Symmetry-Breaking in Bifurcations of Optimal Solutions for Coordinated Nonholonomic Robotic Control

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***Abstract*— This paper considers optimal control of a system of nonholonomic robots. Control effort and deviation from a desired formation are minimized for the system of robots traveling between specified initial and final configurations, with a bifurcation parameter which is the relative weighting assigned to the control effort *versus* the formation constraint. The main contribution of this paper is an extension of our previous work which considered the nature of multiple solutions to a holonomic optimal control problem for a distributed system. An important aspect of the previous work was that symmetries in the holonomic system guaranteed symmetries in the bifurcations of the solutions to the optimization problem. In the current work, a system of relatively simple nonholonomic robots break the symmetry in the system, which results in symmetry-breaking in the bifurcations.**

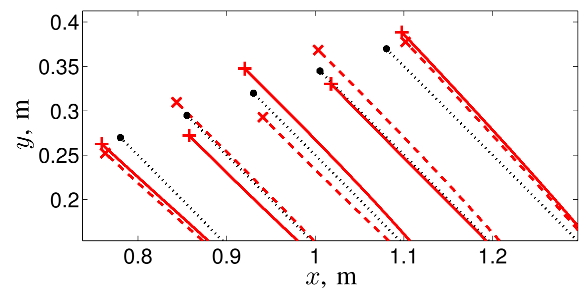

Fig. 1. Bifurcations in optimal trajectories for a formation of mobile nonholonomic robots.

I. INTRODUCTION

Coordinated control of distributed and multi-agent systems is currently topical and the focus of much research effort directed toward various applications. For example, [1] considers control of robotic underwater vehicles, [2] deals with satellite clustering, [3] considers electric power systems, [4] with search and rescue, and so on. For the specific application of formation control for mobile multi-robot systems, the literature can be roughly categorized into three groups: leader-follower methods [5], [6], [7], behavior-based methods [8], [9], [10] and virtual structure methods [11], [12], [13]. Also, more general and powerful results exist, such as in [14], [15].

This paper considers control a formation of robots moving along an optimized trajectory between specified initial and a final configurations, and is an extension of our previous work [16], [17] because it is applied to a system of nonholonomic robots. The cost function contains two types of terms, one type which minimizes the control effort for each robot individually and another type which penalizes deviations from the desired formation. As the relative preference given to the two optimization terms is varied, bifurcations in the nature of the solutions occur, an example of which is illustrated in Figure 1. In Figure 1, intermediate points of three different trajectories, all of which are solutions to the optimal formation problem investigated in this paper, are illustrated, with the $\times$, $+$ and $\cdot$, respectively at the same point in time. Given the fact that optimization in robotic motion planning is common, a complete understanding of the existence and nature of multiple solutions to such problems is of great engineering importance.

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The existence of multiple nontrivial solutions of boundary value problems for nonlinear second order ordinary differential equations has been investigated by some authors. Not surprisingly, however, the results are not as fully developed as the case for the bifurcation of fixed points for ordinary differential equations. For example, for $x'' + a(t)f(x) = 0$, $x(0) = 0$, $x(1) = 0$, the properties of the solutions depend on the limiting behavior of the function $f(x)$. Erbe and Wang [18] studied the existence of positive solutions of the equation with linear boundary conditions. Also, for

$$f_0 = \lim_{s \to +0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{s \to +\infty} \frac{f(s)}{s},$$

they showed the existence of at least one positive solution in two cases, superlinearity ($f_0 = 0, f_\infty = \infty$) or sublinearity ($f_0 = \infty, f_\infty = 0$). In [19], Erbe, Hu and Wang showed that there were at least two positive solutions in the case of superlinearity at one end (zero or infinity) and sublinearity at the other end. Naito and Tanaka [20] and Ma and Thompson [21] established a precise condition concerning the behavior of the ratio $f(s)/s$ for the existence and nonexistence of solutions. Their main result was that the problem had at least k solutions if the ratio $f(s)/s$ crossed the k eigenvalues of the associated eigenvalue problem. For a class of systems of second order ODEs, Marcos do Ó, Lorca and Ubilla [22] used the fixed-point theorem of cone expansion/compression type, the upper-lower solutions method and degree arguments to study the existence, nonexistence, and multiplicity of positive solutions of the boundary value problem. While the problems they address are similar in nature to ours, none of these results are, unfortunately, directly applicable to it.

