An Algorithm for Stopping a Class of Underactuated Nonlinear Mechanical Robotic Systems

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Abstract—We provide a constructive global discontinuous control law with state dependent switches for a class of underactuated nonlinear mechanical robotic systems that will drive the system to an arbitrarily small neighborhood of rest from all initial configurations and velocities in arbitrarily small time. Because all physical mobile robotic systems are mechanical in nature, control methodologies which exploit the fact that the system is governed by principles of mechanics which are particularly important for robotic engineers. The philosophy of the approach is that instead of using control algorithms which start with a completely generic dynamical system, we constrain the structure of the system to be one which is a Lagrangian control system. To the extent the structure of the mechanical system can be exploited, stronger control results are possible to obtain, such as the stopping algorithm in this paper. Specifically, for control of general nonlinear systems, there are many unsolved problems for the case when the system is not at an equilibrium, and the results in this paper are an initial contribution to this area. The robot is assumed to be underactuated by one in the configuration space; hence, in the state space it is underactuated by twice the dimension of the configuration space plus two. Our method can easily be extended to construct a global discontinuous control law with state dependent switches that will drive the system to an arbitrarily small neighborhood of any velocity from any initial configuration and velocity in arbitrarily small time.

I. INTRODUCTION

Nonlinear mechanical control systems form a large and challenging class of control systems and are of particular importance in robotics. The simple example used in this paper is a planar robotic hovercraft model illustrated in Figure 1. We emphasize that while this model has been studied extensively (for example it is differentiably flat [18]), our focus is not to simply control the hovercraft, but rather to illustrate the application of our general results to a specific system. That model has three configuration variables, position and orientation in the plane, and hence in the state space, i.e., converted to a system of first-order equations, is six-dimensional. It has two forces as inputs. Differential and Riemannian geometry provide an elegant framework for modeling, analysis and control for such systems. This framework has given rise to powerful insights into nonlinear controllability in the zero velocity setting motivating stabilization, tracking and motion planning algorithms [2]. A vexing and persistent problem in nonlinear control, including control of mechanical robotic systems, is extending such results to the non-zero velocity case.

We have extended this research on the analysis and control of underactuated mechanical robotic systems to the non-zero velocity setting. Our short term goal has been to develop necessary and sufficient conditions for reaching zero velocity [9] and [10]. We have identified an intrinsic vector-valued symmetric bilinear form that can be associated with an underactuated mechanical control system and developed computable tests dependent upon the definiteness of the symmetric form to determine if the system can or cannot be driven to rest. We have also developed an iterative stopping algorithm that takes advantage of the underlying quadratic structure [11].

As will be shown in this paper, our results are useful design tools which provide constructive strategies for actuator assignment and help to make the control scheme robust to actuator failure [15]. For fully actuated mechanical systems, it is possible to provide a comprehensive solution to the problem of trajectory tracking [2]. In contrast, motion planning algorithm for underactuated mechanical systems is still not well understood [13]. Due to the challenging nature of these problems, many of the existing results have been limited for example to ad hoc gait generation algorithms [7] [3], ad hoc configuration to configuration algorithms with zero-velocity transitions between feasible motions [2] and numerically generated optimal trajectories [12]. The constructive algorithm presented in this paper is formulated in a general setting and applicable to any member of the general class of robotic systems underactuated by one control that satisfy the sufficient conditions given in Theorem 4.5 in Section IV.

For effective control of robotic systems, control methods must take into account the underlying nonlinear geometry of the geometric model, which is made difficult by the nonzero drift. In general, these systems are not full-state feedback
linearizable and thus not amendable to standard techniques in control theory [6]. The last decades have shown a great interest in the analysis and control of nonlinear mechanical control systems designed to be underactuated by one control. Such systems include underactuated ships [4], gymnastic robots [17], the Harrier which is a planar vertical/short take-off and landing aircraft in the absence of gravity [14], a hovercraft type vehicle [16] and a planar rigid body with two thrusters moving on a flat horizontal plane [8].

The class of quadratically coupled mechanical systems underactuated by one control is closely related to the systems considered in [5] where a homogeneous, discontinuous, stabilising feedback controller is designed based on the underlying quadratic structure. Our constructive algorithm can also be applied to their example system. In general, quadratic or symmetric bilinear structures can be found in a variety of areas in control theory which has motivated a new initiative to understand the underlying geometry [1] [2].

