AME 90951 | HW # 2 | 2/19

7 hrs

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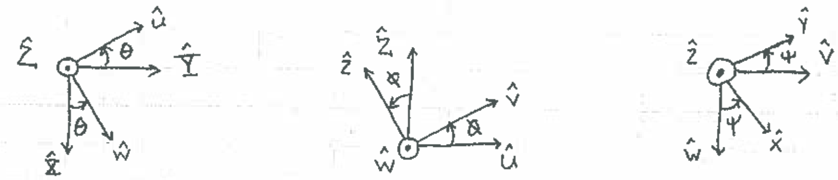
1) $SO(3) = \{R \in \mathbb{R}^{3\times3} \mid RR^T = R^T R = I,\ \det(R) = +1\}$



$$R_1 = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ +s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \qquad R_3 = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$

$$R = R_1 R_2 R_3 = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \cdot R_3 = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \cdot \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi - s_\theta c_\alpha s_\psi & -c_\theta s_\psi - s_\theta c_\alpha c_\psi & s_\theta s_\alpha \\ s_\theta c_\psi + c_\theta c_\alpha s_\psi & -s_\theta s_\psi + c_\theta c_\alpha c_\psi & -c_\theta s_\alpha \\ s_\alpha s_\psi & s_\alpha c_\psi & c_\alpha \end{bmatrix}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

$$\psi = \operatorname{atan2}(R_{31}, R_{32}) \qquad (1)$$

$$\theta = \operatorname{atan2}(R_{13}, -R_{23}) \qquad (2)$$

$$c_\alpha s_\psi = \frac{+R_{21} - s_\theta c_\psi}{c_\theta}$$

$$\alpha = \operatorname{atan2}\left(R_{31}, \frac{R_{21} - s_\theta c_\psi}{c_\theta}\right) \qquad (3)$$

By construction, Eqs. 1, 2, 3 map $SO(3)$ to $\mathbb{R}^3$. Eqs. 1, 2, 3 form α.
The open set u is
$-1 \le R_{31}, R_{32}, R_{13}, R_{23} \le +1$, $R_{23} \ne 0$

In $\mathbb{R}^3$, the open set is
$-180° < \psi, \theta, \alpha \le 180°$ $\theta \ne 0$

atan2 is continuous, bijective mapping from $[-1,1], [-1,1] \to (-\pi, \pi)$
Thus, α is a homeomorphism and $SO(3)$ has a coordinate chart.
The coordinate chart is not global. A second coordinate chart is a rotational order of X-Z-X. Together the charts would form a C^∞ atlas.

need diffeomorphism, which it is.

TOPS 35502

2) If $F: N \to M$ is a diffeomorphism, the rank of the jacobian

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

is equal to n.

- From page 497 in the book we know that the matrix representation of F_* is just the jacobian of F. Since the rank of the jacobian is n, we know that F_* must be bijective and is thus an isomorphism between tangent spaces.

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3. Solution:

(a). Since $\hat{g}_{t*}(v) = dg_t(v) = (-2\pi \sin 2\pi t, 2\pi\cos 2\pi t)\, v$ is full rank for any t

$\Rightarrow g_{t*}(v)$ is an isomorphism from $\mathbb{R}^1$ to S^1

$\Rightarrow$ By the inverse function theorem, $g(t)$ is a local diffeomorphism.

(b). In fact, given diffeomorphisms $f_1: M_1 \to M_1'$ and $f_2: M_2 \to M_2'$, for every $(p_1, p_2) \in M_1 \times M_2$ smooth coordinate charts $(U_1 \times U_2, \varphi_1 \times \varphi_2)$ and $(V_1 \times V_2, \psi_1 \times \psi_2)$, with $(p_1, p_2) \in U_1 \times U_2$ and $(f_1(p_1), f_2(p_2)) = (q_1, q_2) \in V_1 \times V_2$.

Then, the smoothness of

$$(\psi_1 \times \psi_2) \circ (f_1 \times f_2) \circ (\varphi_1 \times \varphi_2)^{-1} = (\psi_1 \circ f_1 \circ \varphi_1^{-1}) \times (\psi_2 \circ f_2 \circ \varphi_2^{-1})$$

and $(\varphi_1 \times \varphi_2) \circ (f_1 \times f_2)^{-1} \circ (\psi_1 \times \psi_2)^{-1} = (\varphi_1 \circ f_1^{-1} \circ \psi_1^{-1}) \times (\varphi_2 \circ f_2^{-1} \circ \psi_2^{-1})$

come from the smoothness of $\psi_i \circ f_i \circ \varphi_i^{-1}$ and $\varphi_i \circ f_i^{-1} \circ \psi_i^{-1}$ $(i = 1, 2)$.

$\Rightarrow \hat{g}: \mathbb{R}^2 \to S^1 \times S^1$, $\hat{g} = g \times g$ is a special case where $M_1 = M_2 = \mathbb{R}^1$, $M_1' = M_2' = S^1$, $f_1 = f_2 = g$

$\Rightarrow \hat{g}$ is a local diffeomorphism.

