Multiagent Compositional Stability Exploiting System Symmetries

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Abstract

This paper considers nonlinear symmetric control systems. By exploiting the symmetric structure of the system, stability results are derived that are independent of the number of components in the system. This work contributes to the fields of research directed toward compositionality and composability of large-scale system in that a system can be "built-up" by adding components while maintaining system stability. The modeling framework developed in this paper is a generalization of many existing results which focus on interconnected systems with specific dynamics. The main utility of the stability result is one of scalability or compositionality. If the system is stable for a given number of components, under appropriate conditions stability is then guaranteed for a larger system composed of the same type of components which are interconnected in a manner consistent with the smaller system. The results are general and applicable to a wide class of problems. The examples in this paper focus on the formation control problems for multi-agent robotic systems.

Keywords: Symmetric systems, Nonlinear analysis, nonlinear control, distributed control

1. Introduction

Recent research efforts have been directed toward the analysis of *composability* and *compositionality* of control systems, and especially cyber-physical systems [1, 2]. These concepts are not equivalent, but each does relate to the nature in which system components affect overall system properties. In this paper, conditions are determined under which a stable *symmetric system* remains stable if additional components are added in a structured manner, particularly, in a manner which maintains the symmetric aspects of the system. While the results in this paper are general, one important application, which is the focus of the examples, is mobile robot formation control.

Control of multi-agent systems is an important area of engineering research which has received much attention for several decades, but most intensively since approximately the mid-1990s (see, for example, [3, 4, 5, 6] and many others). Formation control for multiple mobile robotic systems is a prototypical application and similarly has a long history, with one focus being on the use of potential functions for coordination (see for example [7, 8, 9] and the citations therein). The use of potential functions has an obvious appeal in that they facilitate stability analyses using Lyapunov functions. The drawbacks are well-known also, which include among other things, the existence of multiple local minima in complex environments, the fact that realistic potential functions representing the realities of sensor ranges introduce mathematical limitations which complicate and limit the stability analysis, *etc.* As observed in [10], many of the prior efforts have assumed specific dynamics with the correct observation that they probably generalize. Our approach in this paper is to develop that generalization.

Perhaps the work closest to this present work is that of [10] wherein a control Lyapunov function is assumed to exist for each agent, from which formation functions and bounds on formation speed can be derived to ensure stability. Also, [11] focuses on control synthesis for adding components, which has a similar theme to the results here. However, the results in that paper are limited to the linear case and are focused on decentralized control, rather than the more symmetric aspects of the systems considered. In this paper, our formulation provides the type of cases and underlying structure for systems to which the results in [10] will apply. Furthermore, our results here apply to a broader class of systems, such as fully distributed ones, to which the previous results do not necessarily apply.

The main contributions of the present paper are:

1. a nonlinear extension of the model and results in [12] and [13] with a simpler representation of system symmetries than our previous work,
2. the development of a theoretical framework that is underlying many of the formation control algorithms in the literature,
3. general stability results that are applicable to such systems regardless of the number of components (compositionality), and,
4. robustness results that ensure stability even under certain types of component failures.

These results will allow a control design engineer to focus the analysis on a smaller, more tractable system, with a guarantee that stability will hold for a much larger system. This paper essentially extends the previous work of one of the authors related to the properties of symmetric systems [14, 15, 16, 17, 18] to consider nonlinear system stability. This previous work cited considers system symmetries that are defined by a group action on the configuration manifold for a distributed system that was induced by the action of a permutation group. The main drawback of such an approach is that, in the general case, identifying such symmetries can be problematic. However, in the case of most engineering and robotics systems, where the individual robots are the components that are easily identified, symmetry identification is much less of a problem. Rather than using this prior approach, this paper will introduce a more straight-forward approach which is a nonlinear extension of the approach used in [12] and [13]. However, it is emphasized that the prior approaches [14, 15, 16, 17, 18, 19, 20] and [21] offer a general approach to the problem that can be used in cases more general than the ones addressed here.

The rest of this paper is organized as follows. Section 2 defines a symmetric system, equivalence relations among different symmetric systems and equivalence classes of symmetric systems. It first develops the idea for a simpler case of one-dimensional interconnections between components and then generalizes it based on group theoretic tools. Section 3 presents the nonlinear stability results for symmetric systems. Section 4 presents an example of the application of these results. Section 5 presents an extension of the results from Section 3 to the case of robust stability in the case where an agent or agents in a symmetric system fail. Finally, Section 6 outline conclusions and future work.

2. Symmetric Systems

This section defines symmetric systems and the relationship among symmetric systems with different numbers of components. Symmetry has been previously considered, such as in [22, 23, 24], but it has not yet been fully exploited for mainstream results. As a motivational example, consider a formation of large number of identical mobile robots where each robot has a control law that attempts to control it so that it maintains a desired distance from its neighbors. Intuitively if more of the same type of robots with the same control law are added to the formation, or conversely if some are removed, the properties of the formation as a whole should normally not drastically change. As a step toward formalizing and determining conditions when this holds, we must formulate definitions for systems when more agents are added or some are removed in structured manner. Toward this end, we define *symmetric systems* and *equivalent symmetric systems*.

The first step is to extend the basic system component description from the linear case in [12] to the nonlinear case. The "basic building block" in one spatial dimension (more general interconnection topologies will be considered subsequently) is illustrated in Figure 1. The outputs from the component are $w^-(t)$ and $w^+(t)$, and the inputs are u, $v^-(t)$ and $v^+(t)$. In this paper the signals $v^\pm$ will

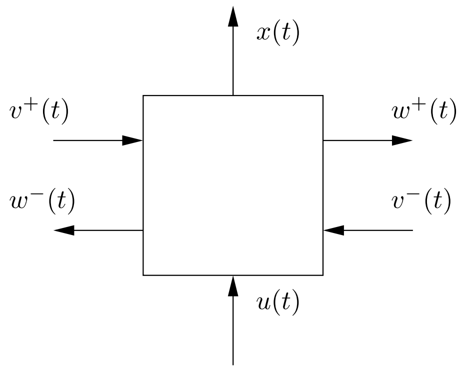

Figure 1: System building block in one spatial dimension.

represent the effects of the coupling with the other components and u are the control inputs. If it is necessary to distinguish between them, the $v^\pm$ signals will be called *coupling inputs*, the u will be called *control inputs* and collectively they will be called the *inputs*. When interconnected in one spatial dimension, a system comprised of a collection of these building blocks is as illustrated in Figure 2.

We wish to express component-by-component, the usual dynamics of a nonlinear control system expressed for the ith component by

$$\dot{x}_i = f_i(x) + \sum_{j=1}^{m_i} g_{i,j}(x) u_{i,j},$$

where $x \in \mathbb{R}^n$, the vector fields $f, g_j \in T\mathbb{R}^n$ and m_i is the number of inputs for the ith component. In order to define a symmetric system that has structure that will be useful, we will consider the following aspects of a system comprised of interacting components:

- the relationship between the nonlinear dynamics of a component and its coupling inputs,
- the structure of how the components are interconnected,
- the dynamics of individual components, and,
- the individual control laws in each component.