A critical tool used to analyze distributions and mechanical control systems is the symmetric product [2]. Given a pair of vector fields $X, Y$, their symmetric product is the vector field defined by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

where $\nabla_X Y$ is the covariant derivative of $Y$ with respect to $X$. In coordinates we have

$$\nabla_X Y = \left( \frac{\partial Y^i}{\partial q^a} X^i + \Gamma^i_{jk} X^j Y^k \right)$$

where $X = X^i \frac{\partial}{\partial q^a}$ and $Y = Y^i \frac{\partial}{\partial q^a}$. The Christoffel symbols are given by

$$\Gamma^i_{jk} = \frac{1}{2} G^{ij} \left( \frac{\partial G_{jk}}{\partial q^a} + \frac{\partial G_{ik}}{\partial q^a} - \frac{\partial G_{jk}}{\partial q^a} \right)$$

where $G^{ij}$ is the inverse of the matrix $G_{ij}$ that represents $G$.

**B. Alternative Representation**

An input distribution $\mathcal{Y}$ on $M$ is a subset $\mathcal{Y} \subset TM$ having the property that for each $q \in M$ there exists a family of vector fields $\{Y_1, \ldots, Y_m\}$ on $M$ so that for each $q \in M$ we have

$$\mathcal{Y}_q = \mathcal{Y} \cap T_q M = \text{span}_\mathbb{R}\{Y_1(q), \ldots, Y_m(q)\}.$$

We refer to the vector fields $\{Y_1, \ldots, Y_m\}$ as generators for $\mathcal{Y}$ and $\{W_1, \ldots, W_m\}$ as the orthonormal generators for $\mathcal{Y}$ where $G(W_a, W_p) = 1$ when $a = p$ and $G(W_a, W_p) = 0$ when $a \neq p$. Let $\mathcal{Y}^\perp$ denote an orthonormal frame $\{Y_1^\perp, \ldots, Y_m^\perp\}$ that generates the $G$-orthogonal complement of the input distribution $\mathcal{Y}$. Note that even though $\mathcal{Y}^\perp$ is canonically defined, we must choose an orthonormal basis. It is clear that $\{Y_q, Y_q^\perp\}$ forms a basis for $T_q M$ at each $q \in M$. Note that $\mathcal{Y} = \{W_1, \ldots, W_m\}$ is a set of $m$ orthonormal vector fields while $\mathcal{Y}^\perp = \{Y_1^\perp, \ldots, Y_m^\perp\}$ is a set of $n - m$ orthonormal vector fields.

Given a basis of $G$-orthonormal vector fields $\{X_1, \ldots, X_n\}$ on $M$, the generalized Christoffel symbols are defined by the $n^3$ functions $\hat{\Gamma}^k_{ij} : M \rightarrow \mathbb{R}$ where

$$\nabla_X Y = \hat{\Gamma}^k_{ij} X_k.$$  

**Definition 2.1:** We define the **generalized symmetric Christoffel symbols** for $\nabla$ with respect to the basis of $G$-orthonormal vector fields $\{X_1, \ldots, X_n\}$ on $M$ as the $n^3$ functions $\hat{\Gamma}^k_{ij} : M \rightarrow \mathbb{R}$ defined by

$$\hat{\Gamma}^k_{ij} X_k = \frac{1}{2} \left( \hat{\Gamma}^k_{ij} + \hat{\Gamma}^k_{ji} \right) X_k = \frac{1}{2} G(\langle X_i, X_j \rangle, X_k) X_k.$$

We may define the velocity vector $\dot{\gamma}(t) = v^i(t) \frac{\partial}{\partial q^i}$ of the curve $\gamma(t)$ in terms of the family of vector fields $\{\mathcal{Y}, J\}$. The new expression for $\dot{\gamma}(t)$ is in the form

