Analysis and Control of Networked Embedded Systems

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Networked control systems

- Networked Control Systems (NCS) are spatially distributed systems where the communication among plants, sensors, actuators and controllers occurs in a shared communication network.
- Many aspects of NCS have been investigated, in particular stability and stabilizability problems.
Part I: Symbolic Control Design of Nonlinear Networked Control Systems
  – Mathematical model of nonlinear NCS
  – Symbolic models for NCS
  – Symbolic control design of NCS
  – Efficient control design algorithms

Part II: Modeling, Analysis and Co-Design of Wireless Multi-hop Control Networks (MCN)
  – Mathematical model of linear MCN implementing time-triggered communication protocols
  – Co-design for asymptotic stability and optimal control
  – Fault tolerant control via FDI methods
Part I: Symbolic Control Design of Nonlinear Networked Control Systems
Networked control systems: Our model

\[ u(st + t) = u(st), \quad t \in [0, \tau], s \in \mathbb{N} \setminus 0 \]

\[ \begin{align*}
\{ & x = f(x(t), u(t)) x \in X_{st}, t(0) = x(st), \\
& t \in [0, \tau], s \in \mathbb{N} \setminus 0 \}
\end{align*} \]
Correct-by-design embedded control software:
1. Construct a finite model $T^*(\Sigma)$ of the plant system $\Sigma$
2. Design a finite controller $C$ that solves the specification $S$ for $T^*(\Sigma)$
3. Design a controller $C'$ for $\Sigma$ on the basis of $C$

Advantages:
- Integration of software and hardware constraints in the control design of purely continuous processes
- Use of computer science techniques to address complex specifications
Correct-by-design controller synthesis

- unstable control systems [IEEE-TAC-2012]
- efficient control algorithms [IEEE-TAC-2012]
- networked control systems [HSCC-2012] [IEEE-CDC-2012]
- stable control systems with disturbances [SICON-2009] [IJC-2012]
- stable switched systems [IEEE-TAC-2010]
- incremental stability [Angeli, IEEE-TAC-2002]
- stable control systems [Automatica-2008]
- stable time-varying delay systems [IEEE-CDC-2010]
- stable time-delay systems [SCL-2010]
- networked control systems [IEEE-TAC-2012]
- stable control systems with disturbances [SICON-2009] [IJC-2012]
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Correct-by-design controller synthesis for NCS

networked control systems [HSCC-2012] [IEEE-CDC-2012]

efficient control algorithms [IEEE-TAC-2012]

stable control systems with disturbances [SICON-2009] [IJC-2012]

stable switched systems [IEEE-TAC-2010]

approximate bisimulation [Girard & Pappas, IEEE-TAC-2007]

incremental stability [Angeli, IEEE-TAC-2002]

stable control systems [Automatica-2008]

stable time-delay systems [SCL-2010]

stable time-varying delay systems [IEEE-CDC-2010]
A Labelled Transition System (LTS) is a tuple

\[ T = (Q, L, \rightarrow, O, H) \]

where:

- \( Q \) is the set of states
- \( L \) is the set of labels
- \( \rightarrow \subseteq Q \times L \times Q \) is the transition relation
- \( O \) is the set of outputs
- \( H: Q \rightarrow O \) is the output function

We denote \((q, l, p) \in \rightarrow\) by \( q \rightarrow l \rightarrow p \)

\( T \) is said to be:

- symbolic/finite when \( Q \) and \( L \) are finite
- countable when \( Q \) and \( L \) are countable
- metric when \( O \) is a metric space
Dealing with heterogeneity

Nonlinear Networked control system as an LTS

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>τ</th>
<th>2τ</th>
<th>3τ</th>
<th>4τ</th>
<th>5τ</th>
<th>6τ</th>
<th>7τ</th>
<th>8τ</th>
<th>9τ</th>
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<tbody>
<tr>
<td>u</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>u↓1</td>
<td>u↓1</td>
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<td>u↓1</td>
<td>u↓1</td>
<td>u↓1</td>
<td>u↓2</td>
</tr>
<tr>
<td>x</td>
<td>x(0)</td>
<td>x(τ)</td>
<td>x(2τ)</td>
<td>x(3τ)</td>
<td>x(4τ)</td>
<td>x(5τ)</td>
<td>x(6τ)</td>
<td>x(7τ)</td>
<td>x(8τ)</td>
<td>x(9τ)</td>
<td>...</td>
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</table>