In the most general case, the vector fields, f_i and $g_{i,j}$ in the equation of motion for the ith component and the outputs w_i^+ and w_i^- for the component may depend on the state of the component, x_i *as well as* the coupling inputs, $v_i^\pm$, so the dynamics of component i are given by

$$\begin{aligned} \dot{x}_i(t) &= f_i\left(x_i(t), v_i^+(t), v_i^-(t)\right) \\ &\quad + \sum_{j=1}^{m_i} g_{i,j}\left(x_i(t), v_i^+(t), v_i^-(t)\right) u_{i,j}(t) \\ w_i^-(t) &= w_i^-\left(x_i(t), v_i^+(t), v_i^-(t)\right) \\ w_i^+(t) &= w_i^+\left(x_i(t), v_i^+(t), v_i^-(t)\right). \end{aligned}$$

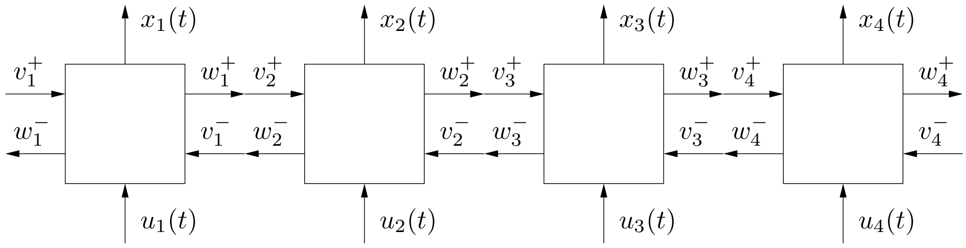

Figure 2: System interconnected in one spatial dimension.

We will consider how the system is interconnected shortly, but for now observe that for a system of interconnected components where the incoming signals, $v^{\pm}(t)$ are from the outgoing signals from the component's neighbors, since the vector fields f_i and $g_{i,j}$ arise from the physical dynamics of the component, if these vector fields can depend on the outputs from the neighbors, this would reflect a change in the physical dynamics of the system due to the coupling between components. The class of the types of coupling that could be represented by this formulation is very broad and could include, for example, when there is a physical joining of agents, as with reconfigurable, modular robots.

For a very large class of problems, including formation control for mobile robots, there normally is no physical contact between the robots and hence the nature of the coupling between the robots is simplified. In particular, it is only through the control inputs that the output from the other components affects the dynamics of an agent, which is expressed by

$$\begin{aligned} \dot{x}_i(t) &= f_i\big(x_i(t)\big) + \sum_{j=1}^{m_i} g_{i,j}\big(x_i(t)\big)u_{i,j}\,(t) \\ w_i^-(t) &= w_i^-\,(x_i(t)) \\ w_i^+(t) &= w_i^+\,(x_i(t))\,. \end{aligned} \tag{1}$$

For the rest of this paper, we will restrict our attention to systems of this type.

Now we consider the nature of the interconnections in the system. For a system with N components, a subset of the components have *periodic interconnections in one dimension* if the inputs and outputs of adjacent components are related by

$$\begin{aligned} w_i^+(t) &= v_{i+1}^+(t), \quad w_i^-(t) = v_{i-1}^-(t), \\ v_i^+(t) &= w_{i-1}^+(t), \quad v_i^-(t) = w_{i+1}^-(t), \end{aligned} \tag{2}$$

for all i in some subset $\mathcal{I} \subset \{1,\dots,N\}$. A set of components that have periodic interconnections is called an *orbit of periodically interconnected components*. Of course, a system may have multiple orbits of periodically interconnected components, and in such a case there will be multiple orbit index sets.

The system illustrated in Figure 2 is of this type for $\mathcal{I} = \{2,3\}$. It is possible for the entire system to have periodic interconnections in one dimension if Equation 2 holds for all $i \in \{1,\dots,N\}$ and for mod(N), or if the system has an infinite number of components on a one-dimensional integer lattice. For the system in Figure 2, if component 4 is connected to component 1 in the same manner that the other components are connected; namely $v_1^+ = w_4^+$, and $v_4^- = w_1^-$ then the whole system has periodic interconnections.

For the set of components with periodic interconnections if the dynamics of the system are further restricted in that the control law for a component is defined by feedback in terms of that component's state and the outputs from the neighbors, then the control inputs for component i in Equation 1 can be written as

$$u_{i,j}(t) = u_{i,j}\left(x_i(t), w_{i-1}^+(x_{i-1}(t)), w_{i+1}^-(x_{i+1}(t))\right). \tag{3}$$

Now we consider the case when the components in an orbit of periodically interconnected components are the same so they have identical dynamics. An *orbit of symmetric components* is an orbit of periodically interconnected components in one dimension if

$$\begin{aligned} f_i(x) = f_k(x), &\quad g_{i,j}(x) = g_{k,j}(x), \\ w_i^-(x) = w_k^-(x), &\quad w_i^+(x) = w_k^+(x) \end{aligned}$$

and $m_i = m_k = m$ for $x \in \mathbb{R}^n$, for all $i,k \in \mathcal{I}$ and for each $j = 1,\dots,m$. Finally, when the components in an orbit of symmetric components have identical control laws, we have a *symmetry orbit* which requires

$$u_{i,j}(x_1, w_{i-1}^+(x_2), w_{i+1}^-(x_3)) = u_{k,j}(x_1, w_{k-1}^+(x_2), w_{k+1}^-(x_3))$$

for $(x_1,x_2,x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, for all $i,k \in \mathcal{I}$ and for each $j = 1,\dots,m$.

The idea behind a symmetry orbit is that the agents in the orbit are identical, have identical control laws and furthermore are identically interconnected. We observe that, in general, it is only necessary for the dynamics of each system to be "identical" in the sense that they are diffeomorphically related, in which case under a coordinate transformation they are identical. In this paper we will restrict our attention to systems with components with identical dynamics with the recognition that the results apply to a broader set of problems.

Of course, systems may be spatially interconnected in dimensions greater than one or with a different type of periodicity, as is illustrated in Figures 3 and 4, respectively.

