Proceedings of the 2002 IEEE
International Conference on Robotics & Automation
Washington, DC • May 2002

Reduction and Nonlinear Controllability of Symmetric Distributed Robotic Systems with Drift

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Abstract—**This paper considers the "reduction" problem for distributed robotic systems. In particular, controllability of systems containing multiple instances of identical robotic systems or components where the overall system is invariant with respect to interchanging these identical robots or components is considered. The main result is a proposition which shows that for an equivalence class of symmetric systems of this type, controllability of the entire class of systems can be determined by analyzing the smallest member of the equivalence class.**

1 Introduction

This paper considers symmetric distributed robotic systems which consist of, perhaps many, interacting robots working together to perform a task. As the number of interacting robots increases, so does the overall dimension and complexity of the system. There have been many efforts exploring high level planning and coordination between groups of robots [1], [2], [3], [4], [5]; however, none of these attempted to directly exploit any of the symmetry properties of distributed systems. The aim of this work is to consider discrete symmetries to "reduce" the order of complexity of large-scale distributed systems. In this paper, we consider the controllability of nonlinear robotic systems with equations of motion of the following, general form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \tag{1}$$

where $f(x)$ and $g(x)$ are smooth analytic vector fields and the u_i's are admissible control inputs. The vector field $f(x)$ is referred to as the *drift* vector field and the $g_i(x)$ are defined as *input* vector fields.

The main development of this paper is a proposition that can be used to check the controllability of large-scale systems by analyzing lower order systems. For multi-robot systems, an analysis of a system containing an arbitrarily large number of identical interacting robots could be undertaken on an much smaller set of robots. In fact, it is shown that the controllability of an entire equivalence class of systems can be determined by checking only one of its members. A graph theoretical representation is developed to aid in the analysis of the system and to clearly illustrate the interaction between robots. Figure 1 displays a graph of a distributed robotic system. Each vertex of the graph represents a robot or a group of robots in the overall system and an edge denotes the fact that the connected nodes affect each other in some manner. If nodes 2-11 are identical the system is characterized by an S_{10} symmetry (the symmetric group of order 10). This is a consequence of the fact that each of the robots 2-11 can be interchanged without altering the system.

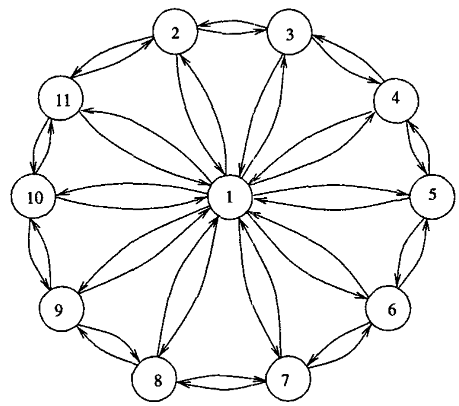


Figure 1. A graph of eleven robots.

The main utility of this work is that it provides a means to make the analysis of very large scale cooperating or interacting robotic systems tractable. While this paper is limited to controllability, efforts are also under way to extend the basic framework of the approach presented here to the analysis of stability of such systems as well as the control synthesis problem. In whole, a complete package of controller design tools is envisioned that are tractable in the sense that the analysis or synthesis problem could be directed toward a smaller-scale system that then could be utilized on an arbitrarily larger-scale system.

There have been many efforts directed toward the reduction of mechanical systems [6], [7], [8], [9], [10], [11] and control systems [12], [13]. However, these results are directed toward cases of Lie group symmetries. In contrast, this paper considers discrete symmetries. Distributed control systems has also been given much attention [14], [15], [16], [17], [18], [19]; however, these results are typically limited to *linear* systems. The method found in [16] considers linear systems with time-varying nonlinear connections, but not fully nonlinear systems and the main results in [17], [18], [19] are not directed at "reduction" but rather determining the fault tolerance of linear systems. The main result of this paper is an extension of previous work [20], which is limited to driftless systems. This work is also closely related to decentralized systems [21], [22], [23]. Again, however, the results in these references are limited to linear systems.