N↓1 = 4  N↓2 = 6
Dealing with heterogeneity

Nonlinear Networked control systems as LTSs

\[(x(0), x(\tau), x(2\tau), x(3\tau)) \xrightarrow{u \downarrow 1} (x(4\tau), x(5\tau), x(6\tau), x(7\tau), x(8\tau), x(9\tau))\]

<table>
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<tr>
<th>t</th>
<th>0</th>
<th>(\tau)</th>
<th>2(\tau)</th>
<th>3(\tau)</th>
<th>4(\tau)</th>
<th>5(\tau)</th>
<th>6(\tau)</th>
<th>7(\tau)</th>
<th>8(\tau)</th>
<th>9(\tau)</th>
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<tr>
<td>u</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>u \downarrow 1</td>
<td>u \downarrow 1</td>
<td>u \downarrow 1</td>
<td>u \downarrow 1</td>
<td>u \downarrow 1</td>
<td>u \downarrow 1</td>
<td>u \downarrow 2</td>
<td>...</td>
</tr>
<tr>
<td>x</td>
<td>x(0)</td>
<td>x(\tau)</td>
<td>x(2(\tau))</td>
<td>x(3(\tau))</td>
<td>x(4(\tau))</td>
<td>x(5(\tau))</td>
<td>x(6(\tau))</td>
<td>x(7(\tau))</td>
<td>x(8(\tau))</td>
<td>x(9(\tau))</td>
<td>...</td>
</tr>
<tr>
<td></td>
<td>N \downarrow 1 = 4</td>
<td></td>
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<td>...</td>
</tr>
<tr>
<td></td>
<td>N \downarrow 2 = 6</td>
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London CPS Workshop, October 20-21, 2012, University of Notre Dame London Centre
Dealing with heterogeneity

Nonlinear Networked control systems as LTSs

\[(x(0), x(\tau), x(2\tau), x(3\tau)) \xrightarrow{u \downarrow 1} (x(4\tau), x(5\tau), x(6\tau), x(7\tau), x(8\tau), x(9\tau)) \quad (N \downarrow 2 = 6)\]

\[(x(4\tau), x(5\tau), x(6\tau), x(7\tau)) \xrightarrow{u \downarrow 1} (x(4\tau), x(5\tau), x(6\tau), x(7\tau)) \quad (N \downarrow 2 = 4)\]

\[(x(4\tau), x(5\tau), x(6\tau), x(7\tau), x(8\tau)) \quad (N \downarrow 2 = 5)\]

Denote by \(T(\Sigma)\) the LTS associated with a NCS \(\Sigma\)

\[
\begin{array}{cccccccccc}
  t & 0 & \tau & 2\tau & 3\tau & 4\tau & 5\tau & 6\tau & 7\tau & 8\tau & 9\tau & \ldots \\
  u & 0 & 0 & 0 & u \downarrow 1 & u \downarrow 1 & u \downarrow 1 & u \downarrow 1 & u \downarrow 1 & u \downarrow 1 & u \downarrow 2 & \ldots \\
  x & x(0) & x(\tau) & x(2\tau) & x(3\tau) & x(4\tau) & x(5\tau) & x(6\tau) & x(7\tau) & x(8\tau) & x(9\tau) & \ldots \\
  N \downarrow 1 = 4 & N \downarrow 2 = 6 & \ldots
\end{array}
\]
Quantifying accuracy

[Pola, Tabuada, SICON-09]

Alternating approximate bisimulation
Given LTSs $T_i = (Q_i, A_i \times B_i, \rightarrow_i, O_i, H_i)$ (i = 1, 2) with $O_1 = O_2$, and a precision $\varepsilon > 0$, consider a relation

$$R \subseteq Q_1 \times Q_2$$

$R$ is an alternating approximate simulation relation of $T_1$ by $T_2$ if for all $(q_1, q_2) \in R$

- $d(H_1(q_1), H_2(q_2)) \leq \varepsilon$

- $\forall a_1 \exists a_2 \forall b_2 \exists b_1$ such that

  $$q_1 \xrightarrow{(a_1,b_1)} p_1 \text{ and } q_2 \xrightarrow{(a_2,b_2)} p_1 \text{ and } (p_1, p_2) \in R$$