$$\dot{\gamma}(t) = w^a(t) W_a(\gamma(t)) + s^b(t) Y^b_\perp(\gamma(t))$$

where

$$w^a = G(\dot{\gamma}(t), W_a(\gamma(t)))$$

and

$$s^b = G(\dot{\gamma}(t), Y^b_\perp(\gamma(t))).$$
Proposition 2.2: Let \( \Sigma = \{ M, G, Y, U \} \) be a simple mechanical control system defined above. The following holds along the curve \( \gamma(t) \) satisfying (1):
\[
\frac{d}{dt} w^l = - \tilde{\Gamma}^l_{ap} w^a w^p - 2 \tilde{\Gamma}^l_{ar} w^a s^r - m \\
- \tilde{\Gamma}^l_{rk} s^{r-m}(t) s^{k-m} + u^a g(Y_a, W_l)
\] (5)
\[
\frac{d}{dt} s^{b-m} = - \tilde{\Gamma}^b_{ap} w^a w^p - 2 \tilde{\Gamma}^b_{ar} w^a s^r - m \\
- \tilde{\Gamma}^b_{rk} s^{r-m}(t) s^{k-m}
\] (6)

where \( a, l, p \in \{ 1, \ldots, m \}, b, k, r \in \{ m + 1, \ldots, n \} \).

Proof: We proceed by substituting Equation 2 into Equation 3 and Equation 4 then differentiating and taking advantage of the compatibility associated with the Levi-Civita connection.

C. Partial Feedback Linearization

By inspection we see that Equation 5 can be written as
\[
\frac{dw}{dt} = \begin{bmatrix} w & s \end{bmatrix} \begin{bmatrix} \tilde{\Gamma} & I \end{bmatrix} \begin{bmatrix} w \\ s \end{bmatrix} + Gu. (7)
\]
We perform feedback linearization on Equation 7 by defining our input to be
\[
u = -G^{-1}F + G^{-1}z
\]
where we assume that \( G \) is full rank. Now substituting \( u \) into Equation 7 to get
\[
\frac{dw}{dt} = z.
\]

III. Quadratically Coupled Control System

Let us consider a class of nonlinear mechanical systems where \( n - m = 1 \) and the right-hand-side of Equation 6 is independent of \( q \) and \( s \). The governing equations in local coordinates are
\[
\frac{d}{dt} q = v, \\ \frac{d}{dt} w = z, \\ \frac{d}{dt} s = Q(w)
\]
where the configuration and pseudo-velocity states are \((q, w, s) \in Y \times Y^l, z \in Z \subseteq \mathbb{R}^m\) is the control, and \( Q: Y_q \to \mathbb{R} \) is a quadratic map, i.e., \( Q(\lambda w) = \lambda^2 Q(w) \) for all \( w \in \mathbb{R}^m \) and \( \lambda \in \mathbb{R} \). We refer to \( w \) and \( s \) to be pseudo-velocities because in general \((w, s) \neq v\). This class of system is clearly not linearly controllable. Let us define the control system to be the tuple \( \Delta = (TM, \frac{d}{dt}, \frac{d}{dt}, V) \) that defines the state space, governing equations of motion and the set of available inputs. A basic property of \( \mathbb{R} \)-valued quadratic forms is that indefiniteness is equivalent to the existence of a basis \( \mathbb{F} \) for \( \mathbb{R}^m \) such that the diagonal entries in the matrix representation of \( Q \) are all zero [2].

In other words, if \( Q \) is indefinite then exists a basis \( \mathbb{F} \) for \( \mathbb{R}^m \) such that the expansion
\[
\frac{ds}{dt} = Q_{ij} w^i w^j \\
= 2Q_{12}w^1 w^2 + \cdots + 2Q_{1m}w^1 w^m + \cdots \\
+ 2Q_{23}w^2 w^3 + \cdots + 2Q_{2m}w^2 w^m + \cdots \\
+ 2Q_{m-1m}w^{m-1} w^m
\]
(9)

where indices \( i, j = 1, \ldots, m \) and \( Q_{ij} = 0 \) when \( i = j \).

It is also clear that if \( Q \) is indefinite then there exists at least 1 symmetric term \( Q_{ij} \neq 0 \) where \( i \neq j \). In order to simplify the derivation and presentation of our algorithm we will refer to the first coefficient \( Q_{ij} \neq 0 \) in Equation 10 as \( Q_c \) and the corresponding \( w \) parameters as \( w_c^1 \) and \( w_c^2 \).

IV. ALGORITHM

This section contains the main result of this paper. We provide a global discontinuous control law with state dependent switches for a class of mechanical control systems that will drive the system to an arbitrarily small neighborhood of \((q, 0) \in TM\) from all initial configurations and initial velocities in arbitrarily small time. This class of mechanical control systems is characterized by a quadratic coupling \( Q \) between the equations of motion for the actuated velocity states and the unactuated velocity state. Our results are applicable to the case when the quadratic coupling is indefinite.