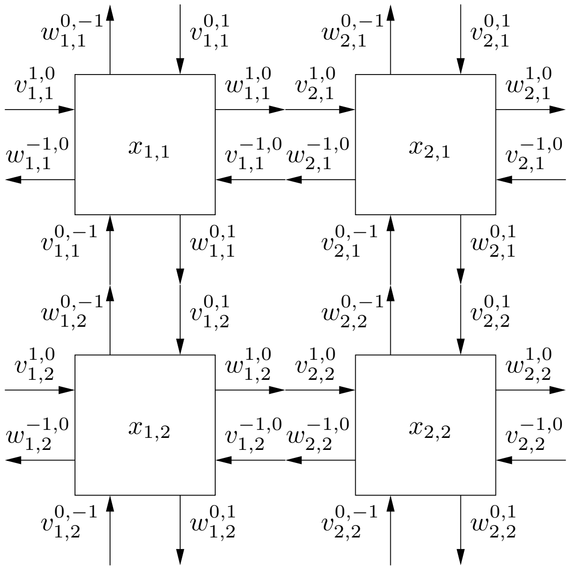

Figure 3: Periodic interconnections in two dimensions.

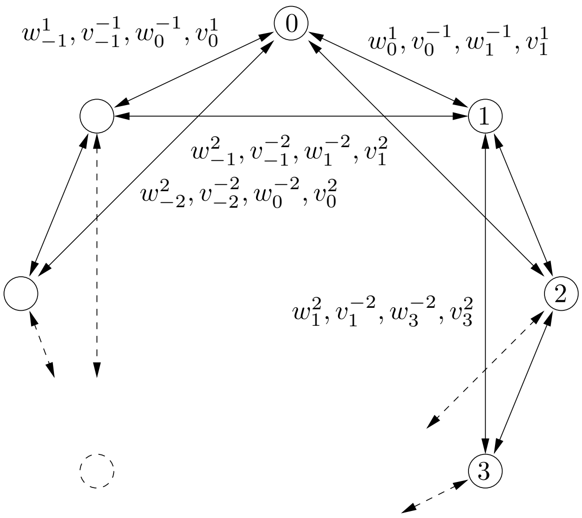

Figure 4: System topology for Example 2.

With respect to the latter notion, interconnections are not necessarily limited to connections with only two neighbors in each dimension, as is illustrated Figure 4. For clarity of presentation, in both figures the control input is not illustrated. Additionally, in Figure 4 the two directed edges connecting each component are represented by one arrow, *i.e.*, all four signals are represented by one edge.

In order to handle these more general cases, we consider the nature of the groups generated by the manner in which components are interconnected. Systems considered in this paper will have components that are members of groups. Recall that a *group* is nonempty set, G with

1. a binary associative operation, $\sigma : G \times G \to G$,
2. an identity element e such that $\sigma(e, g) = \sigma(g, e) = g$ for all $g \in G$, and
3. for every $g \in G$ there exists an element $g^{-1} \in G$ such that $\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = e$.

We use the notation $|G|$ to denote the number of elements in a set G. The rest of this paper considers systems defined on groups for which the one-dimensional case already developed is a special case.

A *subgroup* is a subset of a group that is itself a group. Of particular importance in this paper are elements of a group that *generate* a subgroup. If X is a subset of a group G, then the smallest subgroup of G containing X is called the *subgroup generated by* X. For simplicity, for the rest of this paper we will assume that if $s \in X$, then $s^{-1} \in X$ as well. The idea is that the (sub)group generated by X can be "built up" from the elements of X operating on each other until the set is closed. We will typically use a "multiplication" notation instead of σ for the operation, *i.e.*, $g_1 g_2 = \sigma(g_1, g_2)$. Constraints among the generators are given by *relations* of the form $s_1 s_2 \ldots s_m = e$ for $s_1, \ldots, s_m \in X$. Finally, we will represent systems by a *Cayley graph*, which is a directed graph with vertices that are the elements of a group, G, generated by the subset X, with a directed edge from g_1 to g_2 only if $g_2 = s g_1$ for some $s \in X$. A directed edge from node g_1 to g_2 represents that a coupling input to g_2 is equal to an output from g_1. In general, the edges are directed, an edge from g_1 to g_2 does not necessarily imply an edge is directed from g_2 to g_1. However, because we assumed that if $s \in X$ then $s^{-1} \in X$, it will be the case that if an edge is directed from g_1 to g_2, an edge is also directed from g_2 to g_1. See [25] for a more extensive exposition.

Example 1. *Consider the ring of components illustrated in Figure 4. Each vertex has edges connecting to four other vertices and hence the system is generated by four elements. Let g denote a vertex, i.e.,*

$$g \in \{-2, -1, 0, 1, \ldots, N-3\} = G.$$

Consider the subset of generators $X = \{-2, -1, 1, 2\}$, the group operation to be addition and the relation $s^N = e = 0$. This relation makes the group operation of addition to be $\bmod N$, and hence the group is the quotient of the set of integers $\mathbb{Z}$ where elements of $\mathbb{Z}$ that differ by an integer multiple of N are equivalent. The Cayley graph is illustrated in Figure 4. A vertex is only adjacent to four neighbors because the set of generators has four elements.

For the system illustrated in Figure 3, let $G = \mathbb{Z} \times \mathbb{Z}$ and for $g = (n_1, n_2) \in G$, define the group operation by component-wise addition, i.e., for $g_1 = (n_1, n_2)$ and $g_2 = (m_1, m_2)$, $g_1 g_2 = (n_1 + m_1, n_2 + m_2)$. For the set of generators $s_{1,0} = (1, 0)$ $s_{-1,0} = (-1, 0)$, $s_{0,1} = (0, 1)$ and $s_{0,-1} = (0, -1)$ the Cayley graph is illustrated in Figure 3. With no relation on the generators, the group would be an infinite integer lattice. ◇

For a system on the group G with the set of generators $X = \{s_1, s_2, \ldots, s_{|X|}\}$, denote the state variable corresponding to $g \in G$ by x_g, the set of neighbors of component $g \in G$ by $Xg = \{s_1 g, s_2 g, \ldots, s_{|X|} g\}$, the states of the neighbors by x_{Xg} and the states of the neighbors of the neighbors by x_{X_Xg}. For component g, denote the set of outputs to be $\{w_g^{s_1}, w_g^{s_2}, \ldots, w_g^{s_{|X|}}\}$ and similarly the set of inputs $\{v_g^{s_1}, v_g^{s_2}, \ldots, v_g^{s_{|X|}}\}$. We will consider systems that have the same number of coupling inputs and outputs. Subsequently when we define periodic interconnections, we will impose the structure that w_g^s is the output from g that is taken as an input to component sg.

The dynamics of a component, $g \in G$ are represented by[1]

$$\begin{aligned} \dot{x}_g(t) &= f_g(x_g(t)) \\ &\quad + \sum_{j=1}^{m_g} g_{g,j}(x_g(t))\, u_{g,j}\left(x_g(t), v_g^{s_1}(t), \ldots, v_g^{s_{|X|}}(t)\right) \\ w_g^s(t) &= w_g^s(x_g(t)), \end{aligned} \tag{4}$$

for all $s \in X$. Periodic interconnections and a symmetry orbit are defined in a manner similar to the case of one spatial dimension, leading to the following definition.