$R$ is an alternating approximate bisimulation relation between $T_1$ and $T_2$ if

- $R$ is an alternating approximate simulation relation of $T_1$ by $T_2$
- $R^{-1}$ is an alternating approximate simulation relation of $T_2$ by $T_1$

$T_1$ is $\varepsilon$-alternating simulated by $T_2$, denoted $T_1 \preccurlyeq_\varepsilon T_2$, if $\pi |_{Q_1}(R) = Q_1$

$T_1$ and $T_2$ are $\varepsilon$-alternating bisimilar, denoted $T_1 \equiv_\varepsilon T_2$, if $\pi |_{Q_1}(R) = Q_1$ and $\pi |_{Q_2}(R) = Q_2$.
Symbolic models

**Theorem [HSCC-2012]** For any $\delta$-GAS nonlinear NCS $\Sigma$ with compact state and input spaces,

$$\forall \varepsilon > 0 \; \exists \text{ symbolic transition system } T^*(\Sigma):$$

$$T^*(\Sigma) \models \varepsilon \Sigma$$
Symbolic control design

Problem formulation:
Given a NCS $\Sigma$, a specification LTS $S$ and a desired precision $\varepsilon > 0$, find a symbolic controller $C$ such that:

- $T(\Sigma) \parallel_\mu C \preceq_\varepsilon S$
- $T(\Sigma) \parallel_\mu C$ is non-blocking
Symbolic control design

Solution: $C = Nb \left( T^*(\sum) \parallel_{\mu_x} S \right)$

Drawback:
High computational complexity!

Efficient on-the-fly (off-line) algorithms that integrate the synthesis of $C$ with the construction of $T^*(\sum)$ proposed in:

[Pola, Borri, Di Benedetto, IEEE-TAC-2012]
[Borri, Pola, Di Benedetto, IEEE-CDC-2012]

<table>
<thead>
<tr>
<th>One academic example</th>
<th>Space complexity</th>
<th>Time complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional approaches</td>
<td>2,759,580 data</td>
<td>5,442 sec</td>
</tr>
<tr>
<td>On-the-fly approach</td>
<td>48 data</td>
<td>13 sec</td>
</tr>
</tbody>
</table>
Part II: Modeling, Analysis and co-Design of Wireless Networked Control Systems

- Impact of Delays
- Impact of Scheduling
- Impact of Failures
- Impact of Routing
- Control signals sent to the plant via a controllability network
- Measured data sent to the controller via an observability network
A different level of abstraction

- Network perceived through aggregate performance variables: quantization, packet drops, variable delays and their effect on control system
- Lose information at a lower level of abstraction
A different level of abstraction

- Network perceived through aggregate performance variables: quantization, packet drops, variable delays and their effect on control system
- Lose information at a lower level of abstraction

- Relate network non-idealities to network parameters: topology, transmission power, scheduling, routing:
  - Mathematical model of linear MCN implementing time-triggered communication protocols
  - Co-design for asymptotic stability and optimal control
  - Node failure and malicious intrusion detection, fault tolerant control
WirelessHART MAC layer (scheduling)

- Time is divided in periodic frames, each divided in $\Pi$ time slots, each of duration $\Delta$
- To avoid interference, a periodic scheduling allows each node to transmit data only in a subset of time slots

Diagram:
- Cycle $n-1$, Cycle $n$, Cycle $n+1$
- $T = \Pi \Delta$
- Superframe
WirelessHART network layer (multi-path routing)

- To each pair of nodes source-destination \((v_S, v_D)\) is associated an acyclic graph that defines the set of allowed routing paths
- Redundancy in the routing paths
Multi-hop control networks

Centralized Controller, Relay Network: no data processing (acyclic graph)

Controller Network: linear data processing (cyclic weighted graph)

[Alur, D'Innocenzo, Johansson, Pappas, Weiss, IEEE-TAC-11]

[Pajic, Sundaram, Pappas, Mangharam, IEEE-TAC-11]

Centralized Controller, Relay Network: linear data processing (acyclic weighted graph)

[D'Innocenzo, Di Benedetto, Serra, IEEE-TAC, provisionally accepted, 2012]
Multi-hop control networks model