The remainder of this paper is organized as follows. Section 2 develops a graph theoretic description of nonlinear distributed system with drift, defines the notion of vector field equivalence, which leads to the definition of symmetric systems. Using the symmetry group of a system naturally defines an entire equivalence class of symmetric systems. Section 3 provides a brief review of controllability and accessibility for nonlinear distributed systems and presents the main result of the of the paper. This is followed by an illustrative example in Section 4. Section 5 gives a brief summary of the results and a description of future work.

2 Symmetric Distributed Robotic Systems

This section presents the definitions and notation used in our analysis of nonlinear distributed robotic systems. In this section we define vector field equivalence, define symmetric systems and define an equivalence relation between systems of nonlinear distributed robots. From the equivalences relation, an equivalence class is defined for which controllability is maintained.

2.1 Nonlinear Distributed Nonlinear Systems with Drift

We will consider smooth analytic systems of the form

$$\Sigma: \begin{array}{rcl} \dot{x}_1 & = & f_1(x) + g_{1,1}(x)u_{1,1} + g_{2,1}(x)u_{2,1} + \cdots \\ \dot{x}_2 & = & f_2(x) + g_{1,2}(x)u_{1,2} + g_{2,2}(x)u_{2,2} + \cdots \\ & \vdots & \\ \dot{x}_n & = & f_n(x) + g_{1,n}(x)u_{1,n} + g_{2,n}(x)u_{2,n} + \cdots , \end{array} \tag{2}$$

where M is a smooth manifold, $x \in M$, f_i and $g_{i,j}$ are smooth vector fields on M. The notation $u_{i,j}$ denotes the jth control input associated with robot i and $g_{i,j}$ is the associated input vector field. A drift term that is a function of states in node i is denoted by f_i.

It may be the case that the overall system is characterized by limited interaction among the robots, in which case each robot may be only affected by a small subset of other robots. To help provide a clear representation of such distributed robotic systems, we will utilize a graph-theoretic model of the system. Formally, we define a *digraph* of a nonlinear system, Σ, written as $\mathcal{G}_\Sigma$, to be the pair $(\mathbf{V}, \mathbf{E})$ consisting of a set of vertices $\mathbf{V} = \{V_1, V_2, \ldots, V_m\}$ and a set of directed edges, denoted by $\mathbf{E}$, which are ordered pairs of elements from $\mathbf{V}$. Each *vertex* $V_i \in \mathbf{V}$, represents a regular submanifold of the configuration space, M, such that M is the Cartesian product of the vertices, *i.e.*, $M = V_1 \times V_2 \times \cdots \times V_m = \prod_{i=1}^{m} V_i$. The *edge directed from* V_i to V_j, denoted $E_{i,j} = \{V_i, V_j\} \in \mathbf{E}$, represents the vector fields which map elements of the vertices V_i and V_j to the tangent space of V_j, *i.e.*,

$$E_{i,j} : V_i \times V_j \mapsto TV_j.$$

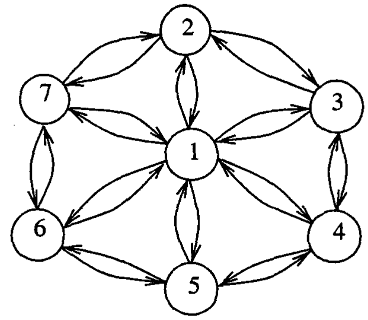


Figure 2. Digraph for eight node system.

The edge $E_{i,j}$ is the jth component of the driftless vector fields $g_{i,k}(x)$ from Equation 2 that multiply the control inputs associated with the node i and the drift vector field $f_{i,j}(x_i, x_j)$, where $f_i(x) = \sum_{j=1}^{n} f_{i,j}(x_i, x_j)$. That is, the $f_{i,j}(x_i, x_j)$ vector field is composed of the components of $f_i(x)$ that takes states from node V_i and V_j and maps them to TV_j.