This paper presents bifurcation results for a specific formation control problem. These solutions were found by using a relaxation method to solve the nonlinear two-point boundary value problem. The existence of multiple solutions

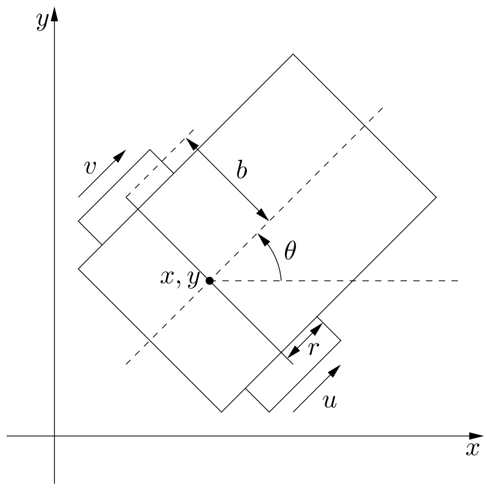

Fig. 2. Kinematic MICAbot model.

and their bifurcation structure is important for robotics and controls engineers who deal with motion planning methods that are based on optimization techniques. Knowledge of the existence and nature of bifurcations of solutions of this type are important for practicing engineers because if a solution is found that is optimal, but not necessarily desirable, it may be the case that a different solution for the same cost function exists and is superior, for example, for obstacle-avoidance reasons. Searching for multiple solutions of an optimization problem may be less costly than reformulating the optimization problem. Also, if the complete bifurcation structure of the system can be constructed, it provides a possible obstacle-avoidance scheme in that it a choice can be made among various locally optimal trajectories.

II. SYSTEM AND SOLUTION METHOD

The nonholonomic model considered in this paper is the MICAbot platform, a centimeter-scale two-wheeled robot [23]. Kinematically, it resembles a wheelchair, having three degrees of freedom and two control inputs. Hence, it is underactuated by one degree of freedom. Figure 2 shows the MICAbot geometry. The state variables are x, y, and θ, which correspond to the three degrees of freedom. The controlled inputs are u and v, the angular velocities of the wheels. For all the simulations in this paper, the wheel radius, r, and half the axle track, b, are assumed to be 2 cm and 5 cm, respectively.

Assuming a rolling without slipping condition for each wheel, the kinematics of the MICAbot are described by three nonholonomic constraints:

$$\begin{aligned}\dot{x} &= \frac{r}{2}\cos\theta(u+v)\\ \dot{y} &= \frac{r}{2}\sin\theta(u+v)\\ \dot{\theta} &= \frac{r}{2b}(u-v).\end{aligned} \tag{1}$$

Using a standard calculus of variations approach with the cost functional which minimizes the control effort

$$\begin{aligned}L^* = &\frac{1}{2}(u^2+v^2) + \lambda_x\left(\dot{x} - \frac{r}{2}\cos\theta(u+v)\right)\\ &+ \lambda_y\left(\dot{y} - \frac{r}{2}\sin\theta(u+v)\right) + \lambda_\theta\left(\dot{\theta} - \frac{r}{2b}(u-v)\right),\end{aligned} \tag{2}$$

we obtain following ordinary differential equations for optimal solutions

$$\begin{aligned}\dot{x} &= \frac{r}{2}\cos\theta(u+v), & \dot{\lambda}_x &= 0\\ \dot{y} &= \frac{r}{2}\sin\theta(u+v), & \dot{\lambda}_y &= 0\\ \dot{\theta} &= \frac{r}{2b}(u-v), & \dot{\lambda}_\theta &= \frac{r}{2}(u+v)(\lambda_x\sin\theta - \lambda_y\cos\theta)\end{aligned}$$

where

$$\begin{aligned}u &= \frac{r}{2}\left(\lambda_x\cos\theta + \lambda_y\sin\theta + \frac{1}{b}\lambda_\theta\right)\\ v &= \frac{r}{2}\left(\lambda_x\cos\theta + \lambda_y\sin\theta - \frac{1}{b}\lambda_\theta\right).\end{aligned}$$