Syntax:
- Linear plant and discrete-time linear controller
- Weight function $W$ determines data processing through the network - reminiscent of network coding
- Communication scheduling $\eta$ assigns transmission of nodes
Data flow dynamics at time scale of slots

\[ \eta \]

\[ \begin{array}{c|c|c|c|c} \cdots & v_1, v_2 & v_3, v_2 & v_2, v_u & \cdots \end{array} \]

\[ B_1 \rightarrow v_1 \rightarrow v_2 \rightarrow 0 \]

\[ B_3 \rightarrow v_3 \rightarrow v_4 \rightarrow v_u \]

\[ w_{1,2}, w_{3,2}, w_{4,2}, w_{2,u}, w_{3,4} \]
Data flow dynamics at time scale of slots

\[ \eta \]

\[ \begin{array}{c|c|c|c|c} & \cdots & V_1, V_2 & V_3, V_2 & V_2, V_u & \cdots \\ \hline \end{array} \]
Data flow dynamics at time scale of slots

\[ \eta \]

\[ ... \quad V_1, V_2 \quad V_3, V_2 \quad V_2, V_u \quad ... \]

\[ B_1 w_{1,2} + B_3 w_{3,2} \]

\[ V_c \quad w_{c,1} \quad v_1 \quad w_{1,2} \quad v_2 \quad w_{2,u} \quad v_u \]

\[ v_3 \quad w_{3,2} \quad v_2 \quad w_{3,4} \quad v_4 \]

\[ B_3 \]
Data flow dynamics at time scale of slots

$$\eta$$

| ... | $v_1, v_2$ | $v_3, v_2$ | $v_2, v_u$ | ... |

$B_1 w_{1,2} + B_3 w_{3,2}$
Asymptotic stabilizability of a MCN

- Model the semantics of MCN by cascade of discrete time MIMO LTI systems, with sampling time equal to the frame duration.

\[ R(z) \] depends on network topology \( G \), weight function \( W \) and scheduling \( \eta \).

**Theorem:** A MCN is controllable if and only if:

1. \((A, B)\) is controllable
2. At least one scheduled path connects controller and actuator (condition on network topology and on scheduling function \( \eta \)).
3. No zero-pole cancelations (algebraic conditions on weight function \( W \)).

[Smarr, D’Innocenzo, Di Benedetto, NecSys’12]

Transient response to unit-step: optimal \( L_2 \)-norm co-design

[Smarr, D’Innocenzo, Di Benedetto, IEEE-CDC-12]
Fault tolerant stabilizability of a MCN

Let $F = 2 \uparrow E \downarrow R \cup E \downarrow O$ be the set of all configurations of links subject to a failure or a malicious intrusion $M_f$.

Assumptions:
- No fault detection algorithms in the network protocol: only use input to and output from the MCN
- Failures are slow with respect to plant time constants

Problem 1: Guarantee existence of a stabilizing controller for the MCN dynamics $M_f$ associated to any $f \in F$

[Di Benedetto, D'Innocenzo, Serra, IFAC World Congress, 2011]
Fault tolerant stabilizability of a MCN

Let $F = \{ \uparrow E \downarrow \mathcal{R} \cup \downarrow E \downarrow \mathcal{O} \}$ be the set of all configurations of links subject to a failure or a malicious intrusion $M_f$.

Assumptions:
- No fault detection algorithms in the network protocol: only use input to and output from the MCN
- Failures are slow with respect to plant time constants

Problem 2: Design a dynamical system (FDI) able to detect and isolate any $f \in F$

[D’Innocenzo, Di Benedetto, Serra, IEEE-CDC-ECC-11]
Conclusions

Part I
- Mathematical model of general class of nonlinear NCS
- Symbolic models for NCS
- Symbolic controllers for NCS
- Efficient control algorithms

Part II
- Mathematical framework for co-design of control networks implementing time-triggered protocols
- Relate properties of multi-hop control networks and network configuration (topology, scheduling and routing)
- Fault tolerant control:
  - Permanent failures and malicious attacks via FDI
  - Transient failures (packet losses): work in progress
"Symbolic control design of Cyber-Physical systems"