EXAMPLE 2.1 Consider the system,

$$\Sigma: \quad \dot{x} = f(x) + g_2(x)u_2 + \cdots + g_7(x)u_7,$$

described by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_1x_2 + x_3 \\ x_1x_3 + x_4 \\ x_1x_4 + x_5 \\ x_1x_5 + x_6 \\ x_1x_6 + x_7 \\ x_1x_7 + x_2 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \\ x_2^2 \\ 0 \\ 0 \\ 0 \\ x_2 \end{bmatrix} u_2 + \cdots + \begin{bmatrix} x_7 \\ x_7^2 \\ 0 \\ 0 \\ 0 \\ x_7 \\ 1 \end{bmatrix} u_7$$

where $x_i \in V_i$ and each vertex has one associated input u_i for $i = \{2, \ldots, 7\}$. (Note that in general each vertex can be represent more than one state and associated with which could be more than one input). Figure 2 displays the digraph of this system. For example, $f_2(x) = x_1x_2 + x_3$ which can be written as $f_2(x) = f_{2,1}(x_1, x_2) + f_{2,2}(x_2) + \cdots + f_{2,n}(x_2, x_n)$, where $f_{2,1}(x_1, x_2) = x_1x_2$, $f_{2,2}(x_2) = 0$, $f_{2,3}(x_2, x_3) = x_3$, and $f_{2,4}(x_2, x_4) = \cdots = f_{2,n}(x_2, x_n) = 0$. Furthermore, the first slot in the vector field $g_2(x)$ is, $g_{2,1}(x) = x_2$, so the in the graphical representation of this system, $\mathcal{G}_\Sigma$, the edge directed from V_1 to V_2 is given by,

$$E_{2,1} = f_{2,1}(x) + g_{2,1}(x)u_{2,1},$$

which is given by,

$$E_{2,1} = x_1x_2 + x_2u_2.$$

A more rigorous definition of symmetric nodes follows shortly, but at this point note that the vertices V_2 and V_7 are symmetric to the extent that, for example, the way node 2 affects node 3, $E_{2,3} = x_2^2u_2$, is the same as the way node 7 affects node 2, $E_{2,7} = x_7^2u_7$. Because of symmetry, these two nodes can be interchanged without affecting the dynamics of the system. By "interchanged" we mean a physical interchange, *i.e.,* if, V_2 and V_7 are interchanged they affect and are affected by their neighbors in the same way that they affect and are affected by their new neighbors. The goal of this paper is to exploit this type of symmetry to formulate a simpler reduced order analyzes of such systems.

Since it is possible to have multiple inputs associated with one node, it may be necessary to further distinguish edges by representing to which vector field within the subsystem an edge is associated. To do this, a third subscript can be added, *i.e.,*

$$E_{i,j,k} : V_i \times V_j \mapsto TV_j.$$

This edge, $E_{i,j,k}$ still maps between the same spaces, but the third subscript indicates that it is the jth component of $g_{i,k}$. To avoid unnecessary notational complexity, we will often drop the third subscript and use $E_{i,j}$ to represent the ordered set of vector fields, $E_{i,j} = \{E_{i,j,1}, E_{i,j,2}, \ldots\}$.