Our method can easily be extended to construct a global discontinuous control law with state dependent switches that will drive the system to an arbitrarily small neighborhood of any velocity from any initial configuration and velocity in arbitrarily small time. Our procedure for demonstrating the existence of such a control law involves three basic steps.

1) Drive \( \dot{w} \to 0 \) in time \( \kappa \).
2) Drive \( s \to 0 \) in time \( \gamma \).
3) Drive \( w \to 0 \) while keeping \( s \in B(0, \epsilon) \) in time \( \tau \).

We begin by introducing a series of lemmas which contain the control laws corresponding to each step in our basic procedure. The lemmas will be used to prove our main result.

Lemma 4.1: For all \( w(t_0) \) and \( \kappa > 0 \) if \( z \) defined on \([t_0, t_1]\) is \( z(t) = -w(t_0)/\kappa \) where \( t_1 = t_0 + \kappa \) then \( w(t_1) = 0 \).

Proof: We substitute the control law \( v(t) \) into Equation 8 to get.
\[
\frac{dw}{dt} = -\frac{w(t_0)}{\kappa}.
\]
We can apply separation of variables and integration to get
\[
w(t) = w(t_0) - \frac{w(t_0)}{\kappa}(t - t_0)
\]
on \([t_0, t_1]\). It’s clear from \( w(t) \) that \( w(t_1) = 0 \) holds.

Remark 4.2: For the remainder of the proofs, we will only need the first nonzero term \( Q_c \) in the expansion of Equation 9 and the corresponding \( w \) terms \( w_c^1 \) and \( w_c^2 \). Once the \( w \) is driven to zero then we will only utilize the \( z \) inputs \( z_c^1 \) and \( z_c^2 \) corresponding to \( Q_c, w_c^1 \) and \( w_c^2 \). The remaining \( z \) inputs will be set to zero. Again, this simplifies our derivation and presentation of the stopping algorithm.

Lemma 4.3: For all \( s(t_1), w(t_1) = 0 \) and \( \gamma > 0 \) if \( Q : \mathbb{R}^m \to \mathbb{R} \) is indefinite and \( z \) defined on \([t_1, t_2]\) is
\[
z_c^1 = -sgn(Q_c) |M|, \quad z_c^2 = sgn(s(t_1)) |M|
\]
where \( t_2 = t_1 + \gamma \) and
\[
M = \left( \frac{3}{2} s(t_1) Q_c \gamma^3 \right)^{1/2}
\]
then \( s(t_2) = 0 \).

**Proof:** We substitute the control law \( z(t) \) into Equation 8 to get
\[
\frac{dw_1}{dt} = -\text{sgn}(Q_c)|M|, \quad \frac{dw_2}{dt} = \text{sgn}(s(t_1))|M|
\]
with
\[
M = \left( \frac{3 s(t_1)}{2 Q_c \gamma^3} \right)^{1/2}.
\]
We can apply separation of variables and integration to get
\[
w_1^1(t) = -\text{sgn}(Q_c)|M|(t - t_1)
w_2^1(t) = \text{sgn}(s(t_1))|M|(t - t_1)
\]
on \([t_1, t_2]\). We also know that
\[
s(t_2) = s(t_1) + A \left( \frac{ds}{dt} \right) \gamma
\]
where
\[
A \left( \frac{ds}{dt} \right) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} 2Q_c w_1^1(t) w_2^2(t) dt
= \frac{2Q_c z_c^1 z_c^2}{\gamma} \int_0^\gamma t^2 dt = \frac{2Q_c z_c^1 z_c^2}{3} \gamma^2.
\]
This gives us
\[
s(t_2) = s(t_1) + \frac{2Q_c z_c^1 z_c^2}{3} \gamma^3.
\]
Now substitute for \( z_c^1 \) and \( z_c^2 \) to get
\[
s(t_2) = s(t_1) + \frac{2Q_c(-\text{sgn}(Q_c)|M|)(\text{sgn}(s(t_1)))|M|}{3} \gamma^3.
\]
Now substitute for \( M \) and simplify to get
\[
s(t_2) = s(t_1) - |s(t_1)| \text{sgn}(s(t_1))
\]
\[
s(t_2) = s(t_1) - |s(t_1)| \text{sgn}(s(t_1))
\]
\[
s(t_2) = 0.
\]