29/04/2013 – 03/05/2013

Istanbul (Turkey)

www.eeci-institute.eu
Given a NCS $\Sigma$ define the LTS $T(\Sigma) = (Q \downarrow \tau, Q \downarrow 0, \tau, L \downarrow \tau, \rightarrow \downarrow \tau, O \downarrow \tau, \mathit{H} \downarrow \tau)$ where:

- $Q \downarrow \tau \subseteq Q \downarrow 0 \cup Q \downarrow e$ where $Q \downarrow e := UN = N \downarrow \mathit{min} \uparrow N \downarrow \mathit{max} \uplus Q \uparrow N$ and for any $q = (x_1, x_2, ..., x_{\downarrow N}) \in Q \uparrow N, x \downarrow i + 1 = x(\tau, x \downarrow i, u \uparrow -), i \in [1; N - 2]$, and $x \downarrow N = x(\tau, x \downarrow N - 1, u \uparrow +)$ for some control inputs $u \uparrow -, u \uparrow +$

- $Q \downarrow 0, \tau = Q \downarrow 0$

- $L \downarrow \tau = [U] \downarrow \mu \downarrow U$

- $q \uparrow 1 \uparrow \downarrow \tau \downarrow \uparrow q \uparrow 2$ where, for some $N \downarrow 1, N \downarrow 2 \in [N \downarrow \mathit{min}; N \downarrow \mathit{max}]$
  
  $x \downarrow i + 1 \uparrow 1 = x(\tau, x \downarrow i \uparrow 1, u \downarrow 1 \uparrow -), i \in [1; N \downarrow 1 - 2]$
  
  $x \downarrow N \uparrow 1 = x(\tau, x \downarrow N \downarrow 1 - 1 \uparrow 1, u \downarrow 1 \uparrow +)$
  
  $x \downarrow i + 1 \uparrow 2 = x(\tau, x \downarrow i \uparrow 2, u \downarrow 2 \uparrow -), i \in [1; N \downarrow 2 - 2]$
  
  $x \downarrow N \uparrow 2 = x(\tau, x \downarrow N \downarrow 2 - 1 \uparrow 2, u \downarrow 2 \uparrow +)$
  
  $u \downarrow 2 \uparrow - = u \downarrow 1 \uparrow +$
  
  $u \downarrow 2 \uparrow + = u$

- $O \downarrow \tau = X \downarrow \tau$

- $H \downarrow \tau$ is the identity function
\( T(\Sigma) \) collects all the information of the NCS \( \Sigma \) available at the sensor, but it is not a symbolic model. We therefore propose a symbolic model by quantizing the state space \( X \) of the plant \( P \)

Given \( x \in X \) let \( [x] \downarrow \mu \downarrow X \) \( \overline{[X]} \downarrow \mu \downarrow X \) be such that \( || x - [x] \downarrow \mu \downarrow X || \leq \mu \downarrow X \)
Appendix B (2/4)

Define the system $T^*(\Sigma) = (Q\downarrow*, Q\downarrow0,*, L\downarrow*, \rightarrow, \downarrow*, O\downarrow*, H\downarrow*)$

where:

- $Q\downarrow* \subseteq [Q\downarrow0 \cup Q\downarrow e] \downarrow \mu \downarrow x$ s.t. for any $q^\uparrow* = (x\downarrow1^\uparrow*, x\downarrow2^\uparrow*, ..., x\downarrow N^\uparrow*) \in Q\downarrow*$, $x\downarrow i+1^\uparrow* = \lfloor x(\tau,x\downarrow i^\uparrow*, u\downarrow*^\uparrow-) \rfloor \downarrow \mu \downarrow x$, $i \in [1; N-2]$, and $x\downarrow N^\uparrow* = \lfloor x(\tau,x\downarrow N -1^\uparrow*, u\downarrow*^\uparrow+) \rfloor \downarrow \mu \downarrow x$ for some $u\downarrow*^\uparrow-$, $u\downarrow*^\uparrow+$

- $Q\downarrow0,* = \lfloor X\downarrow0 \rfloor \downarrow \mu \downarrow x$

- $L\downarrow* = \lfloor U \rfloor \downarrow \mu \downarrow u$

- $q^\uparrow1 \rightarrow \downarrow*^\uparrow u\downarrow*^\uparrow q^\uparrow2$ where, for some $N\downarrow1$, $N\downarrow2$