Interchanging nodes is more than interchanging vector fields. We introduce notation that will makes interchanging nodes a straightforward process. Let $\tilde{V}_i = V_i = \{V_{\tilde{i}_1}, V_{\tilde{i}_2}, \ldots, V_{\tilde{i}_n}\}$ be an ordered set of vertices which are connected to V_i and directed to the elements of $\tilde{V}_i$ and let $\tilde{E}_i = \{E_{i,\tilde{i}_1}, \ldots, E_{i,\tilde{i}_M}\}$ be an ordered set of edges directed from V_i to elements of $\tilde{V}_i$. The manner in which $\tilde{V}_i$ and $\tilde{E}_i$ are ordered is determined by interactions and/or communications between the nodes. To illustrate the use of this notation, consider the system given in Example 2.1. The ordered set of vertices for , V_2, and V_7 are $\tilde{V}_2 = \{V_{\tilde{2}_1} = V_1, V_{\tilde{2}_2} = V_2, V_{\tilde{2}_3} = V_3, V_{\tilde{2}_4} = V_7\}$ and $V_7 = \{V_{\tilde{7}_1} = V_1, V_{\tilde{7}_2} = V_7, V_{\tilde{7}_3} = V_2, V_{\tilde{4}} = V_6\}$, respectively. Furthermore, the corresponding sets of ordered edges are $\tilde{E}_2 = \{E_{2,\tilde{2}_1} = E_{2,1}, E_{2,\tilde{2}_2} = E_{2,2}, E_{2,\tilde{2}_3} = E_{2,3}, E_{2,\tilde{2}_4} = E_{2,7}\}$ and $\tilde{E}_7 = \{E_{7,\tilde{7}_1} = E_{7,1}, E_{7,\tilde{7}_7} = E_{7,7}, E_{7,\tilde{7}_3} = E_{7,2}, E_{7,\tilde{7}_4} = E_{7,6}\}$.

2.2 Symmetric Distributed Nonlinear Systems

This section defines what it means for a distributed robotic system to be symmetric. Recall, the motivating idea is that there is a subset of individual robots that can be interchanged without changing the dynamics of the system. Mathematically, this will be represented by the fact that vector fields from various nodes will be "equivalent." Since the vector fields directed from different nodes are defined on different spaces, we need a definition of equivalence which is more than just requiring them to be "identical."

DEFINITION 2.2: (VECTOR FIELD EQUIVALENCE) Two vector fields, g_1 And g_2 are *equivalent,* denoted $g_1 \sim g_2$, if there exists a diffeomorphism, $\psi : M \mapsto M$, such that

$$\psi_* \circ g_1 = g_2.$$

Equivalently, we can define $E_{i,j} \sim E_{k,l}$ by only considering the jth and lth components of g_i and g_k, respectively. Recall the definition of the push forward of a vector field is $\psi_* g = T\psi \circ g \circ \psi^{-1}$.

Typically equivalence is related to a permutation of coordinates. Recall that the symmetric group of order $p!$, denoted S_p, is the group of permutations of p objects. The group S_p is commonly referred to as the symmetry group. From Example 2.1, the vector fields g_2 and g_7 are equivalent. To see this, let ψ be the diffeomorphic permutation of states given by

$$\psi\left([x_1, x_2, x_3, x_4, x_5, x_6, x_7,]^T\right) = [x_1, x_3, x_4, x_5, x_6, x_7, x_2]^T$$

then $\psi_* \circ g_2 = g_7$ since

$$\psi^{-1}g_2(x) = [x_7, 1, x_7^2, 0, 0, 0, x_7,]^T$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_7 \\ 1 \\ x_7^2 \\ 0 \\ 0 \\ 0 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_7 \\ x_7^2 \\ 0 \\ 0 \\ 0 \\ x_7 \\ 1 \end{bmatrix} = g_7. \tag{3}$$

Using this group, we can now define a symmetric distributed system as follows.