**Lemma 4.4:** Let us assume that
\[
w_1^1(t_2) = -\text{sgn}(Q_c)|M| \gamma
\]
and
\[
w_2^1(t_2) = \text{sgn}(s(t_1))|M| \gamma.
\]
For all \( s(t_1), \epsilon, \gamma > 0 \) and \( s(t_2) = 0 \), if
\[
z_c^1 = \frac{\text{sgn}(Q_c)}{\tau} |M|, \quad z_c^2 = -\frac{\text{sgn}(s(t_1))}{\tau} |M|
\]
where
\[
\tau < \frac{\epsilon \gamma^3}{|s(t_1)|} \quad \text{and} \quad M = \left( \frac{3 s(t_1)}{2 Q_c \gamma^3} \right)^{1/2}
\]
then \((w(t_3), s(t_3)) \in B(0, \epsilon)\).

**Proof:** We substitute the control law \( z(t) \) into Equation 8 to get
\[
\frac{dw_1}{dt} = -\frac{\text{sgn}(Q_c)}{\tau} |M|, \quad \frac{dw_2}{dt} = -\frac{\text{sgn}(s(t_1))}{\tau} |M|
\]
with
\[
\tau < \frac{\epsilon \gamma^3}{|s(t_1)|} \quad \text{and} \quad M = \left( \frac{3 s(t_1)}{2 Q_c \gamma^3} \right)^{1/2}
\]
We can apply separation of variables and integration to get
\[
w_1^1(t) = w_1^1(t_2) + \frac{\text{sgn}(Q_c)}{\tau} |M|(t - t_2)
w_2^1(t) = w_2^1(t_2) - \frac{\text{sgn}(s(t_1))}{\tau} |M|(t - t_2)
\]
on \([t_2, t_3]\). It is clear that \( w(t_3) = 0 \). Now we need to check that \(|s(t_3)| < \epsilon\). Since \( s(t_2) = 0 \) then we also know that
\[
s(t_3) = A \left( \frac{ds}{dt} \right) \gamma
\]
where
\[
A \left( \frac{ds}{dt} \right) = \frac{1}{t_3 - t_2} \int_{t_2}^{t_3} 2Q_c w_1^1(t) w_2^2(t) dt
= \frac{2Q_c z_c^1 z_c^2}{\tau} \int_0^\gamma t^2 dt = \frac{2Q_c z_c^1 z_c^2}{3} \gamma^3.
\]
This gives us
\[
s(t_3) = \frac{2Q_c z_c^1 z_c^2}{3} \gamma^3.
\]
Now substitute for \( z_c^1 \) and \( z_c^2 \) to get
\[
s(t_3) = \frac{2Q_c(\text{sgn}(Q_c)|M|)(\text{sgn}(s(t_1)))|M|}{3} \gamma^3.
\]
Now substitute for \( M \) and simplify to get
\[
|s(t_3)| = \left| \frac{2Q_c}{3} \left( \frac{3 |s(t_1)|}{2 Q_c \gamma^3} \right) \gamma \right|
\]
\[
|s(t_3)| = |s(t_1)| \frac{\tau}{\gamma^3}.
\]
It is clear that if
\[
\tau < \frac{\epsilon \gamma^3}{|s(t_1)|}
\]
then \(|s(t_3)| < \epsilon\). Recall by construction we pick
\[
\tau < \frac{\epsilon \gamma^3}{|s(t_1)|}
\]
which implies \(|s(t_3)| < \epsilon\).

**Theorem 4.5:** Given the control system \( \Delta = (TM, \frac{dw}{dt}, \frac{ds}{dt}, Z) \). For all initial conditions \( w(t_0) \) and \( s(t_0) \) and constants \( \epsilon, \delta > 0 \) if \( Q \) is indefinite and \( z(t) \) is defined on \([t_0, t_3]\) to be

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>( z_c^1(t) )</th>
<th>( z_c^2(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>([t_0, t_1])</td>
<td>(-\frac{w_1(t_0)}{\kappa})</td>
<td>(-\frac{w_2(t_0)}{\kappa})</td>
</tr>
<tr>
<td>([t_1, t_2])</td>
<td>(-\text{sgn}(Q_c)</td>
<td>M</td>
</tr>
<tr>
<td>([t_2, t_3])</td>
<td>\frac{\text{sgn}(Q_c)}{\tau}</td>
<td>M</td>
</tr>
</tbody>
</table>
where 
\[
M = \left( \frac{3}{2Q_c \gamma} t_1 \right)^{1/2} \quad \text{and} \quad \tau < \frac{\epsilon \gamma^3}{|s(t)|}
\]
then \((w(t_3), s(t_3)) \in B(0, \epsilon)\) and \(|t_3 - t_0| < \delta\).