Definition 1. *Let G be a group with a set of generators, X. A system with components $g \in \mathcal{I} \subset G$ with dynamics given by Equation 4 has* periodic interconnections on $\mathcal{I}$ *if*

$$v_g^s(t) = w_{s^{-1}g}^s\left(x_{s^{-1}g}(t)\right), \tag{5}$$

for all $g \in \mathcal{I}$ and $s \in X$. Furthermore, if

$$\begin{aligned} f_{g_1}(x) = f_{g_2}(x), &\quad g_{g_1,j}(x) = g_{g_2,j}(x), \\ w_{g_1}^s(x) = w_{g_2}^s(x), &\quad m_{g_1} = m_{g_2} = m \end{aligned} \tag{6}$$

for all $s \in X$, $g_1, g_2 \in \mathcal{I}$, $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$, then $\mathcal{I}$ forms an orbit of symmetric components. *Finally, if the control laws also satisfy*

$$\begin{aligned} u_{g_1,j}\left(x_1, w_{s_1^{-1}g_1}^{s_1}(x_2), \ldots, w_{s_{|X|}^{-1}g_1}^{s_{|X|}}(x_{|X|+1})\right) = \\ u_{g_2,j}\left(x_1, w_{s_1^{-1}g_2}^{s_1}(x_2), \ldots, w_{s_{|X|}^{-1}g_2}^{s_{|X|}}(x_{|X|+1})\right) \end{aligned} \tag{7}$$

for all $(x_1, \ldots, x_{|X|+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$, $g_1, g_2 \in \mathcal{I}$, $j \in \{1, \ldots, m\}$ and $s \in X$, then the elements of $\mathcal{I}$ form a symmetry orbit. *Such a system with a symmetry orbit is called a* symmetric system on $\mathcal{I}$. *If $\mathcal{I} = G$ it is called a* symmetric system on G. ▷

In general the control inputs for different components, *e.g.*, u_{g_1} and u_{g_2}, are functions on different domains. Specifically, the domain for u_{g_1} contains $(x_{g_1}, x_{X g_1})$ and correspondingly the domain for u_{g_2} contains $(x_{g_2}, x_{X g_2})$. However, an important aspect of the following results is that Equation 7 requires that u_{g_1} and u_{g_2} be equal *as functions.* In other words, for a symmetric system all the control inputs are functions from $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($1+|X|$ copies) to $\mathbb{R}$, and these are equal if, when evaluated at the same point in the domain, give the same value in the range. Of course, in the control system, different inputs take values in different domains corresponding to different components and neighbors; however, if we are able to make statements about the behavior of one of the function on a given domain, if the domains of the other functions are restricted to have the same range of values, then the same statements hold for other functions that are equal.

Example 2. *A recurring example in this paper is a system of $N+1$ planar agents and is a variation of that in [9]. We will first show that this specific example fits within the general framework we are developing. Each robot has a position and velocity in $\mathbb{R}^2 \times \mathbb{R}^2$, with equations of motion for the ith robot given by*

$$\frac{d}{dt}\begin{bmatrix} x_i \\ \dot{x}_i \\ y_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \dot{x}_i \\ 0 \\ \dot{y}_i \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{i,1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{i,2}. \tag{8}$$

All computations are $\operatorname{mod}(N+1)$. *The goal formation is a regular $(N+1)$-polygon centered at the origin, hence the desired formation distance between components i and j is*

$$d_{ij} = \begin{cases} 1, & |i-j| = 1 \\ \frac{\sin\left(\frac{2\pi}{N+1}\right)}{\sin\left(\frac{\pi}{N+1}\right)}, & |i-j| = 2 \end{cases}$$

and the desired distance of robot i to the origin is

$$r_i = \frac{1}{2\sin\frac{\pi}{N+1}}.$$

As is common in formation control problems, note that there are an infinite number of configurations which satisfy the conditions for "the desired formation" because "the" formation may be rotated about the origin. Take the control law to be

$$\begin{aligned} \begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix} = &-\sum_j \begin{bmatrix} \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}(x_i - x_j) \\ \frac{\left(\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}-d_{ij}\right)}{\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}}(y_i - y_j) \end{bmatrix} \\ &- k_d \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} - \begin{bmatrix} k_o \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} x_i \\ k_o \frac{\sqrt{x_i^2+y_i^2}-r_i}{\sqrt{x_i^2+y_i^2}} y_i \end{bmatrix} \end{aligned} \tag{9}$$

where $j \in \{i-2, i-1, i+1, i+2\}$ and k_d and k_o are positive constant gains.

To show that this system has a symmetry orbit where the orbit contains all the robots in the system, we need to show it satisfies all the elements of Definition 1. First, observe that this system can be represented by the graph illustrated in Figure 4 with $G = \{-2, -1, 0, 1, 2, \ldots, N-3\}$, the group operation to be addition, let $X = \{-2, -1, 1, 2\}$ and the relation $s^N = 0$, $N \geq 5$. With these definitions,

[1]The symbol g will be used in two ways, both as the vector field in $\dot{x} = f(x) + g(x)u$ and also in the sense of $g \in G$, where the distinction should always be clear from the context.

the Cayley graph for the system is as illustrated in Figure 4. Also, observe from the control law in Equation 9, the control for robot i depends on its own state as well as the states for robots $i-2$, $i-1$, $i+1$ and $i+2$, which are equivalent to the four generators. Hence, define each of the outputs for robot i to be the vector of the robot's position, i.e.,

$$w_i^s = \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{10}$$

where $s \in X = \{-2, -1, 1, 2, \}$.

Define the inputs to component i to be

$$v_i^s = \begin{bmatrix} x_{i-s} \\ y_{i-s} \end{bmatrix}, \quad s \in \{-2, -1, 1, 2\},$$

which satisfies Equation 5. The dynamics, as given in Equation 8 satisfy Equation 6. Finally, the feedback law given in Equation 9 satisfies Equation 7. Because these hold for all $i \in \{-2, -1, 0, \ldots, N-3\}$ the system has an orbit of symmetric components which contains all the components in the system. ◇

The utility of the definition of a symmetric system is that it is possible to "build up" an equivalent system by adding components to it and requiring that they be interconnected in a manner equivalent to the original system. We will define two systems to be *equivalent* if they have symmetry orbits with identical components which are interconnected in the same manner, but they possibly have a different number of components in the symmetry orbit. The means by which this can be done is to have the systems have the same generators, but possibly different relations which can result in a different group.