DEFINITION 2.3: (SYMMETRIC DISTRIBUTED SYSTEM)
Let a *symmetry orbit*, $\mathbf{O} \subset \mathbf{V}$, be a subset of $\mathbf{V}$ containing p vertices, *i.e.*, $\mathbf{O} = \{V_{k_1}, V_{k_2}, \ldots, V_{k_p}\}$, let $\mathbf{F} \subset \mathbf{V}$ be the subset of $\mathbf{V} \setminus \mathbf{O} \subset \mathbf{V}$ containing $n-p$ *fixed* vertices, *i.e.*, $\mathbf{F} = \{V_{f_1}, \ldots, V_{f_{n-p}}\}$, let $\tilde{V}_{k_l}$ be the ordered set of vertices connected to V_{k_l}, and let $\rho \in S_p$. The system Σ is a *symmetric nonlinear distributed system* if

$$g_{k_i} \sim g_{\rho(k_i)} \qquad \forall i \in \{1, \ldots, p\} \text{ and } \forall \rho \in S_p.$$

Equivalently, a system is a symmetric nonlinear distributed system if

$$E_{k,\tilde{k}_l} \sim E_{\rho(k),\rho(\tilde{k})_l} \quad \text{and} \quad E_{\tilde{k}_l,k} \sim E_{\rho(\tilde{k})_l,\rho(k)},$$

$\forall k \in \{k_1, \ldots, k_p\}$, $\forall l \in \{1, \ldots, (\tilde{k}_l)_m\}$, and $\forall \rho \in S_p$.

The system from Example 2.1 is a symmetric distributed system where the symmetry orbit contains robots 2 through 7 and the system is characterized by an S_6 symmetry corresponding to all permutations of the indices $\{2, \ldots, 7\}$, which can be verified by way of computations similar to that in Equation 3.

2.3 Equivalent Symmetric Nonlinear Distributed Systems

We will now define an equivalence relation among symmetric nonlinear distributed systems. An equivalence class of systems can be defined using the developed equivalence relation. It will be shown that for this class of systems, controllability of the entire equivalence class can be determined by determining controllability of just one its members; in particular, the member that has the fewest vertices in its symmetric orbit.

Before we define nonlinear equivalence for distributed systems, we must develop a method that allows us to compare relative size of two systems. Let Σ_1 and Σ_2 be symmetric nonlinear distributed systems and let $\mathcal{G}_{\Sigma_1} = \{\mathbf{V_1}, \mathbf{E_1}\}$ and $\mathcal{G}_{\Sigma_2} = \{\mathbf{V_2}, \mathbf{E_2}\}$ denote their corresponding digraphs. We say $\mathcal{G}_{\Sigma_1} \geq \mathcal{G}_{\Sigma_2}$ if the number of vertices in $\mathcal{G}_{\Sigma_2}$ is greater than the number of vertices in $\mathcal{G}_{\Sigma_1}$.

DEFINITION 2.4: (EQUIVALENT NONLINEAR DISTRIBUTED ROBOTIC SYSTEMS)
Let Σ_1 and Σ_2 be symmetric nonlinear distributed robotic systems and $\mathcal{G}_{\Sigma_1} \geq \mathcal{G}_{\Sigma_2}$. Since each system is a symmetric nonlinear distributed system there exist symmetry orbits $\mathbf{O_1} \subset \mathbf{V_1}$ and $\mathbf{O_2} \subset \mathbf{V_2}$ containing p and q $(p \geq q)$ vertices, respectively, *i.e.*,

$$\mathbf{O_1} = \{V_{(k_1)_1}, V_{(k_1)_2}, \ldots, V_{(k_1)_p}\}$$

and

$$\mathbf{O_2} = \{V_{(k_2)_1}, V_{(k_2)_2}, \ldots, V_{(k_2)_q}\}.$$

The systems Σ_1 and Σ_2 are *equivalent symmetric nonlinear distributed systems* if

1. $E_{k,(\tilde{k}_l)_1} \sim E_{k,(\tilde{k}_l)_2} \quad \forall k \in \{k_1, \ldots, k_q\}, \forall l \in \{1, \ldots, (\tilde{k}_l)_m\}$;
2. $\mathbf{F_1} = \mathbf{V_1} \setminus \mathbf{O_1}$ and $\mathbf{F_2} = \mathbf{V_2} \setminus \mathbf{O_2}$ contain the same number of vertices, *i.e.*, $\mathbf{F_1} = \mathbf{F_2} = \{V_1, \ldots, V_m\}$; and,
3. $E_{k,(\tilde{k}_l)_1} \sim E_{k,(\tilde{k}_l)_2} \quad \forall k \in \{1, \ldots, m\}, \forall l \in \{1, \ldots, (\tilde{k}_l)_m\}$.