We use a relaxation method to determine numerical solutions with specified boundary conditions for these equations. For the system of first-order ODEs, $\vec{x}'(t) - f(t,\vec{x}) = 0$ and the finite difference approximation define

$$\vec{E}_k \equiv \vec{x}_k - \vec{x}_{k-1} - h_k f\left(\tfrac{1}{2}(t_k + t_{k-1}), \tfrac{1}{2}(\vec{x}_k + \vec{x}_{k-1})\right) = 0, \quad k = 2, 3, ..., M \tag{3}$$

where $h_k = t_k - t_{k-1}$ and M is the number of mesh points. If n_1 is the number of boundary conditions at the first boundary and n_2 is the number of boundary conditions at the second boundary, then $\vec{E}_1$ will have n_1 nonzero entries and $\vec{E}_{M+1}$ will have n_2 nonzero entries. Since the relaxation method is iterative, incremental changes of each dependent variable, $\Delta x_{j,k}$, between iterations must be determined. A Taylor series expansion of Equation 3 results in

$$\begin{aligned}\vec{E}_k(\vec{x}_k + \Delta\vec{x}_k, \vec{x}_{k-1} + \Delta\vec{x}_{k-1}) &\approx E_{j,k}(\vec{x}_k, \vec{x}_{k-1})\\ &+ \sum_{n=1}^{N}\frac{\partial E_{j,k}}{\partial x_{n,k-1}}\Delta x_{n,k-1} + \sum_{n=1}^{N}\frac{\partial E_{j,k}}{\partial x_{n,k}}\Delta x_{n,k},\end{aligned} \tag{4}$$

where $j = 1, 2, ..., N$, which gives $M \times N - (n_1 + n_2)$ equations representing the interior points. For the first and second boundary conditions, respectively,

$$\vec{E}_1(\vec{x}_1 + \Delta\vec{x}_1) \approx E_{j,1}(\vec{x}_1) + \sum_{n=1}^{N}\frac{\partial E_{j,1}}{\partial x_{n,1}}\Delta x_{n,1}, \tag{5}$$

$$\vec{E}_{M+1}(\vec{x}_M + \Delta\vec{x}_M) \approx E_{j,M+1}(\vec{x}_M) + \sum_{n=1}^{N}\frac{\partial E_{j,M+1}}{\partial x_{n,M}}\Delta x_{n,M}, \tag{6}$$

where $j = 1, 2, ..., n_1$ for Equation 5 and $j = 1, 2, ..., n_2$ for Equation 6. For the solution to converge, the left hand sides of Equations 4-6 obviously should approach zero. These equations are linear even if the differential equation is nonlinear, and hence, $\Delta x_{j,k}$, can be solved for using standard methods from linear algebra such as Gaussian elimination.

Now, considering a fleet or MICAbots operating in a coordinated manner instead of a single robot, consider a

MICAbot	Initial, $t = 0$ s		Final, $t = 1$ s	
	(x, y), m	θ, deg	(x, y), m	θ, deg
1	(1.0, 0)	90	(0, 1.0)	180
2	(1.1, 0)	90	(0, 1.1)	180
3	(1.2, 0)	90	(0, 1.2)	180
4	(1.3, 0)	90	(0, 1.3)	180
5	(1.4, 0)	90	(0, 1.4)	180

TABLE I

BOUNDARY CONDITIONS FOR 5-MICABOT FORMATION.

system with n MICAbots where the desired distance between neighboring robots is specified. In that case, if we consider the cost functional

$$J = \int_0^{t_f} \Bigg[\frac{1}{2} \sum_{i=1}^{n} u_i^2 + v_i^2 + \sum_{i=1}^{n} \Big[\lambda_{x_i} \left(\dot{x}_i - \frac{r}{2} \cos\theta_i (u_i + v_i) \right) + \lambda_{y_i} \left(\dot{y}_i - \frac{r}{2} \sin\theta_i (u_i + v_i) \right) + \lambda_{\theta_i} \left(\dot{\theta}_i - \frac{r}{2b} (u_i - v_i) \right) \Big] + k \sum_{i=1}^{n-1} (d_{i,i+1} - \tilde{d})^2 \Bigg] dt \qquad (7)$$

where $d_{i,i+1} = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$ and $\tilde{d}$ is the desired distance between adjacent robots. The differential equations are relatively straight-forward to derive from calculus of variations; however, they are omitted in complete detail due to space limitations. The solutions to this set of differential equations are, of course, trajectories which are extrema of J. In this set of equations, the parameter k determines the relative importance of minimizing each robot's control effort compared to the importance of maintaining the desired formation. The rest of this paper investigates the manner in which solutions bifurcate as the values of k are varied and compare the nature of those solutions to our prior work with a system of holonomic robots.

