system is finite gain (or small-signal finite-gain) $\mathcal{L}_2$ stable and find an upper bound on the $\mathcal{L}_2$ gain. Assume that ψ is a positive constant.

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\psi^2(1 + x_1^2)x_2 - x_1^3 + \psi x_1 u \\ y &= \psi x_1 x_2.\end{aligned}$$

Problem 4: Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1^2 x_3 - x_2 + u \\ \dot{x}_3 &= -x_1 + u \\ y &= x_3.\end{aligned}$$

Show that this system is input-state linearizable. Find a feedback control and change of variables that linearize the state equations.
Show that the system is input-output linearizable. Transform it to its normal form and specify the region over which the transformation is valid. Is this system minimum phase?

Problem 5: Consider the third order system

$$\begin{aligned}\dot{\eta} &= -\frac{1}{2}(1 - \xi_2)\eta^3 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v,\end{aligned}$$

where the states are η, ξ_1, ξ_2, and the control input is v.
Let $\gamma > 0$ be specified. Design a controller $v = c_0\xi_1 + c_1\xi_2$ so that the eigenvalues of the lower linear system are $-|\gamma| < 0$ with multiplicity 2.
Show that if γ is large enough, then the entire nonlinear system is unstable for certain choices of initial conditions.

[1] Inspiration for the problems are from homeworks and exams from Dr. Lemmon's Nonlinear Control course