Compositional Boundedness of Solutions for Symmetric Nonautonomous Control Systems

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***Abstract*— This paper presents boundedness results for symmetric nonautonomous control systems. The results are Lyapunov-based and exploit the symmetric structure present in many compositional or distributed control system. The work extends some of our prior work which defines an equivalence class of symmetric control systems by determining conditions for boundedness of solutions for such systems. The extension is along two lines. First, the prior work was focused on stability for autonomous symmetric systems, and this work extends it to the nonautonomous case. Second, the prior work required exact symmetry in the system whereas the results in this paper allow for symmetry-breaking in the nonautonomous terms, which significantly broadens the class of systems to which these results will apply. These results will be useful for robotics and control engineers dealing with large-scale and distributed systems which are composed of many similar components because it will enable closed-form analysis on a very small system, and then guarantee system properties for much larger equivalent systems. Other potential application areas would be, for example, the design of control algorithms for fleets of autonomous robotic vehicles acting in a coordinated manner.**

I. INTRODUCTION

Many current research efforts concerning cyber-physical systems focus on *composability* and *compositionality* of control systems [22], [8]. The idea in each case is to be able to infer or ensure overall system properties based on the properties of the individual components of the system. Some of the author's prior work has focused on stability of so-called *symmetric systems* wherein a system is comprised of many identical individual components interconnected in a structured manner (this will be rigorously defined subsequently in Section II).

More importantly, an *equivalence relation* among symmetric systems with different numbers of components can be defined, hence leading to the notion of an equivalence class of symmetric systems. Given an equivalence class, then, it makes sense to investigate what important control-theoretic properties remain invariant across the equivalence class. If present, identification of such invariant properties will be very important for controls engineers because an analysis of only one system in the equivalence class (presumably the simplest) leads to proven conclusions regarding the properties of *all* the different systems in the equivalence class.

The author's prior work has focused on stability of symmetric systems in [5] and stability of *approximately* symmetric systems in [4], [3]. The results in this paper are an important extension of these prior results in that they focus on the complementary problem of *boundedness*. A prototypical problem in boundedness is that an otherwise asymptotically stable system is subjected to a persistent bounded nonautonomous input. If this input is "small" then for large values of the state, the system behaves, intuitively, as if the origin is asymptotically stable. However, as trajectories approach the origin, the inputs become significant, destroying the asymptotic stability of the origin. If the magnitude of the nonautonomous inputs is bounded, trajectories remain within a bound of the equilibrium point.

This paper considers the application of boundedness results to symmetric systems with the goal of determining conditions under which bounds for solutions for an entire equivalence class of systems can be inferred. As mentioned, this work is an extension of the work presented in [5], but it is also related to other efforts reported in [14], [12], [13], [15], [16], [11]. The main application focus in those results and the results in this paper is formation control of mobile robotic systems, of which there is a vast literature such as in [7], [19], [2], [20], [10], [17] and many others. Much more limited focus has been addressed toward symmetries, such as in [1], [23], [6]. The results in this paper are also closely related to Input to State Stability, and the technical relationship with those results are discussed subsequently.

The main idea of the result in this paper is that if a system is characterized by a symmetry, then there is quite a bit if structure present in the equations of motion that may be exploited for control and analysis purposes, which was the basis for our prior results in [5]. The results in that paper formalized the definition of a symmetric system, and based on that definition defined an equivalence relation among symmetric systems with a different number of components. Then it determined conditions under which stability is a property that is invariant across the equivalence class of systems defined by the equivalence relation. In this paper, we start with the basis of the equivalence class of symmetric system and determined conditions for boundedness of solutions *across the equivalence class*, hence having the results hold for many systems with different numbers of agents.

The rest of this paper is organized as follows. Section II reviews, from [5], the definition of a symmetric system, equivalence relations among different symmetric systems and equivalence classes of symmetric systems. Section III presents the boundedness result. Section IV presents an example of the application of these results. Finally, Section V outline conclusions and future work.