III. BIFURCATION OF OPTIMAL TRAJECTORIES

Now, consider a system of five MICAbots and a coordinate system where the robots are initially in a line evenly spaced between $x = 1.0$ m and $x = 1.4$ m along the x-axis, each with an orientation of $\theta_i = \pi/2$, and assume the final formation is where the robots are evenly distributed between $y = 1.0$ m and $y = 1.4$ m along the y-axis with an orientation of $\theta_i = \pi$, as summarized in Table I.

With the formation weighting parameter, k, set to zero, the solutions are as illustrated in Figure 3 and because $k = 0$, the desired distance between the robots is not maintained other than at the boundaries. In the interior of the trajectories, because of the geometry of the problem, the actual distance between neighboring MICAbots is less than the desired distance of $\tilde{d} = 0.1$.

Remark 1: This paper considers how the solutions for the system bifurcate as the parameter k varies. It is emphasized that such bifurcations have important distinctions from standard bifurcations from dynamical systems theory. Specifically, the latter considers bifurcations of *fixed points* of a dynamical system. For the systems considered in this paper, we are considering solutions to *boundary value problems*, in contrast to initial value problems. Of course, solutions to the boundary value problem are fixed points of variations of the cost function; however, the entire solution is the fixed point and our desire is to quantify the bifurcations in a physically-meaningful way. Hence, it is necessary from the beginning to define the manner in which we are quantifying the bifurcations.

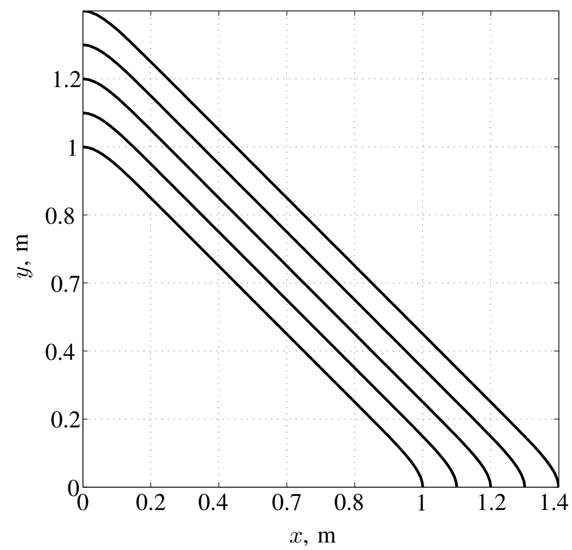

Fig. 3. Optimal solution for five-MICAbot system when $k = 0$.

A. Simulations and Bifurcations of Solutions

The quantity we use to quantify differences between solutions is taken to be the difference between two solutions at a specified value of time. The $\mathcal{L}_2$ norm would appear to perhaps be a better choice; however, we wish to use a measure that will represent a signed difference between solutions, which in the case at hand will indicate whether one solution is "above" or "below" the other. Specifically, in the bifurcation diagrams presented subsequently in this paper, the quantity used is the distance between a solution for a specified k-value and the $k = 0$ solution.

If the value of the bifurcation parameter is increased to $k = 8 \times 10^5$, multiple solutions exist, five of which are illustrated in Figures 4-8. The dotted line in each of those figures represents the $k = 0$ solution and the solid lines are the solution for $k = 8 \times 10^5$. The crosses indicate the each of the solutions at $t = 0.25, 0.50$ and 0.75 and the dots indicated the $k = 0$ solutions at those same points in time. As is especially clear in Figure 1, but also in Figures 4-8, part of nature of the multiplicity of solutions is that neighboring robots can get "ahead" or "behind" its neighbors. This allows each robot to track the $k = 0$ trajectory more closely, which minimizes the control effort, but also more closely maintain the formation distance constraint.

Figures 4-8 illustrate multiple solutions for a fixed value of the bifurcation parameter, k. Now, we construct bifurcation diagrams by tracking the solutions as k is varied. The relaxation method is particularly efficient for this because solutions that have already been determined may be used as the initial condition for the method. These diagrams illustrate

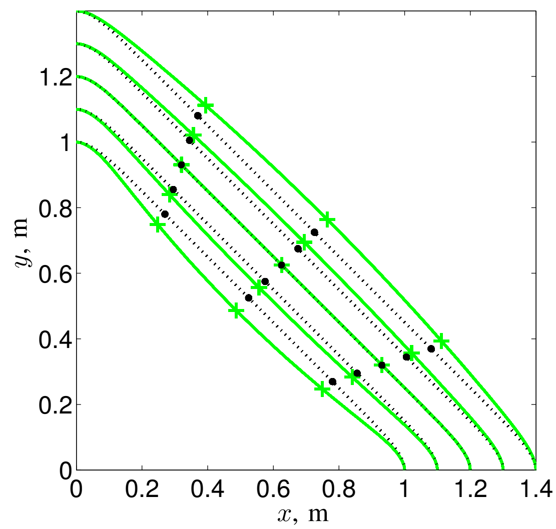

Fig. 4. Solution 1, positions marked at $t = 0.25$, 0.50, and 0.75 s.