Denote the equivalence class of systems defined by this equivalence relation by $\bar{\Sigma}$.

Basically, this definition simply requires that all of the robots in the symmetry orbit of the smaller system be equivalent to robots in the symmetry orbit of the larger system. Since it is a symmetry orbit, then the robots from the orbit from the smaller system are automatically similar to *all* the robots in the orbit of the larger system. Intuitively, all that has happened to go from the smaller to larger system is that identical robots have been inserted into the symmetry orbit of the smaller system to create the larger system.

3 Reduction and Controllability

First, it is necessary recall the definition of the term "controllable" for the system

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m.$$

DEFINITION 3.1: (CONTROLLABILITY)
A system is *small time locally controllable* ("STLC," or simply "controllable") if the set of states that are reachable in time T contains a neighborhood of x_0 for all $T > 0$.

Let $\mathcal{C}$ denote the smallest subalgebra of $V^\infty(M)$ (the Lie algebra of smooth vector fields on a manifold M whose product is the Lie bracket, $[\cdot,\cdot]$) that contains $f, g_1, \ldots, g_m$. If $\dim(\mathcal{C}) = \dim(M)$ at a point x, then the system described by Equation 2 satisfies the *Lie Algebra Rank Condition* ("LARC") at x. The following is well known as "Chow's Theorem."

THEOREM 3.2 *If the system described by Equation 2 is such that $f_i(x) \equiv 0$ $\forall i$ and satisfies the LARC at a point x_0 then it is STLC from x_0.*

If an edge contains a drift vector field, then Theorem 3.2 only proves accessibility, *i.e.*, the set of reachable points from x_0 is open, but may not contain an neighborhood of x_0. Controllability for systems with drift requires a stronger result. First, distinguish between "good" brackets and "bad" brackets as follows. Call a Lie bracket "bad" if it contains an odd number of vector fields and an even number of each of the driftless vector fields, $g_{i,j}$, where the term "even" includes zero. If a Lie bracket is not "bad", then it is "good." For example,

f	(1 total vector field) is bad.
$[g_1, g_2]$	(2 total vector fields) is good.
$[f, [g_1, g_2]]$	(3 total vector fields) is bad.
$[g_1, [g_1, g_2]]$	(3 total vector fields) is good.

Note that even though $[g_1, [g_1, g_2]]$ has an odd number of vector fields it is still a good Lie bracket because it also contains an odd number of driftless vector fields. The following is a general theory for local controllability [24].

THEOREM 3.3 *If a system, Σ, satisfies LARC and all bad brackets are spanned by lower order good brackets, then Σ is STLC.*

The following proposition is the main result of the paper.

PROPOSITION 3.4 *If any one member, Σ_n, of the equivalence class of symmetric distributed control systems, $\overline{\Sigma}$ satisfies Theorem 3.3, then all larger members of the equivalence class, $\Sigma_i \in \overline{\Sigma}$ where $i > n$ of symmetric distributed control systems are STLC.*

Proof: First we prove accessibility. Assume that $\Sigma_n \in \overline{\Sigma}$, satisfies the LARC. Partition the configuration manifold into submanifolds corresponding to each node in the control system, *i.e.,* let $M = \Pi_{i=1}^{n} V_i$. Let Σ_n contain n nodes with m nodes in the symmetry orbit, and denote Δ_i to be the subdistribution of $\overline{\Delta}_n$ which spans the tangent bundle to the submanifold associated with node i. We will show that if Σ_n is satisfies the LARC, then Σ_{n+1} satisfies the LARC, and then the result follows by induction.