The support of the US National Science Foundation under the CPS Large Grant No. CNS-1035655 is gratefully acknowledged.

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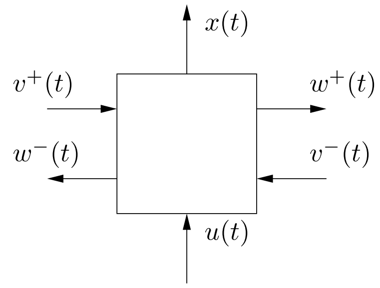

Fig. 1. System building block in one spatial dimension.

II. Symmetric Systems

This section gives an overview of symmetric systems and the relationship among symmetric systems with different numbers of components. Then it extends the definition of a symmetric system from [5] to the nonautonomous case.

As a simple starting point, consider the "basic building block" in one spatial dimension illustrated in Figure 1. The $w^-(t)$ are the outputs from the component and u, $v^-(t)$ and $v^+(t)$ are the inputs. The signals $v^\pm$ represent coupling with the other components and u are the control inputs. We consider nonlinear systems of the usual form

$$\begin{aligned} \dot{x}_i(t) &= f_i\big(x_i(t)\big) + \sum_{j=1}^{m_i} g_{i,j}\big(x_i(t)\big)u_{i,j}\,(t) \\ w_i^-(t) &= w_i^-\,(x_i(t)) \\ w_i^+(t) &= w_i^+\,(x_i(t))\,. \end{aligned} \tag{1}$$

If the system is controlled via feedback, this is manifested in that the outputs from the neighbors appear in the control input for component i in Equation 1, *i.e,*,

$$u_{i,j}(t) = u_{i,j}\left(x_i(t), w_{i-1}^+(x_{i-1}(t)), w_{i+1}^-(x_{i+1}(t))\right). \tag{2}$$

We will consider a general form in order to have some control inputs defined via feedback and some inputs defined open loop and only depending on time, *i.e.*, the nonautonomous terms:

$$\begin{aligned} &u_{i,j}(t) = \\ &\quad u_{i,j}\left(x_i(t), w_{i-1}^+(x_{i-1}(t)), w_{i+1}^-(x_{i+1}(t))\right) + \hat{u}_{i,j}(t). \end{aligned}$$

The $\hat{u}_{i.j}$ term will be the symmetry-breaking term that this paper addresses.

In order to deal with systems with a more general interconnection topology, we consider systems defined on groups. Recall, *group* is a set, G with

1) a binary associative operation, $\sigma : G \times G \to G$,
2) an identity element e such that $\sigma(e,g) = \sigma(g,e) = g$ for all $g \in G$, and
3) for every $g \in G$ there exists an element $g^{-1} \in G$ such that $\sigma(g, g^{-1}) = \sigma(g^{-1}, g) = e$,

where $|G|$ denotes the number of elements in a set G. We will normally write $g_1 g_2$ instead of $\sigma(g_1, g_2)$.

An important aspect of this work is that the interconnection structure of the system will be represented by a set of *generators*, denoted by X, for the group upon which the system is defined. If X is a subset of a group G, then the smallest subgroup of G containing X is called the *subgroup generated by* X. For the rest of this paper we will assume that if $s \in X$, then $s^{-1} \in X$ as well. *Relations* define constraints among the generators, and are of the form $s_1 s_2 \ldots s_m = e$ for $s_1, \ldots, s_m \in X$. Finally, we represent systems by a *Cayley graph*, which is a directed graph with vertices that are the elements of a group, G, generated by the subset X, with a directed edge from g_1 to g_2 only if $g_2 = sg_1$ for some $s \in X$. A directed edge from node g_1 to g_2 represents that a coupling input to g_2 is equal to an output from g_1, (see [21], [18] for a more extensive exposition) .