(c). Let $L = \{(t,s) \in \mathbb{R}^2, s = at\}$ where a is an irrational number. Since

$\hat{g}(t,s) = (\cos 2\pi t, \sin 2\pi t, \cos 2\pi s, \sin 2\pi s)$

If $\hat{g}(t_0, s_0) = \hat{g}(t_1, s_1)$, then $\hat{g}(t_0, at_0) = \hat{g}(t_1, at_1)$

$\Rightarrow t_0 = t_1 + m, \quad at_0 = at_1 + n$, where m and n are some integer.

Since a is irrational $\Rightarrow m = n = 0 \Rightarrow t_0 = t_1$

$\Rightarrow \hat{g}$ is one-to-one.

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Problem 4:

Everyone's is different.

Controls Course Homework #2

Ashley Kulczycki

5) Isidori, page 481. Show that the helix example is an embedding.

Consider the mapping $F: \mathbb{R} \to \mathbb{R}^3$ given by

$$x_1(t) = \cos 2\pi t$$
$$x_2(t) = \sin 2\pi t$$
$$x_3(t) = t$$

whose image is an "helix" winding on an infinite cylinder whose axis is the x_3 axis.

pg 479 Definition: Let $F: N \to M$ be a smooth mapping of manifolds.
(i) F is an immersion if rank(F) = dim(N) for all $p \in N$
(ii) F is an univalent immersion if F is an immersion and is injective (one-to-one)
(iii) F is an embedding if F is an univalent immersion and the topology induced on F(N) by one of N coincides with the topology of F(N) as a subset of M.

Definition: Let N and M be smooth manifolds. A mapping $F: N \to M$ is a smooth mapping if for each $p \in N$ there exists coordinate charts (U, ϕ) of N and (V, ψ) of M, with $p \in U$ and $F(p) \in V$, such that the expression of F in local coordinates is C^∞.

Definition: A smooth or C^∞ manifold is a manifold equipped with a complete C^∞ atlas.

Definition: A manifold N of dimension n is a topological space which is locally Euclidean of dimension n, is Hausdorff and has a countable basis.

The rank of a mapping $F: N \to M$ at a point $p \in N$ is the rank of the jacobian matrix $\frac{\partial f}{\partial x}$ at $x = \phi(p)$.

Theorem: Let N and M be smooth manifolds, both of dimension n. A mapping $F: N \to M$ is a diffeomorphism if and only if F is bijective, F is smooth, and rank(F) = n at all points of N.

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Controls course Homework #2

Ashley Kulczycki S.2

5) cont.

$$\operatorname{rank}(F) = \operatorname{rank}\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -2\pi \sin 2\pi t \\ 2\pi \cos 2\pi t \\ 1 \end{pmatrix} = 1 \text{ for all } t.$$

The mapping $F: N \to M$, where $N = \mathbb{R}$, $M = \mathbb{R}^3$

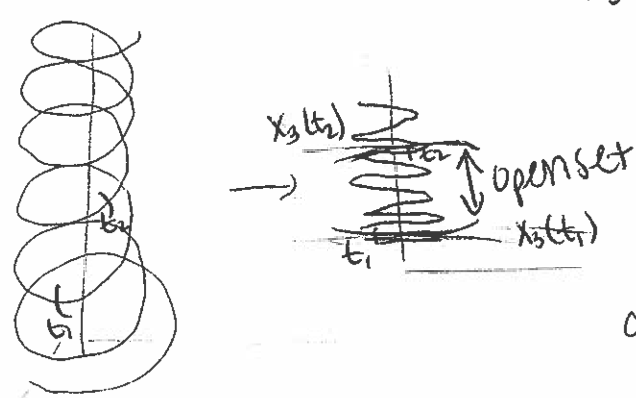
$\dim(N) = 1$ Therefore $\operatorname{rank}(F) = \dim(N) = 1$! (for all $t \to$ all points $p \in N$)

$\Rightarrow$ F is an immersion

F is an univalent imersion since the mapping is one-to-one ($x_3(t)$ is the main reason why (given a time, $t \Rightarrow 1\ x_3(t)$, given an $x_3(t) \to 1$ t value, unique) ($F(t_1) = F(t_2) \Rightarrow t_1 = t_2$)

Now to check on the topology of $F(N)$ as a subset of M.

Take a domain (t_1, t_2) (an open set), of N, which is mapped by F onto a subset U' of $F(N)$. This subset is open by definition of the topology (intersection $\to$ openset). Note that U' is also an intersection of $F(N)$ with an open set in M or $\mathbb{R}^3$.



$x_3(t)$ is the easiest way to look at it.
An open set (t_1, t_2) maps to the openset $(x_3(t_1), x_3(t_2))$, a three dimensional open set free in $x_1(t)$ and $x_2(t)$, but bounded by the plane $x_3(t_1)$ and the plane $x_3(t_2)$.

$\Rightarrow$ F is an embedding since it is an univalent immersion and the topology induced on $F(N)$ by one of N coincides with the topology of $F(N)$ as a subset of M $\square$