University of Notre Dame
Aerospace and Mechanical Engineering

AME 90951: Geometric Nonlinear Control Theory
Homework 3

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Spring 2013

Issued: February 27, 2013
Due: March 7, 2013

This homework is to basically verify many of the theorems considered in the course so far. I could just give you such a system, but it will be better to work you through the process by which I would make it up. It is basically by working through the results "backwards" to get a system, and then verify the system has the properties predicted by the theorem.

1. **Lemma 1.6.1:** If Δ is non-singular, involutive and invariant under f, then there is a coordinate transformation under which

$$\overline{f}(z) = \begin{bmatrix} \overline{f}_1(z_1, z_2, \dots, z_d, z_{d+1}, \dots, z_n) \\ \overline{f}_2(z_1, z_2, \dots, z_d, z_{d+1}, \dots, z_n) \\ \vdots \\ \overline{f}_d(z_1, z_2, \dots, z_d, z_{d+1}, \dots, z_n) \\ \overline{f}_{d+1}(z_{d+1}, \dots, z_n) \\ \vdots \\ \overline{f}_{d+1}(z_n, \dots, z_n) \end{bmatrix}.$$

So, let's do this backwards. Make up

$$\overline{f}(z) = \begin{bmatrix} z_1^2 + 2z_2 + z_1 z_3^3 + z_4 \\ z_1^3 + z_4 \\ z_3 + z_4^2 \\ z_4^3 \end{bmatrix}$$

which has the right decomposition properties for $\overline{f}$ if $d = 2$ and $n = 4$.

Also make up a coordinate transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2z_1 + z_4 \\ z_2 + 3z_3 \\ 2z_1 + 3z_2 + 5z_3 - 6z_4 \\ z_1 + z_2 \end{bmatrix} \qquad \Longleftrightarrow \qquad x = \Phi^{-1}(z).$$

Now, you have to start doing the work.

(a) Is Φ a valid coordinate transformation? Is it a diffeomorphism everywhere in $\mathbb{R}^4$, or only locally?

(b) What is $f(x)$?

(c) Determine two vector fields, $\overline{\tau}_1$ and $\overline{\tau}_2$ in the z coordinates such that $\Delta_z = \text{span}\{\overline{\tau}_1, \overline{\tau}_2\}$ is invariant under $\overline{f}(z)$ and such that Δ_z is involutive.

(d) What is $\Delta_x = \text{span}\{\tau_1, \tau_2\}$?

(e) Is Δ_x invariant under $f(x)$?

2. **Lemma 1.6.2:** Take Δ, f, v_1 and v_2 from Example 1.6.4. Let $f_1 = f$ and make up a f_2 such that Δ is invariant under f_2. Then verify by computation that Δ is invariant under $[f_1, f_2]$. You will receive 90% credit if your made-up f_2 is either trivial itself or if the bracket with f_1 is zero. You will receive 100% credit if things are nontrivial.

3. **Example 1.6.4:** Numerically verify the result illustrated by Figure 1.6 using the vector field from Example 1.6.4. Do this both for the original coordinates, x and the transformed coordinates z. Would the level sets be planes in one or the other sets of coordinates?

4. **Example 1.8.4:** Make up a new example 1.8.4 for Alberto. Start on p. 61 and make up a new $\dot{z}$ system of the right form (keep in mind what this example is trying to demonstrate). The coordinate transformation is right above that, and make up a new one of those as well. From that, back out the system in $\dot{x} = f(x) + g(x)u$ form, and then verify it has the right properties for the decomposition we want.

   Hint: don't go crazy with complexity, especailly in the final form of the system in the z coordinates and with the coordinate transformation. Having a lot of complexity in either of those may make the expression for the original system *very* messy!