Definition 2. *Two symmetric systems on the finite groups G_1 and G_2 are* equivalent *if G_1 and G_2 are generated by the same set of generators, X,*

$$\begin{aligned} f_{g_1}(x) = f_{g_2}(x), \quad g_{g_1,j}(x) = g_{g_2,j}(x), \\ w^s_{s^{-1}g_1}(x) = w^s_{s^{-1}g_2}(x) \end{aligned} \tag{11}$$

and

$$u_{g_1,j}\left(x_1(t), w^{s_1}_{s_1^{-1}g_1}(x_2(t)), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_1}(x_{|X|+1}(t))\right) = u_{g_2,j}\left(x_1(t), w^{s_1}_{s_1^{-1}g_2}(x_2(t)), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g_2}(x_{|X|+1}(t))\right) \tag{12}$$

for all $\left(x_1, x_2, \ldots, x_{|X|+1}\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$, $x \in \mathbb{R}^n$, $g_1 \in G_1$, $g_2 \in G_2$, $s \in X$, and $j \in \{1, \ldots, m\}$ where $m = m_{g_1} = m_{g_2}$. ▷

Example 3. *Returning to Example 2, consider two systems with components that satisfy Equation 8 and components belonging to two groups,*

$$\begin{aligned} G_1 &= \{-2, -1, 0, 1, 2, \ldots, N-3\} \\ G_2 &= \{-2, -1, 0, 1, 2, \ldots, M-3\} \end{aligned}$$

where $M > N$. These systems are equivalent because the dynamics of all the components are identical, the feedback definitions are identical. Both groups are generated by $X = \{-2, -1, 1, 2\}$. The only difference is the relation for G_1 is $s^N = 0$ and the relation for G_2 is $s^M = 0$. ◇

For notational convenience, we will concatenate all the states and vector fields from each component into one system description of the form, $\dot{x}_G = f_G(x_G) + g_G(x_G)u(t)$ where

$$x_G = \begin{bmatrix} x_{g_1} \\ x_{g_2} \\ \vdots \\ x_{g_{|G|}} \end{bmatrix}, \quad f_G(x_G) = \begin{bmatrix} f_{g_1}(x_{g_1}) \\ f_{g_2}(x_{g_2}) \\ \vdots \\ f_{g_{|G|}}(x_{g_{|G|}}) \end{bmatrix},$$

etc. The $x_{g_i} \in \mathbb{R}^n$ are the states of the g_ith component in the symmetry orbit.

3. Stability of Symmetric Systems

This section presents the compositionality stability results. The results are directed toward being able to infer stability of a whole equivalence class of systems based on the stability of one of the members of the class. The results are Lyapunov-based and the first result, Proposition 1 concerns negative (semi)definiteness of the derivative of a Lyapunov function for each member of an equivalence class of symmetric systems. Then Proposition 2 builds on it for Lyapunov stability results as does Proposition 3 for "stability" in the context of LaSalle's invariance principle.

Proposition 1. *Given a symmetric system on a finite group G with generators X, assume there is a function $V_G : \mathcal{D}_G \to \mathbb{R}$ that is smooth on some open domain $\mathcal{D}_G \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($|G|$ times) such that*

1. *V_G may be expressed as the sum of terms corresponding to each component where*

$$V_g : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{1+|X| times} \to \mathbb{R}$$

$$\begin{aligned} V_G(x_G) &= \sum_{g \in G} V_g\left(x_g, x_{Xg}\right) \\ &= \sum_{g \in G} V_g\left(x_g, w^{s_1}_{s_1^{-1}g}(x_{s_1^{-1}g}), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g}(x_{s_{|X|}^{-1}g})\right), \end{aligned} \tag{13}$$

   for all $x \in \mathcal{D}_G$,

2. *the individual functions corresponding to each component in G are equal as functions, i.e.,*

$$V_{g_1} = V_{g_2} = V \tag{14}$$

   for all $g_1, g_2 \in G$, and

3. *for* ***any one of the*** $g \in G$*,*

$$\frac{\partial V_G}{\partial x_g}(x_G)\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right) \leq 0 \tag{15}$$

for all $x_G \in \mathcal{D}_G$.

Then

1. $\dot{V}_G(x) \leq 0$ *for all* $x \in \mathcal{D}_G$ *and*
2. *for any equivalent symmetric system on* $\hat{G}$*, there is a* $V_{\hat{G}}$ *such that* $\dot{V}_{\hat{G}} \leq 0$ *on some open domain,* $\mathcal{D}_{\hat{G}}$*.*

We discuss a few important points related to this Proposition before presenting the proof.

- The utility of this proposition is that the behavior of $\dot{V}_G$ with respect to the dynamics of **only one component**, g, needs to be checked.
- Equation 13 requires that the Lyapunov function corresponding to component g only depend on the states of g, x_g and the states of its neighbors, x_{Xg}.
- One may naively hope that we could simply say that because $\dot{V}_G \leq 0$, then $\dot{V}_g \leq 0$ for any of the components. This is, in fact, **not** the case. Subsequently we present some examples and, as can be seen in Figure 8, which plots the individual Lyapunov functions for a five-robot system, it is **not** the case that each Lyapunov function is negative (semi)definite. This is in contrast to the overall Lyapunov function, which is the sum of the individual Lyapunov functions, which is negative semidefinite, as is illustrated in Figure 7. Hence, the test for stability is not based on each individual $\dot{V}_i$, but rather is given by Equation 15, which depends on the entire V_G but only computations based on the states of an individual component, x_g.

Now we prove Proposition 1.

PROOF. First we show that $\dot{V}_G \leq 0$ and then we will show that any equivalent system on $\hat{G}$ is such that $\dot{V}_{\hat{G}} \leq 0$.

Because the Lyapunov functions corresponding to each component are identical, we may take

$$\mathcal{D}_G = \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{|G|\text{times}} \tag{16}$$

for some subset $\mathcal{D} \subset \mathbb{R}^n$. Note that for $h \in G$, because only V_h and its neighbors depend on x_h,

$$\begin{aligned}\frac{\partial V_G}{\partial x_h}(x_G) &= \frac{\partial}{\partial x_h}\left(\sum_{g \in G} V_g\left(x_g, x_{Xg}\right)\right) \\ &= \frac{\partial}{\partial x_h}\left(\sum_{s=e, s \in X} V\left(x_{sh}, x_{Xsh}\right)\right)\end{aligned}$$

where e is the identity element in G. Hence,

$$\dot{V}_G(x_G) = \sum_{g \in G}\left[\frac{\partial}{\partial x_g}\left(\sum_{s=e, s\in X} V\left(x_{sg}, x_{Xsg}\right)\right)\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right)\right]. \tag{17}$$

By hypothesis, one of the terms in the sum is negative semidefinite, and we will show this implies that all of the terms in the sum are negative semindefinite.

For a given g, the term in square brackets is a function with a domain that is the Cartesian product among the states of g, the states of the neighbors of g and the states of the neighbors of the neighbors of g, which is a set of the form $\mathcal{D} \times \cdots \times \mathcal{D}$. We will show every term in the series is equal to every other term *as functions.* Hence, because the domains of each function are restricted to the same range of values, then negative semidefiniteness of one of them implies the same for all of them.