-x\downarrow i+1^\uparrow1 = \lfloor x(\tau,x\downarrow i^\uparrow1, u\downarrow1^\uparrow-) \rfloor \downarrow \mu \downarrow x$, $i \in [1; N-2]$

-x\downarrow N^\uparrow1 = \lfloor x(\tau,x\downarrow N\downarrow1 -1^\uparrow1, u\downarrow1^\uparrow+) \rfloor \downarrow \mu \downarrow x$

-x\downarrow i+1^\uparrow2 = \lfloor x(\tau,x\downarrow i^\uparrow2, u\downarrow2^\uparrow-) \rfloor \downarrow \mu \downarrow x$, $i \in [1; N-2]$

By construction $S\downarrow^*(\Sigma)$ is a symbolic model!
Given a nonlinear control system \( x = f(x, u) \), a smooth function \( V: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be a \( \delta \)-GAS Lyapunov function for \( P \) if there exist \( \lambda \in \mathbb{R}^+ \) and \( K_\infty \) functions \( \alpha_1, \alpha_2 \) such that, for any \( x_1, x_2 \in \mathbb{R}^n \) and any \( u \in U \)

1) \( \alpha_1(||x_1 - x_2||) \leq V(x_1,x_2) \leq \alpha_2(||x_1 - x_2||) \);

2) \( \frac{\partial V}{\partial x_{11}} f(x_{11},u) + \frac{\partial V}{\partial x_{22}} f(x_{22},u) \leq -\lambda V(x_1,x_2) \).

Theorem [Angeli, IEEE-TAC-2002]

A nonlinear control system \( x = f(x, u) \) is \( \delta \)-GAS if it admits a \( \delta \)-GAS Lyapunov function.
Theorem 1 [HSCC-2012]

Consider the NCS Σ and suppose that the plant nonlinear control system P enjoys the following properties:

1. There exists a δ-GAS Lyapunov function for Σ, hence there exists $\lambda \in \mathbb{R}^+$ s.t. for any $x_1, x_2 \in X$, and any $u \in U$

   \[
   \frac{\partial V}{\partial x_1} f(x_1, u) + \frac{\partial V}{\partial x_2} f(x_2, u) \leq -\lambda V(x_1, x_2).
   \]

2. There exists a $K_{\infty}$ function $\gamma$ such that

   \[
   V(x, x^\uparrow') \leq V(x, x^\uparrow'') + \gamma(\|x' - x''\|)
   \]

   for every $x, x^\uparrow', x'' \in X$.

Then for any desired precision $\varepsilon > 0$, any sampling time $\tau > 0$, and any state quantization $\mu \downarrow x > 0$ such that
How to capture interaction between the symbolic model and the symbolic controller?

Approximate parallel composition

**Def [Tabuada, IEEE-TAC-2008]**

Given $T_1 = (Q_1, L_1, O_1, H_1)$ and $T_2 = (Q_2, L_2, O_2, H_2)$, with $O_1 = O_2$, and a precision $\theta > 0$, the approximate composition of $T_1$ and $T_2$ is the system

$$T_1 \parallel_{\theta} T_2 = (Q, L, O, H)$$

where:

- $Q = \{(q_1, q_2) \in Q_1 \times Q_2 : d(H_1(q_1), H_2(q_2)) \leq \theta\}$
- $L = L_1 \times L_2$
- $(q_1, q_2) \xrightarrow{(l_1, l_2)} (q'_1, q'_2)$, if $q_1 \xrightarrow{l_1} q'_1$ and $q_2 \xrightarrow{l_2} q'_2$
- $O = O_1$
- $H(q_1, q_2) = H_1(q_1)$
Synthesis through a three-step process:

1. Compute the symbolic model $T^*(\Sigma)$ of $\Sigma$

2. Compute the approximate parallel composition $C^* = T^*(\Sigma) || \mu_x S$

3. Compute the maximal robust non-blocking part $\text{Nb}(C^*)$ of $C^*$
Synthesis through a three-step process:

1. Compute the symbolic model $T^*(\Sigma)$ of $\Sigma$
2. Compute the approximate parallel composition $C^* = T^*(\Sigma) \parallel_{\mu x} S$
3. Compute the maximal robust non-blocking part $\text{Nb}(C^*)$ of $C^*$
Appendix D (3/5)