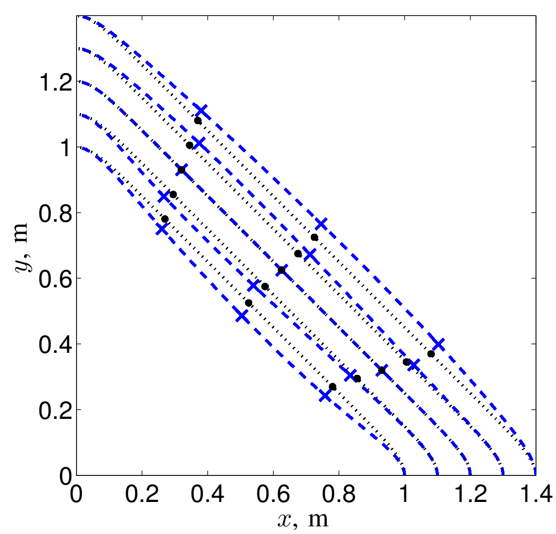

Fig. 5. Solution 2, positions marked at $t = 0.25$, 0.50, and 0.75 s.

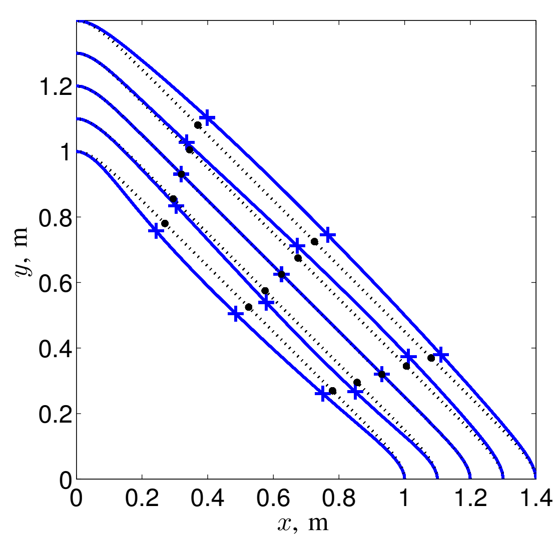

Fig. 6. Solution 3, positions marked at $t = 0.25$, 0.50, and 0.75 s.

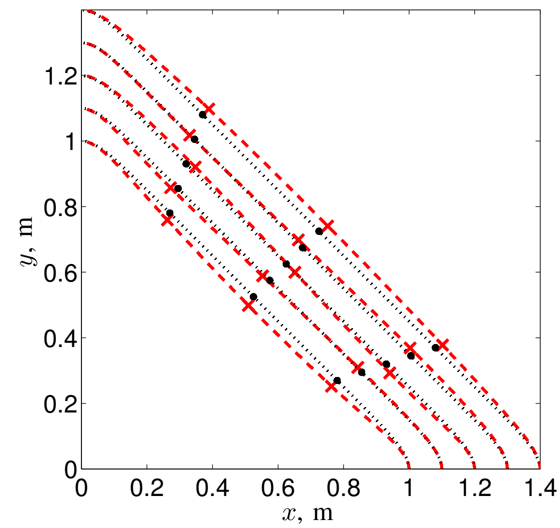

Fig. 7. Solution 4, positions marked at $t = 0.25$, 0.50, and 0.75 s.