Consider any two $g_1, g_2 \in G$. Because of the definition of a symmetric system, $f_{g_1} = f_{g_2}$ and $g_{g_1,j} = g_{g_2,j}$ as vector fields (Equation 11) and $u_{g_1,j} = u_{g_2,j}$ as functions (Equation 12). Finally, if we define the mappings corresponding to the differentials by

$$D_g V : \mathcal{D} \times \cdots \times \mathcal{D} \to \mathbb{R}^n$$

$$D_g V(x_g, x_{Xg}, x_{XXg}) = \frac{\partial}{\partial x_g}\left(\sum_{s=e, s\in X} V\left(x_{sg}, x_{Xsg}\right)\right),$$

the differentials corresponding to different components are equal *as differentials i.e.*, $D_{g_1}V = D_{g_2}V$. Hence, as functions, each term in the square brackets are equal, and because the domain of each is restricted to the same set of values, each term is negative semidefinite.

Now, consider any equivalent system. For any equivalent system on $\hat{G}$, define $\mathcal{D}_{\hat{G}} = \mathcal{D} \times \cdots \times \mathcal{D}$ ($\left|\hat{G}\right|$ times) and $V(x) = \sum_{g \in \hat{G}} V\left(x_g, x_{Xg}\right)$ for $x \in \mathcal{D}_{\hat{G}}$. Then

$$\begin{aligned}\dot{V}_{\hat{G}}(x) &= \sum_{g \in \hat{G}} \dot{V}_G\left(x_g, x_{Xg}\right) \\ &= \sum_{g \in \hat{G}} \frac{\partial V_g}{\partial x_g}\left(x_g, x_{Xg}\right)\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right) \\ &= \sum_{g \in \hat{G}} \frac{\partial}{\partial x_g}\left(\sum_{s=e, s\in X} V(x_{sg}, x_{Xsg})\right)\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right)\end{aligned}$$

and each term in the sum is negative semidefinite by the same arguments as for the system on G. □

This proposition gives a computational means to determine stability for an entire equivalence class of systems based on a simple computation. The computation is even of lower order than the usual computations need to determine Lyapunov stability for the system on G itself and furthermore extends to any equivalent symmetric system. The utility of this Proposition is that if $\dot{V} \leq 0$ for a symmetric system, then we can conclude that $\dot{V} \leq 0$ for any equivalent system. This is consistent with the intuitive notion that we should be able to add or remove identical components as long as they interact similarly with their neighbors. The "similar" interaction is enforced by the requirement that the group structure of equivalent symmetric systems be generated by the same set of generators.

This proposition only considers the properties of $\dot{V}$, so we must add the necessary additional conditions to the system to be able to infer stability. The following two propositions complete the picture with respect to Lyapunov stability (Proposition 2) and LaSalle's invariance principle (Proposition 3).

Proposition 2. *Let $x_G = 0 \in \mathcal{D}_G$ be an equilibrium point for a symmetric system on G. Assume there exists a V_G that satisfies the hypotheses of Proposition 1, and furthermore assume that each V_g in $V_G = \sum_{g \in G} V_g$ satisfies $V_g(0) = 0$ and $V_g(x_g, x_{Xg}) > 0$ for components of $x \in \mathcal{D} - \{0\}$. Then the origin is stable for the system on G and stable for any equivalent system on $\hat{G}$. Moreover, if $\dot{V}_G(x_G) < 0$ for $x_G \in \mathcal{D}_G - \{0\}$, then the origin is asymptotically stable for the system on G and any equivalent system on $\hat{G}$.*

Proof. These conditions along with Proposition 1 provides the necessary conditions on V_G in order to infer stability or asymptotic stability, as the case may be, from standard Lyapunov theory, such as Theorem 4.1 from [26]. By construction, $V_{\hat{G}}$ is such that $V_{\hat{G}}(0) = 0$ and $V_{\hat{G}}(x) > 0$ for $x \neq 0$, and hence $V_{\hat{G}}$ also has the required properties from which to conclude stability of the origin for the system on $\hat{G}$. □

The utility of Proposition 2 is that if we can prove with a Lyapunov function that the origin of a symmetric system is stable, then it follows that the origin of any equivalent system is also stable. Furthermore it is stable in the same sense, *i.e.*, stable or asymptotically stable.

Combining the results of Proposition 1 and Lasalle's invariance principle leads to the following.

Proposition 3. *Given a symmetric system on G and a function V_G that satisfies the hypotheses of Proposition 1, assume that there exists a positive constant c such that $\Omega_G = \{x_G \in \mathcal{D} | V_G(x_G) \leq c\} \subset \mathcal{D}$ is bounded. Also assume there exists $x_G \in \Omega$ such that for the components (x_g, x_{Xg}, x_{XXg}) of x corresponding to each of the $g \in G$*

$$\frac{\partial V_G}{\partial x_g}(x_G) \left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g) u_{g,j}(x_g, x_{Xg}) \right) = 0. \tag{18}$$

Then,

1. *for the system on G, any solution starting in Ω_G approaches the largest invariant set in the set of points in Ω_G where $\dot{V}_G = 0$ as $t \to \infty$,*
2. *for any equivalent system on $\hat{G}$, there exists an $\Omega_{\hat{G}}$ such that as $t \to \infty$ any solution starting in $\Omega_{\hat{G}}$ approaches the largest invariant set in the set of points in $\Omega_{\hat{G}}$ where $\dot{V}_{\hat{G}} = 0$.*

Proof. The first result directly follows from Proposition 1 (which ensures $\dot{V} \leq 0$) and Lasalle's invariance principle. The second result also follows directly from Proposition 1 and Lasalle's invariance principle as long as there exists the set $\Omega_{\hat{G}}$ that is compact that contains some points where $\dot{V} = 0$. Define $\mathcal{D}_{\hat{G}}$ and $V_{\hat{G}}$ as in the proof to Proposition 1 and let $\Omega_{\hat{G}} = \{x \in \mathcal{D}_{\hat{G}} | V_{\hat{G}} \leq c\}$. This set bounded because each individual component V_g, of $V_G = \sum_{g \in G} V_g$ must be bounded in order for V_G to be bounded. By definition it is also closed and hence it is compact. Also $\Omega_{\hat{G}}$ contains points where $\dot{V}_{\hat{G}} = 0$ by Equation 18. Thus, the conditions on $\Omega_{\hat{G}}$ necessary to apply Lasalle's invariance principle are met, and with the properties of $\dot{V}_{\hat{G}}$ which follow from Proposition 1, the result follows. □

4. Example

This section will complete Example 2.