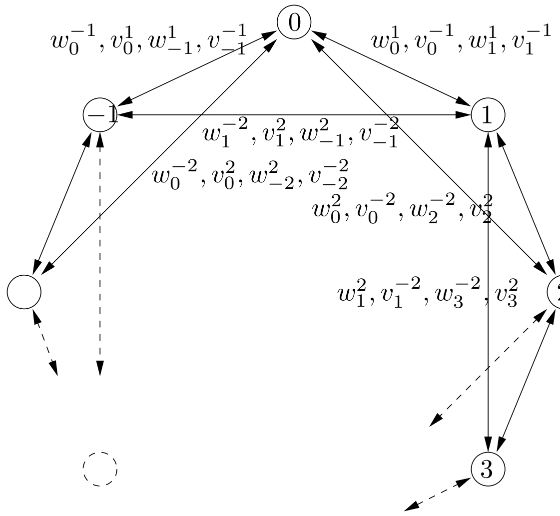

Fig. 2. System topology for Example 2.

Example 1: Consider the ring of components illustrated in Figure 2. Each vertex has edges connecting to four other vertices and hence the system is generated by four elements. Let g denote a vertex, *i.e.*, $g \in \{-2, -1, 0, 1, \ldots, N-3\} = G$. Consider the subset of generators $X = \{-2, -1, 1, 2\}$, the group operation to be addition and the relation $s^N = e = 0$. This relation makes the group operation of addition to be $\bmod N$, and hence the group is the quotient of the set of integers $\mathbb{Z}$ where elements of $\mathbb{Z}$ that differ by an integer multiple of N are equivalent. The Cayley graph is illustrated in Figure 2. ◇

Notationally, for a system on the group G with the set of generators $X = \left\{s_1, s_2, \ldots, s_{|X|}\right\}$, we let x_g denote the state variable corresponding to $g \in G$, $Xg = \left\{s_1 g, s_2 g, \ldots, s_{|X|} g\right\}$ denote the set of neighbors of component $g \in G$, x_{Xg} denote the states of the neighbors of $g \in G$, and x_{XXg} denote the states of the neighbors of the neighbors. For a component g, $\left\{w_g^{s_1}, w_g^{s_2}, \ldots, w_g^{s_{|X|}}\right\}$ denotes the set of outputs, and similarly $\left\{v_g^{s_1}, v_g^{s_2}, \ldots, v_g^{s_{|X|}}\right\}$ denotes the set of inputs. In this more general context, the dynamics of a component, $g \in G$ are represented by

$$\begin{aligned}
\dot{x}_g(t) &= f_g\left(x_g(t)\right) \\
&\quad + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), v_g^{s_1}(t), , \ldots, v_g^{s_{|X|}}(t)\right) \\
&\quad + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) \hat{u}_{g,j}(t) \\
w_g^s(t) &= w_g^s\left(x_g(t)\right),
\end{aligned} \tag{3}$$

for all $s \in X$.

Definition 1: Let G be a group with a set of generators, X. A system with components $g \in \mathcal{I} \subset G$ with dynamics given by Equation 3 has *periodic interconnections on* $\mathcal{I}$ if

$$v_g^s(t) = w_{s^{-1}g}^s\left(x_{s^{-1}g}(t)\right), \tag{4}$$

for all $g \in \mathcal{I}$ and $s \in X$. Furthermore, if

$$\begin{aligned}
f_{g_1}(x) &= f_{g_2}(x), & g_{g_1,j}(x) &= g_{g_2,j}(x), \\
w_{g_1}^s(x) &= w_{g_2}^s(x), & m_{g_1} &= m_{g_2} = m
\end{aligned} \tag{5}$$

for all $s \in X$, $g_1, g_2 \in \mathcal{I}$, $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$, then $\mathcal{I}$ forms an *orbit of symmetric components*. Finally, if the feedback part of the control laws also satisfy

$$\begin{aligned}
&u_{g_1,j}\left(x_1, w_{s_1^{-1}g_1}^{s_1}(x_2), \ldots, w_{s_{|X|}^{-1}g_1}^{s_{|X|}}(x_{|X|+1})\right) = \\
&\qquad u_{g_2,j}\left(x_1, w_{s_1^{-1}g_2}^{s_1}(x_2), \ldots, w_{s_{|X|}^{-1}g_2}^{s_{|X|}}(x_{|X|+1})\right)
\end{aligned} \tag{6}$$