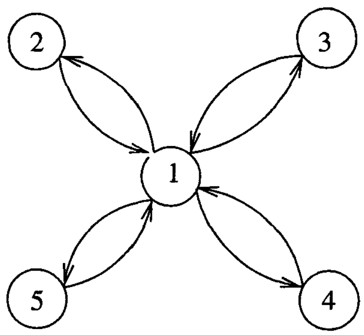
For Σ_{n+1}, recall that the relationship between Σ_n and Σ_{n+1} was that a vertex, V_j, was added to the symmetry orbit. We will consider separately the fixed nodes, nodes in the symmetry orbit not V_j and not adjacent to V_j, the nodes adjacent to V_j and V_j itself. For a node, V_i, in the orbit not adjacent to V_j, denoted by Δ_i, let the collection of vector fields $\{X_1, \ldots, X_q\} \in \overline{\Delta}_n$ span Δ_i. Then in Σ_{n+1} the same set of vector fields will span Δ_i in Σ_{n+1} where the vector fields, X_i are defined *relative* (in position) to V_i. If an adjacent node, say V_k, is a fixed node, *i.e.,* not in the symmetry orbit, then Δ_k will still span TV_k because vector fields have only been *added* to that node, and the existing vector fields are unchanged. For V_j and the adjacent vertices, by symmetry $\exists \rho \in S_{m+1}$ such that $V_j = V_{\rho(l)}$, where V_l is a nonadjacent node. Let the collection of vector fields $\{X_1, \ldots, X_q\}$ spans TV_l, then $\{\nu_* X_1, \ldots, \nu_* X_q\}$ must span TV_l because of the invariance of the system. Recall that if some of the X_i are Lie brackets, the result still holds since Lie brackets are natural with respect to the push forward of diffeomorphisms (see Proposition 4.2.23 of [25]). Similarly, for the vertices adjacent to V_j, $\exists \nu \in S_{m+1}$ such that $V_{\nu(l)}$ is mapped to them, which shows that Δ for the adjacent nodes in the symmetry orbit must span their tangent spaces. Since $\exists \Delta_i$ such that span$\Delta_i = TV_i$, $\Delta_{n+1} = \sum_{i=1}^{n+1} \Delta_i$ spans $TM = \prod_{i=1}^{n+1} TV_i$. Since M is a manifold, by Frobenius' theorem, Δ_{m+1} is involutive. Therefore, $\overline{\Delta}_{m+1}$ is full rank, and hence accessible.

For controllability, we must show that all bad brackets in the larger system must be spanned by lower order good brackets. This follows from an argument very similar to that which proved accessibility. In Σ_{n+1} one drift vector field has been added to the system relative to Σ_n, and all new bad brackets in $\overline{\Delta}_{n+1}$ must contain this drift term. However, by symmetry, if X is a bad bracket, $\exists \nu \in S_{m+1}$ such that $\nu_* X = \tilde{X}$, where $\tilde{X}$ is a bad bracket that is already spanned by lower order good brackets, and since ν is a homomorphism with respect to the Lie bracket product, there must exist good brackets, Y_i such that the set $\{\nu_* Y_i\}$ span the bad bracket X. Therefore, all bad brackets in $\overline{\Delta}_{n+1}$ are spanned by lower order good brackets. Therefore, Σ_{n+1} is STLC. ∎

Since the main good bracket, bad bracket controllability result from [24] only provides a sufficient condition for controllability, Proposition 3.4 requires *more* than the smaller system being STLC, it must satisfy the more restrictive condition that it satisfy the good bracket, bad bracket condition. For driftless systems, STLC of the smallest system is sufficient for controllability, however.