Example 4. *Continuing Example 2, for a fleet of 5 agents, note that $X = \{-2, -1, 0, 1, 2\}$ is a group with the group operation of addition and the relation $s^5 = 0$. Define the Lyapunov function on $G = X$ as*

$$\begin{aligned} V_G(x_G) &= \sum_{i=1}^{5} V_i(x_i, x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}) \\ &= \sum_{i=1}^{5} \frac{1}{2} \left[(\dot{x}_i^2 + \dot{y}_i^2) + k_o \left(\sqrt{x_i^2 + y_i^2} - r_i \right)^2 \right. \\ &\quad \left. + \sum_j \left(\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} - d_{ij} \right)^2 \right], \end{aligned} \tag{19}$$

where $j \in \{i-2, i-1, i+1, i+2\}$, d_{ij} is the desired distance between robots and r_i is the desired distance of robot i from the origin, as defined previously. Note that V_G is smooth everywhere, by construction, V_G is the sum of individual terms of the form $V_i(x_i, x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2})$, and by construction, $V_i = V_j$ as functions.

Next we show that Equation 15 is satisfied. By abuse of notation, let $x_i = (x_i, \dot{x}_i, y_i, \dot{y}_i)$, and computing (tedious) $\frac{\partial V_G}{\partial x_i}(f_i + \sum_j g_{i,j} u_{i,j})$ gives

$$\frac{\partial V_G}{\partial x_i}(f_i + \sum_j g_{i,j} u_{i,j}) = -k_d \left(\dot{x}_i^2 + \dot{y}_i^2 \right),$$

which is clearly negative semidefinite. Hence, by Proposition 1, $\dot{V}_G$ is negative semidefinite as is $\dot{V}_{\hat{G}}$ for any equivalent system.

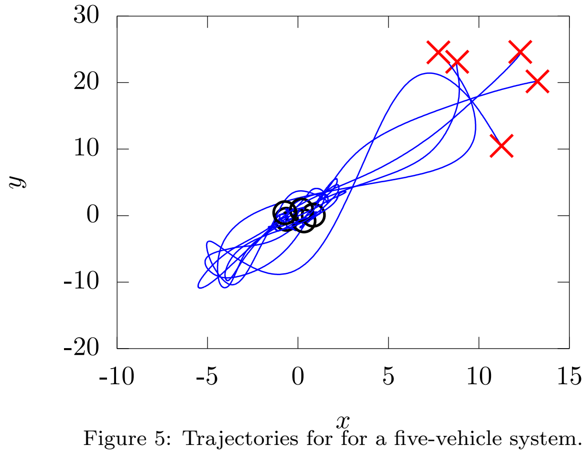

Figure 5: Trajectories for for a five-vehicle system.

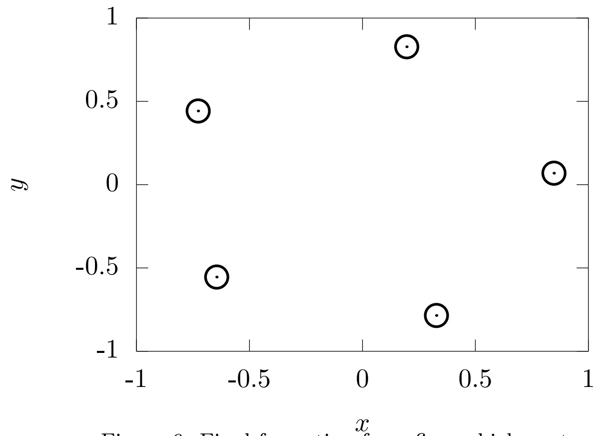

Figure 6: Final formation for a five-vehicle system.

Now, we show that the hypotheses of Proposition 3 are met. Because of the first two terms in V_i, each V_i is radially unbounded. Hence, for any finite initial conditions, there exists a constant, c, such that the initial conditions are in the set Ω_G as defined in Proposition 3. Any state with all robots at rest are such that $\dot{V}_G = 0$. Finally, Equation 15 is satisfied everywhere. Hence, by Proposition 3, the system approaches the largest invariant set such that $\dot{V} = 0$, which is the set that contains the desired formation. The same is true for any equivalent system.

Simulation results for a five-agent system are illustrated in Figures 5 and 6 with $k_d = 0.5$ and $k_o = 0.01$. Figure 5 shows the trajectories for the individual agents (with an x indicating the initial position of a robot and a $\circ$ indicating the steady-state position) and Figure 6 shows the final configuration.

Simulation results for a 17-agent system are illustrated in Figures 9 and 10 with $k_d = 0.5$ and $k_o = 0.01$. Figure 9 shows the trajectories for the individual agents, and Figure 10 shows the final configuration, illustrating convergence to the desired formation for the system independent of the number of agents. Figure 8 shows the evolution of V_1 through V_5 in time, illustrating that they do not individually satisfy $\dot{V}_g \leq 0$. Figure 7 shows the evolution of $V = \sum_{i=1}^{5} V_i$, which does satisfy $\dot{V} \leq 0$. $\diamond$

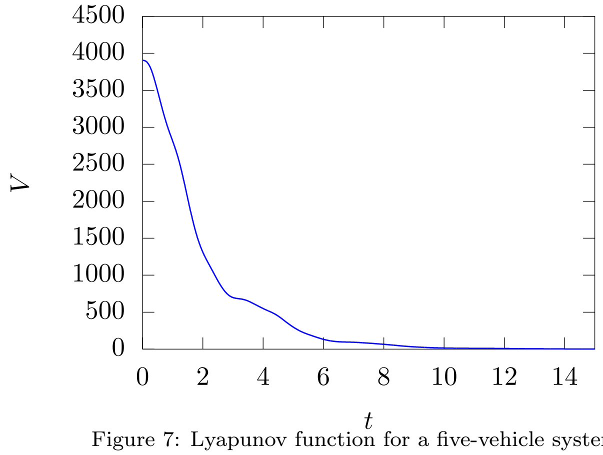

Figure 7: Lyapunov function for a five-vehicle system.

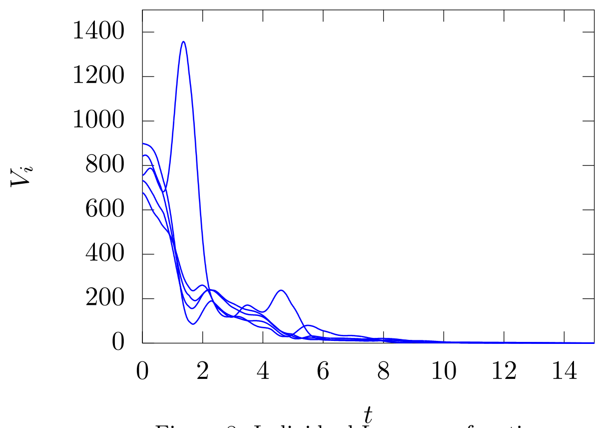

Figure 8: Individual Lyapunov functions.