for all $g_1, g_2 \in \mathcal{I}$, $j \in \{1, \ldots, m\}$, $s \in X$ and $(x_1, x_2, \ldots, x_{|X|+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ then the elements of $\mathcal{I}$ form a *symmetry orbit*. Such a system with a symmetry orbit is called a *symmetric system with nonautonomous inputs on* $\mathcal{I}$. If $\mathcal{I} = G$ it is a *symmetric system with nonautonomous inputs on* G. ◇

Our prior work in [5] presented stability results when the nonautonomous inputs, $\hat{u}_{i,j}$ were identically zero. This paper extends those results to the case where these terms are not zero.

Example 2: Returning to the example, consider each agent to have the dynamics

$$\dot{x}_i = -x_1 + \sum_{j \in \mathcal{N}_i} k_{i,j} \sin \omega_{i,j} t \tag{7}$$

where $|k_i| < \alpha$ is a bounded coefficient (which also may be zero and possibly unknown). This is a symmetric system with

$$\begin{aligned}
f_i &= -x_i \\
g_{i,j} &= k_{i,j} \\
u_{i,j} &= 0 \\
\hat{u}_{i,j} &= \sin \omega_{i,j} t.
\end{aligned}$$

Note that the definition of symmetric system does not include the nonautonomous input term, $\hat{u}_{i,j}$, so these terms may be different, and hence in this specific example, the k_i and ω_i may be different for different nodes.

Now, we will define two systems to be *equivalent* if they have symmetry orbits with identical components which are interconnected in the same manner, but they possibly have a different number of components in the symmetry orbit.

Definition 2: Two symmetric systems on the finite groups G_1 and G_2 are *equivalent* if G_1 and G_2 are generated by the same set of generators, X,

$$\begin{aligned}
f_{g_1}(x) &= f_{g_2}(x), & g_{g_1,j}(x) &= g_{g_2,j}(x), \\
w_{s^{-1}g_1}^s(x) &= w_{s^{-1}g_2}^s(x)
\end{aligned} \tag{8}$$

and the feedback part of the control laws satisfy

$$\begin{aligned}
&u_{g_1,j}\left(x_1(t), w_{s_1^{-1}g_1}^{s_1}(x_2(t)), \ldots, w_{s_{|X|}^{-1}g_1}^{s_{|X|}}(x_{|X|+1}(t))\right) = \\
&u_{g_2,j}\left(x_1(t), w_{s_1^{-1}g_2}^{s_1}(x_2(t)), \ldots, w_{s_{|X|}^{-1}g_2}^{s_{|X|}}(x_{|X|+1}(t))\right)
\end{aligned} \tag{9}$$

for all $g_1 \in G_1$, $g_2 \in G_2$, $s \in X$, $x \in \mathbb{R}^n$, $\left(x_1, x_2, \ldots, x_{|X|+1}\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ and $j \in \{1, \ldots, m\}$ where $m = m_{g_1} = m_{g_2}$. ▷

Example 3: Continue Example 2 and consider two symmetric systems with component dynamics given by Equation 7 and belonging to

$$\begin{aligned}
G_1 &= \{-2, -1, 0, 1, 2, \ldots, N-3\} \\
G_2 &= \{-2, -1, 0, 1, 2, \ldots, M-3\}
\end{aligned}$$

where $M > N$. Because the dynamics of all the components are identical and the feedback definitions are identical, these systems are equivalent. Both have generating sets $X = \{-2, -1, 1, 2\}$ with the only difference being the relation for G_1 is $s^N = 0$ and the relation for G_2 is $s^M = 0$. ◇

III. BOUNDEDNESS OF SOLUTIONS FOR NONAUTONOMOUS SYMMETRIC SYSTEMS

This section presents the main result, Proposition 1. It is based upon the following standard result from [9] (Theorem 4.18), which we restate here for the basis of the proof of Proposition 1.