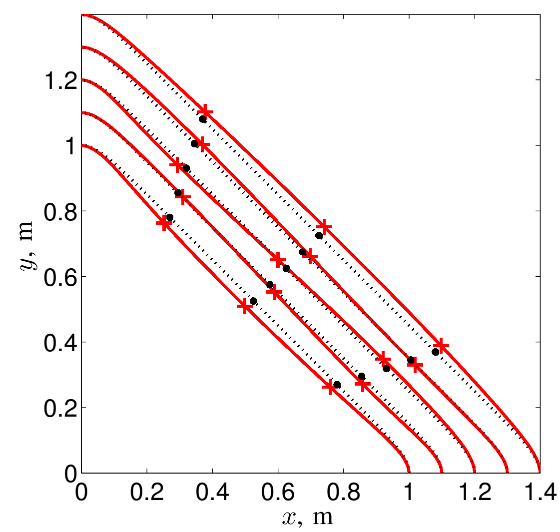

Fig. 8. Solution 5, positions marked at $t = 0.25$, 0.50, and 0.75 s.

the difference between solutions and the $k = 0$ solution for a range of k-values.

It makes sense that as k is increased the number of solutions to the boundary value problem will increase. This is because in the limit as $k \to \infty$, only the maintaining the formation matters compared to the control effort. Hence, in the limit, one would expect that any trajectory which maintains distance between the robots is a solution. Figures 9 through 13 are consistent with this.

B. Discussion

These bifurcation results illustrate a subtle, but important distinction relative to our previous results. Specifically, in [16], [17], which considered *holonomic* systems, we showed that the bifurcation diagrams must be symmetric in that, for the five-robot formation problem like the one considered in this paper, the bifurcation diagrams for robots one and five must be symmetric in that they are reflections of each other, the diagrams for robots two and four must be similarly

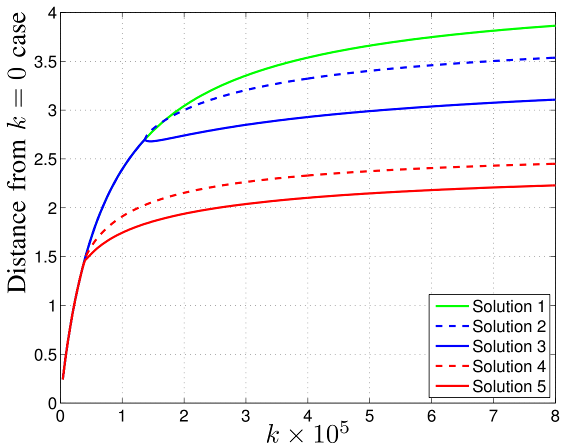

Fig. 9. Bifurcations at $t = 0.25$ s for MICAbot 1.

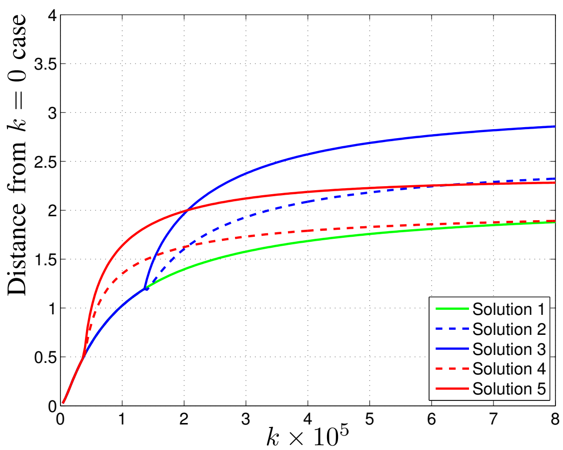

Fig. 10. Bifurcations at $t = 0.25$ s for MICAbot 2.

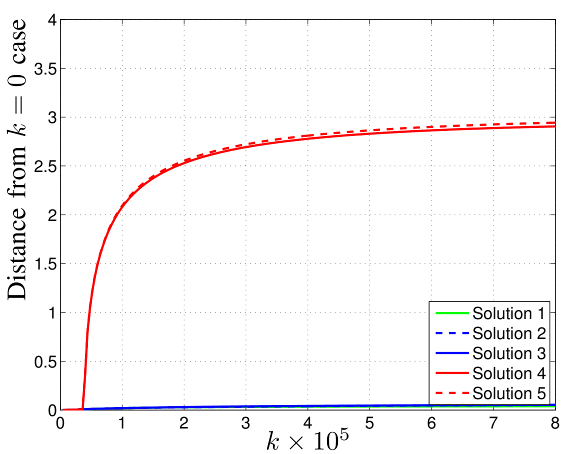

Fig. 11. Bifurcations at $t = 0.25$ s for MICAbot 3.