**Proof:** It follows from Lemma 4.1 that the control law defined on the interval \([t_0, t_1]\) will give rise to \(w(t_1) = 0\).

Now we apply Lemma 4.3 to see that given \(w(t_1) = 0\) and any \(s(t_1)\) that the control law defined on the interval \([t_1, t_2]\) will result in \(s(t_2) = 0\). \(w_1^2(t_2) = -\text{sgn}(Q_c)|M|\gamma\) and \(w_2^2(t_2) = \text{sgn}(s(t_1))|M|\gamma\). Finally, we can appeal to Lemma 4.4 to show that given \(s(t_2) = 0\), \(w_1^2(t_2) = -\text{sgn}(Q_c)|M|\gamma\) and \(w_2^2(t_2) = \text{sgn}(s(t_1))|M|\gamma\) that the control law defined on the final interval \([t_2, t_3]\) will drive the system to \((w(t_3), s(t_3)) \in B(0, \epsilon)\). In order to ensure that \(|t_3 - t_0| < \delta\) pick \(\kappa, \gamma, \tau\) such that \(\kappa + \gamma + \tau < \delta\).

**Remark 4.6:** Theorem 4.5 can be easily extended to construct a discontinuous control law with state dependent switches that will drive the system to an arbitrarily small neighborhood of any velocity from any \((q, w, s) \in TM\) in arbitrarily small time. In addition, it can be shown that as long as the controls for \(\Delta\) take values in a subset \(Z \subset \mathbb{R}^m\) for which \(\Theta \in \text{int}(\text{conv}(U))\) then we can still drive \(\Delta\) to an arbitrarily small neighborhood of rest.

**V. EXAMPLE: FORCED PLANE RIGID BODY HOVERCRAFT ROBOT**

In this section we review the geometric model, construct the alternative representation of the equations of motion, perform partial feedback linearization and construct the stopping algorithm for the planar rigid body (Figure 1). The configuration manifold for the system is the Lie group \(SE(2)\). Use coordinates \((x, y, \theta)\) for the planar robot where \((x, y)\) describes the position of the center of mass and \(\theta\) describes the orientation of the body frame \(\{b_1, b_2\}\) with respect to the inertial frame \(\{e_1, e_2\}\). In these coordinates, the Riemannian metric is given by
\[
\mathcal{G} = m \, dx \otimes dx + m \, dy \otimes dy + J \, d\theta \otimes d\theta,
\]
where \(m\) is the mass of the body and \(J\) is the moment of inertia about the center of mass. Let us analyze the set of inputs that consist of the force \(F^1\) applied to a point and a torque \(F^3\) about the center of mass. We assume that the point of application of the force is a distance \(b > 0\) from the center of mass along the \(b_1\) body-axis. The input force can represent a variable-direction thruster on the body which can be resolve into components along the \(b_1\) and \(b_2\) directions. The control inputs are given by
\[
F^1 = \cos \theta \, dx + \sin \theta \, dy, \quad F^3 = d\theta.
\]

**Problem Statement 5.1:** Given the planar rigid body with input forces \(\{F^1, F^3\}\), construct the control law \(z(t)\) that will drive the system from any \((q(t_0), w(t_0), s(t_0)) \in TM\) to \((q, B(0, \epsilon)) \in TM\) in time \(\delta\).