Theorem 1: Consider

$$\dot{x} = f(t, x). \tag{10}$$

Let $\mathcal{D} \subset \mathbb{R}^n$ be a domain that contains the origin and $V : [0, \infty) \times \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1\left(\|x\|\right) \leq V(t, x) \leq \alpha_2\left(\|x\|\right) \tag{11}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0 \tag{12}$$

$\forall t \geq 0$ and $\forall x \in \mathcal{D}$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset \mathcal{D}$ (B_r is the ball of radius r) and suppose that

$$\mu < \alpha_2^{-1}\left(\alpha_1(r)\right).$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}\left(\alpha_1(r)\right)$, there is a

$T \geq 0$ (dependent on $x(t_0$ and μ) such that the solution of Equation 10 satisfies

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t-t_0\right), \quad \forall t_0 \leq t \leq t_0+T \quad (13)$$
$$\|x(t)\| \leq \alpha_1^{-1}\left(\alpha_2(\mu)\right), \quad \forall t \geq t_0+T. \quad (14)$$

Moreover, if $\mathcal{D} = \mathbb{R}^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then Equations 13 and 14 hold for any initial state $x(t_0)$, with no restriction on how large μ is. ◁

The following is the main result in this paper.

Proposition 1: Given a symmetric system on a finite group G with generators X, assume there is a function $V_G : \mathcal{D}_G \to \mathbb{R}$ that is smooth on some open domain $\mathcal{D}_G \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($|G|$ times) such that

1) V_G may be expressed as the sum of terms corresponding to each component where

$$V_g : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{1+|X|\text{times}} \to \mathbb{R}$$

$$V_G(x_G) = \sum_{g\in G} V_g\left(x_g, x_{Xg}\right)$$
$$= \sum_{g\in G} V_g\left(x_g, w^{s_1}_{s_1^{-1}g}(x_{s_1^{-1}g}), \ldots, w^{s_{|X|}}_{s_{|X|}^{-1}g}(x_{s_{|X|}^{-1}g})\right), \quad (15)$$

for all $x \in \mathcal{D}_G$, where the individual functions corresponding to each component in G are equal as functions, *i.e.*,

$$V_{g_1} = V_{g_2} = V \quad (16)$$

for all $g_1, g_2 \in G$, and where V_g satisfies Equation 11,

2) there exists a positive constant, c such that

$$\left\|\sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right)\hat{u}_{g,j}(t)\right\| < c \quad (17)$$

for all $g \in \hat{G}$, and,

3) for **any one of the** $g \in G$,

$$\frac{\partial V_G}{\partial x_g}(x_g, x_{X_g})\left[f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right]$$
$$\leq -W_4(x_g, x_{X_g}) - c\left\|\frac{\partial V_G}{\partial x_g}(x_g, x_{X_g})\right\| \quad (18)$$

for all $x_G \in \{\mathcal{D}_G | \, \|x_G\| > \mu_G\|$ where $W_4(x)$ is a positive definite function.

Then

1) the system satisfies Theorem 1, and
2) for any equivalent symmetric system on $\hat{G}$, if Equation 17 is satisfied for all the nonhomogeneous terms, then the equivalent system also satisfies Theorem 1.

◁

Remark 1: The utility of this Proposition is that if a symmetric nonautonomous system satisfies the requirements of this Proposition for bounded solutions, so will any equivalent symmetric system as long as the nonautonomous terms are bounded. Furthermore, an additional benefit of this proposition is that the computations are simplified, even for the original system.

Proof: First we show that $\dot{V}_G \leq -W_3(x)$ and then we will show that any equivalent system, $\hat{G}$ is such that $\dot{V}_{\hat{G}} \leq -\hat{c}W_4(x) = -W_3(x)$, where $\hat{c}$ is a positive constant.