COROLLARY 3.5 *If any one member, Σ_n of the equivalence class of symmetric distributed systems, $\overline{\Sigma}$ is STLC and is driftless, i.e., $f_i \equiv 0\ \forall i$, then all larger members of $\Sigma_i \in \overline{\Sigma}$ where $i > n$ are STLC.*

The necessary condition requires further assumptions regarding the largest degree of Lie bracket necessary for $\overline{\Delta}$ to be involutive. Detailed computations illustrate that states from one node can "propagate" to affect other nodes via Lie brackets. In fact, they can "propagate" one node for each order of Lie bracket. We note that even though this proposition does not provide the necessary condition for controllability of a larger system, the sufficient condition is of greater engineering utility due to the fact that it can be used to determine controllability of a larger system by analyzing a smaller system.



4 Example

This section provides a simple example that demonstrates the utility of the main result. Consider a group of 5 mobile robots moving in formation, where each robot is described by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

where u_1 is the linear velocity input and u_2 is the angular velocity input from [26]. The group of robots will follow the nominal trajectory of a leading robot, which appears as a drift vector field in the system. The equation describing this system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \\ \dot{x}_{10} \\ \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{14} \\ \dot{x}_{15} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_6 - x_3 \\ 0 \\ 0 \\ x_9 - x_3 \\ 0 \\ 0 \\ x_3 - x_{12} \\ 0 \\ 0 \\ x_3 - x_{15} \end{bmatrix} + \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

$$+ \begin{bmatrix} 1 \\ 1 \\ 0 \\ \cos x_6 \\ \sin x_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_4 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ \cos x_9 \\ \sin x_9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_5 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_6$$

$$+ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cos x_{12} \\ \sin x_{12} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_7 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_8 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cos x_{15} \\ \sin x_{15} \\ 0 \end{bmatrix} u_9 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{10}.$$

Figure 4 displays a digraph of this system. This equation is a symmetric nonlinear robotic system with drift. Now, consider the reduced order system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x_3 - x_6 \\ 0 \\ 0 \\ x_3 - x_9 \end{bmatrix} + \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

$$+ \begin{bmatrix} 1 \\ 1 \\ 0 \\ \cos x_6 \\ \sin x_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_4 + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \cos x_9 \\ \sin x_9 \\ 0 \end{bmatrix} u_5 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_6$$

Let f, g_1, g_2, and g_3, and g_4 be the vector fields given in the reduced equation. Taking Lie brackets

$$g_8 = [g_1, g_2] = \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad g_9 = [f, g_2] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$g_{10} = [g_2, g_3] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sin x_6 \\ \cos x_6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad g_{11} = [g_2, g_5] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\sin x_9 \\ \cos x_9 \\ 0 \end{bmatrix},$$

we find that $\dim(\mathrm{span}\{g_1, \ldots, g_{11}\}) = 9$ and $f(x) = 0$ at any point, x, where the angular positions of the robots are the same, so by Theorem 3.3 the reduced system is controllable from these same points. Furthermore, by Proposition 3.4 implies that the full system is also controllable from these points as well as is the entire equivalence class of systems.

5 Conclusions and Future Work

In this paper, we have considered the controllability of nonlinear distributed robotic systems with drift. The main result was proving that controllability of large-scale systems can be determined on a reduced order system. In fact, it was shown that the controllability of an entire equivalence class of robotic systems can be determined from testing only one of its members, namely its smallest member.

A group of unicycle type mobile robots was used to demonstrate the main proposition of this paper. Extending this work to the formation control of symmetric robotic systems is currently being explored. The goal of this research is to design formation control algorithms (see [27], [28], [29]) on reduced order systems, which can be extended to symmetrically equivalent large-scale systems. Another avenue of future work is to examine stability properties for symmetric robotic systems. Additionally, we would like to examine the more useful *control synthesis* problem. We would like to exploit the symmetry properties presented in this paper to formulate "reduced order" methods to synthesize controllers.

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