Synthesis through a three-step process:

1. Compute the symbolic model $T^*(\Sigma)$ of $\Sigma$
2. Compute the approximate parallel composition $C^* = T^*(\Sigma) \parallel \mu_x S$
3. Compute the maximal robust non-blocking part $Nb(C^*)$ of $C^*$
Appendix D (4/5)

Synthesis through a three-step process:

1. Compute the symbolic model $T^*(\Sigma)$ of $\Sigma$

2. Compute the approximate parallel composition $C^* = T^*(\Sigma) \parallel_\mu S$

3. Compute the maximal robust non-blocking part $Nb(C^*)$ of $C^*$
Synthesis through a three-step process:

1. Compute the symbolic model $T^*(\Sigma)$ of $\Sigma$

2. Compute the approximate parallel composition $C^* = T^*(\Sigma) \parallel_{\mu_\times} S$

3. Compute the maximal robust non-blocking part $\text{Nb}(C^*)$ of $C^*$
Appendix E – Joint connectivity

**Definition:** Given a multi-hop network $G$ and the associated scheduling $\eta$, we define $G(\eta)$ the subgraph of $G$ induced by the set of all edges scheduled by $\eta$ during the whole frame.

**Definition:** We say that $G$ is jointly connected by the scheduling $\eta$ if and only if there exists a path from the source node $v_S$ to the destination node $v_D$ in $G(\eta)$.
Appendix F (1/2) – Conditions on $W \downarrow R$

Consider a discrete-time MIMO LTI system described by the $l \times m$ transfer function matrix $H(z)$.

$$\Theta \doteq \{(J, K) : J \subseteq m, K \subseteq l, |J| = |K| \geq 1\}$$

Set of all combinations of rows and columns of a $l \times m$ matrix such that the number of rows is equal to the number of columns.

$$\left\{\left|H_{J,K}(z)\right| : (J, K) \in \Theta\right\}$$

Set of all minors of $H(z)$.

$$\psi^H_{J,K}(z) = \delta_H(z) \left|H_{J,K}(z)\right|$$

For $(J, K) \in \Theta$, are the zero polynomials of $H(z)$, where $d_H(z)$ is the characteristic polynomial of $H$.

$$\psi^H(z) = \gcd\left(\psi^H_{J,K}(z), \forall (J, K) \in \Theta\right)$$

Least zero polynomial, namely the greatest common divisor of all zero polynomials of $H(z)$.

**Theorem [Tarokh, ACC-1986]:** A MIMO LTI system with transfer function matrix $H(z)$ is controllable and observable if and only if the scalar transfer function $\psi^H(z)/\delta_H(z)$ has no pole-zero cancelations.
Lemma: The zero polynomials of a MCN M are given by the following expression:

\[
\psi_{J,K}^M(z) = \delta_M(z) \cdot O_{J,J}(z) \cdot P_{J,K}(z) \cdot R_{K,K}(z)
\]

where \( \delta_M(z) = \delta_O(z) \delta_P(z) \delta_R(z) \), \( \forall (J,K) \in \Theta \).

Theorem: A MCN M is controllable and observable if and only if the following hold:

(5a) for all \( i \in \mathcal{M} \) and for all \( j \in \mathcal{L} \), the pairs \( (G\downarrow R, \eta\downarrow R\downarrow i) \) and \( (G\downarrow O, \eta\downarrow O\downarrow i) \) are jointly connected;

\[
\exists (J,K) \in \Theta \text{ s.t. } \psi_{J,J}^O(\bar{p}) \neq 0 \land \psi_{J,K}^P(\bar{p}) \neq 0 \land \psi_{K,K}^R(\bar{p}) \neq 0;
\]

(5b) for each root \( p \) of \( \delta\downarrow P(\bar{z}) \),

(5c) \( m = \mathcal{L} \) and \( \psi_{\downarrow M, \uparrow (0)} \neq 0 \)

Corollary: A MCN M is controllable and observable if Conditions (5a),(5c) hold and:

(6b) for each root \( p \) of \( \delta\downarrow P(\bar{z}) \), the numerators of \( R\downarrow i(\bar{z}) \) and of \( O\downarrow i(\bar{z}) \) do not have roots in \( p \) for all \( i \in \mathcal{M} \) and for all \( j \in \mathcal{L} \).