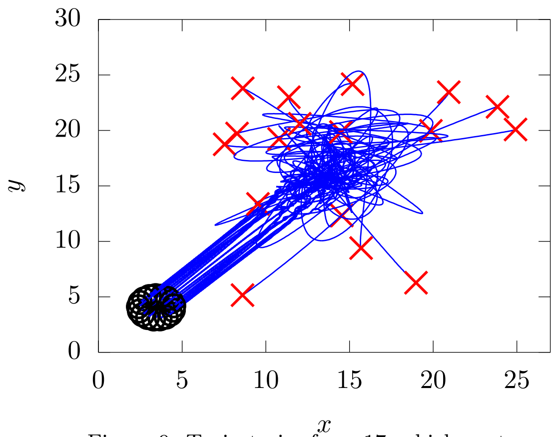

Figure 9: Trajectories for a 17-vehicle system.

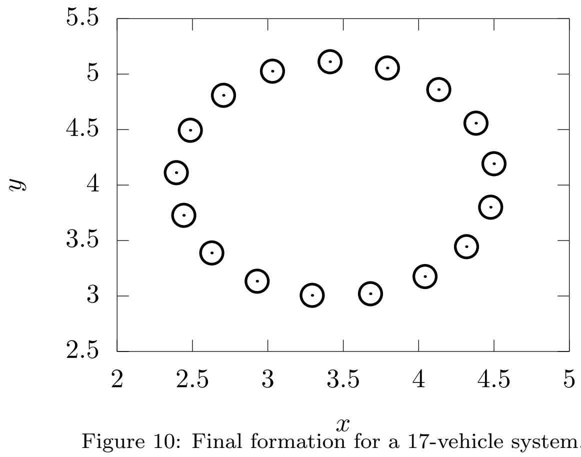

Figure 10: Final formation for a 17-vehicle system.

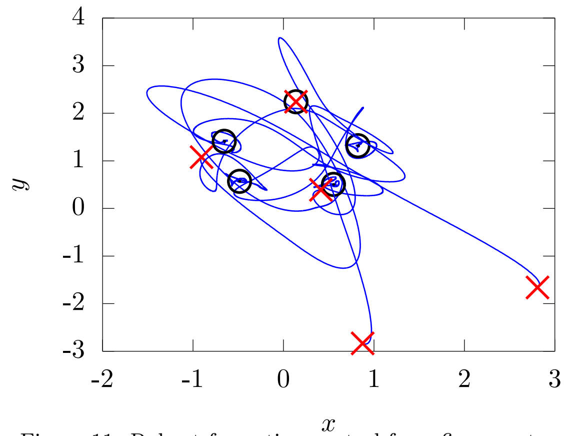

Figure 11: Robust formation control for a five-agent system.

5. Formation Robustness under Agent Failures

The results in the previous sections may be used to formulate some robustness results. First these results are motivated by an example which illustrates the type of system behavior we want to prove.

Example 5. *Consider the system from Examples 2 and 4 with five agents and assume that agent 5 fails in a manner that it has zero velocity and is completely unresponsive to any control input. One would intuitively presume that the rest of the formation will converge to a formation that accommodates such a failure. In fact, this does happen, as is illustrated in Figures 11 and 12. Figure 11, illustrates the trajectories of the agents when agent five fails and remains stationary. Figure 12 illustrates the initial and final configurations for that system. The failed agent has initial (and final) conditions near the point* $(x, y) = (0, 2)$. ⋄

Clearly it is not *a priori* necessary that solutions will remain bounded when an agent fails. In fact, in general it would not be expected because the system being controlled is not the same one for which the controller was designed. Also, consistent with the theme of this paper, we would like results to apply to an entire equivalence class of systems

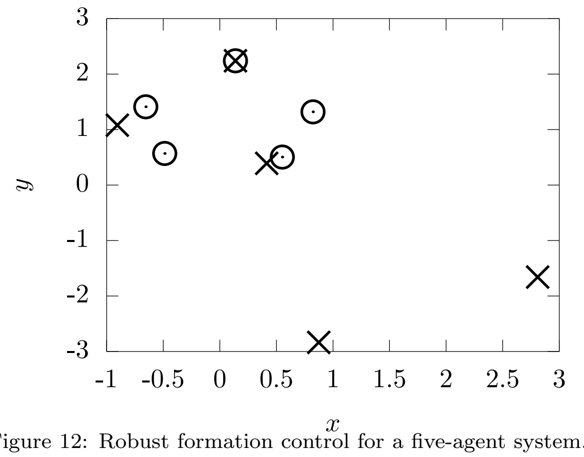

Figure 12: Robust formation control for a five-agent system. Initial conditions are indicated by a $\times$ and final configurations by a $\circ$.

as well. The following corollary to Proposition 3 provides the desired result.

Corollary 1. *If a symmetric distributed system on G satisfies the conditions of Proposition 3, then if any number of agents fail with zero velocity then then conclusions of Proposition 3 still hold.*

Proof. This follows directly from Equation 17. If an agent fails with zero velocity, the term in Equation 17 will have a value of zero, while the other terms are still negative semidefinite. □

6. Conclusions

This paper considers stability and robust stability of symmetric coordinated and distributed systems, with an application focus on coordinated control of systems of mobile robots. The goal is to develop a framework used for spatially periodic systems "built-up" from periodically interconnected components. Observing that many of the formation control algorithms in the literature are not limited by the number of components, but often are limited by assuming specific dynamics, the main contribution of this paper is to formulate a theoretical framework in which stability of many distributed systems can be considered which relies on the symmetric nature of many such systems.

The main contributions are a set of propositions under which stability of an entire class of equivalent systems can be determined from an analysis of just one member of the class. These results are based on formalizing the intuitive notion that if a system contains many similar components with a regular interconnection structure, then adding or removing some components should not drastically change the system properties. Based on this, definitions of symmetric systems and equivalent symmetric systems are defined, leading to the main results. Also, which the results in this paper are limited to systems with identical components, clearly the results are not limited to such cases because seemingly different components may be the same under a nonlinear change of coordinates. While the main example was for mobile robotic formation control, the results are of general applicability.

Current and future efforts related to this work focus on determining boundedness results for symmetric nonautonomous systems. Also, determining a means to allow for slight symmetry breaking is clearly of engineering importance, and hence efforts directed toward developing results for "approximately symmetric" systems are under consideration. Additionally, emergent behavior, such as standard bifurcations of fixed points of differential equations [27], are also expected as system size grows or shrinks. The current efforts can be characterized as developing conditions guaranteeing the absence of emergent behavior. The converse problem of determining when qualitative changes in the dynamics are guaranteed when agents are added or removed is also an area of current focus [28, 29].

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