**Solution 5.2:** We use the metric \(\mathcal{G}\) to compute the input vector fields \(Y_a = \mathcal{G}^g(F^a)\) to be
\[
Y_1 = \frac{1}{m} \cos \theta \, \frac{\partial}{\partial x} + \frac{1}{m} \sin \theta \, \frac{\partial}{\partial y}, \quad Y_3 = \frac{1}{J} \frac{\partial}{\partial \theta}.
\]

Now let us construct the orthonormal basis vector fields \(W_1, W_2\) that generate \(\mathcal{G}\). It’s clear that \(Y_1\) and \(Y_3\) are orthogonal with respect to \(\mathcal{G}\). All we need to do is normalize \(Y_1\) and \(Y_3\) with respect to \(\mathcal{G}\) and set them equal to \(W_1\) and \(W_2\) respectively. This gives us
\[
W_1 = \sqrt{m} \cos \theta \, \frac{\partial}{\partial x} + \sqrt{m} \sin \theta \, \frac{\partial}{\partial y}, \quad W_2 = \sqrt{J} \frac{\partial}{\partial \theta}.
\]

Now we construct the single vector field
\[
Y_{\perp} = -\sqrt{m} \sin \theta \, \frac{\partial}{\partial x} + \sqrt{m} \cos \theta \, \frac{\partial}{\partial y}
\]
where by inspection we see that it is orthogonal to \(\mathcal{G}\) and normalized. The only non-zero generalized symmetric Christoffel symbols are
\[
\hat{\Gamma}^1_{23} = -\sqrt{\frac{J}{2}}, \quad \hat{\Gamma}^3_{21} = -\sqrt{\frac{J}{2}}.
\]

The resulting equations of motion are
\[
\begin{align*}
\frac{dx}{dt} &= \frac{1}{m} \sqrt{m} \cos \theta - s \frac{\sqrt{m}}{m} \sin \theta, \\
\frac{dy}{dt} &= \frac{1}{m} \sqrt{m} \sin \theta + s \frac{\sqrt{m}}{m} \cos \theta, \\
\frac{d\theta}{dt} &= u \sqrt{\frac{J}{m}}, \\
\frac{dw^1}{dt} &= \frac{\sqrt{J}}{J} w^2 s + u \frac{\sqrt{m}}{m}, \\
\frac{dw^2}{dt} &= u \frac{\sqrt{J}}{J}, \\
\frac{ds}{dt} &= \frac{\sqrt{J}}{J} w^2 w^1.
\end{align*}
\]

Now we set \(u\) equal to
\[
\begin{align*}
u^1 &= \frac{m}{\sqrt{m}} \sqrt{\frac{1}{m}} z^1 - \frac{m}{\sqrt{m}} \frac{\sqrt{J}}{J} w^2 w^{1}, \\
u^2 &= \frac{1}{\sqrt{J}}.
\end{align*}
\]

Substitute \(u\) into Equation 11 to get
\[
\begin{align*}
\frac{dw^1}{dt} &= z^1, \\
\frac{dw^2}{dt} &= z^2, \\
\frac{ds}{dt} &= \frac{\sqrt{J}}{J} w^2 w^1.
\end{align*}
\]

By inspection, we have \(Q_c = \sqrt{J}\). Since \(J > 0\) we conclude that \(Q = \frac{\sqrt{J}}{J} w^2 w^1\) is indefinite.
VI. Conclusions and Future Work

This paper presented an algorithm to bring a simple mechanical system underactuated by one to rest from an arbitrary initial velocity. The method relies upon the analysis of an intrinsic bilinear symmetric quadratic form which naturally is present in mechanical systems of this form and the resulting control law is discontinuous with state-dependent switches. Because this form represents the degree to which the controlled and uncontrolled degrees of freedom are coupled, when the form is indefinite it corresponds to the “ability” of the controlled degrees of freedom to both increase and decrease the uncontrolled velocity, which is needed for arbitrary initial conditions. Our result is an important and novel contribution to the state of the art because in the general nonlinear controls context, analysis and synthesis away from equilibria is generally difficult. Our approach exploits the structure of the system as a mechanical system as the basis for the stopping algorithm.

Some preliminary work has been done to extend our results to systems where \( Q \) depends on \( w, s \) and \( q \). We are also working on an iterative algorithm under bounded control inputs. Finally, we are less optimistic about systems where \( Q \) is vector-valued instead of real-valued. A computationally efficient method for determining the indefiniteness of a vector-valued quadratic form does not exist [2]. This has been given some initial consideration in control literature [1]. Additional lines of future inquiry are into mechanics and control in robotic legged locomotion with the goal of extending analyses such as in [19] to the more general context.

**References**