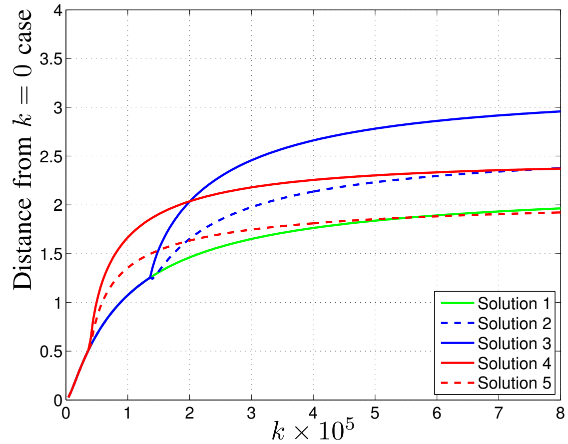

Fig. 12. Bifurcations at $t = 0.25$ s for MICAbot 4.

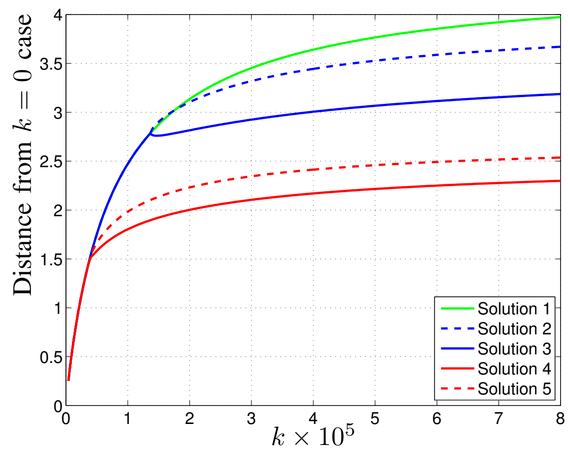

Fig. 13. Bifurcations at $t = 0.25$ s for MICAbot 5.

symmetric and the bifurcation diagram for robot three must be symmetric with respect to itself.

For the current system, this result does *not* hold. This is most easily seen for robots two and four at the right end (high k-values) of the bifurcation diagram where the branches cross for solutions 1 and 5 and also for solutions 2 and 4, but not at the same k-value (other differences are similarly evident). This is *not* a numerical artifact, because the persistence of this difference was investigated by a grid resolution convergence study by increasingly refining the finite difference meshes. In fact, the symmetry of the system is broken by the MICAbot itself because the left and right wheels travel different distances along most trajectories that are not straight lines. The order of the differences between the bifurcation diagrams appears to be on the order of a couple percent, which is also approximately the order of the spacing between the wheels on the MICAbot relative to the overall length of the trajectory. An interesting area of current work is to determine whether the differences between the bifurcated solutions may be bounded, and if so, what sorts of computational savings may be obtained therefrom.

IV. CONCLUSIONS AND FUTURE WORK

A. Conclusions

This paper presented an investigation of bifurcations of solutions of an optimal control problem for the coordinated motion of a fleet of nonholonomic robots. The equations of motion are determined from standard calculus of variations methods and the resulting boundary value problem is solved using a relaxation method. The cost functional includes terms minimizing the control effort of each individual robot as well as terms penalizing deviation from a desired formation.

Increasing numbers of solutions are expected as increasing weight is given to the formation terms in the cost functional because in the limit, if the only terms that matter are the formation terms, then any one of an infinite number of solutions is valid. The bifurcation diagrams are consistent with this in that increasing the bifurcation parameter results in an increased number of numerically-determined solutions. An important result which contrasts with our previous work is that the bifurcation diagrams are not symmetric. Small differences appear, which are on the order of the ratio of the length scale of the wheel base of the robots to the overall trajectory lengths. This symmetry-breaking is important because from our previous work, we were able to construct symmetric solutions from other solutions which resulted in significant computational savings.

B. Future Work

Current and future work is directed toward two areas.

1) Because the differences in the bifurcation diagrams does seem to be related to the magnitude by which the symmetry of the system is broken by the robots considered in this problem, we are focusing on using this length scale to bound the differences between the bifurcation diagrams. If such a bound can be determined, then significant computational savings may result because the existence of one solution may be used to imply the other and furthermore, in many engineering problems, small variations in solutions may be tolerated and hence the original solution may be used as an approximate solution for the approximately symmetric one.
2) Many formation control problems of this type do not scale well with system size. Numerical methods such as algebraic homotopy methods utilized in our prior work will be investigated to determine the extent to which all solutions have been determined numerically.

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