Because the Lyapunov functions corresponding to each component are identical, we may take

$$\mathcal{D}_G = \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{|G|\text{times}} \quad (19)$$

for some subset $\mathcal{D} \subset \mathbb{R}^n$. Note that for $h \in G$, because only V_h and its neighbors depend on x_h,

$$\frac{\partial V_G}{\partial x_h}(x) = \frac{\partial}{\partial x_h}\left(\sum_{g\in G} V_g\left(x_g, x_{Xg}\right)\right)$$
$$= \frac{\partial}{\partial x_h}\left(\sum_{s=e, s\in X} V\left(x_{sh}, x_{Xsh}\right)\right)$$

where e is the identity element in G. Hence,

$$\dot{V}_G(x) = \sum_{g\in G}\left[\frac{\partial}{\partial x_g}\left(\sum_{s=e,s\in X} V\left(x_{sg}, x_{Xsg}\right)\right)\right.$$
$$\left.\cdot\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)\left(u_{g,j}\left(x_g, x_{Xg}\right) + \hat{u}_{g,j}(t)\right)\right)\right]$$

However, by Equation 17

$$\dot{V}_G = \sum_{g\in G}\left[\frac{\partial V_G}{\partial x_g}\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}u_{g,j}\left(x_g, x_{Xg}\right)\right)\right.$$
$$\left. + \frac{\partial V_G}{\partial x_g} g_{g,j}(x_g)\hat{u}_{g,j}(t)\right]$$
$$\leq \sum_{g\in G}\left[\frac{\partial V_G}{\partial x_g}\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}u_{g,j}\left(x_g, x_{Xg}\right)\right)\right.$$
$$\left. + c\left\|\frac{\partial V_G}{\partial x_g}\right\|\right].$$

By hypothesis, one of the terms in the sum is bounded by a negative definite function, and we will show this implies that all of the terms are similarly bounded and hence Equation 12 is satisfied by V_G.

For a given g, the first term in square brackets (corresponding to the autonomous portion due to state feedback) is a function with a domain that is the Cartesian product among the states of g, the states of the neighbors of g and the states of the neighbors of the neighbors of g, which is a set of the form $\mathcal{D} \times \cdots \times \mathcal{D}$, which is a subset of $\mathcal{D}_G$ (not necessarily a proper subset). We will show for these autonomous terms every term in the series is equal to every other term *as functions*. Hence, because the domains of each function is restricted to the same range of values, and the nonautonomous terms are each bounded, then each term must be bounded by a negative definite function and hence the sum is as well.

Consider any two $g_1, g_2 \in G$. Because of the definition of a symmetric system, $f_{g_1} = f_{g_2}$ and $g_{g_1,j} = g_{g_2,j}$ as vector fields (Equation 8) and $u_{g_1,j} = u_{g_2,j}$ as functions (Equation 9). Finally, if we define the mappings corresponding to the differentials by

$$D_g V : \mathcal{D} \times \cdots \times \mathcal{D} \to \mathbb{R}^n$$
$$D_g V(x_g, x_{Xg}, x_{XXg}) = \frac{\partial}{\partial x_g} \left(\sum_{s=e, s \in X} V(x_{sg}, x_{Xsg}) \right),$$

the differentials corresponding to different components are equal *as differentials i.e.*, $D_{g_1} V = D_{g_2} V$. Hence, as functions, each of the first terms in the square brackets are equal, and because the domain of each is restricted to the same set of values, each term has the same bound. Hence, by Equations 17 and 18, we have

$$\dot{V}_G \leq - |G| \, W_4(x).$$

Furthermore, because each V_g individually satisfies Equation 11, V_G satisfies $|G| \, \alpha_1 (\|x\|) \leq V_G(x) \leq |G| \, \alpha_2 (\|x\|)$, and hence also satisfies Equation 11.

For any equivalent system, the argument follows in exactly the same manner. ■

Remark 2: This result is very similar to input to state stability, and we could appeal to results in that area for our proof. However, note that the majority of the work was to show that the Lyapunov function had the correct properties, which would be very similar to the work necessary to show that the overall Lyapunov function satisfied the conditions for ISS. Either route to the result is valid and we decided to directly prove it to help elucidate the structure of symmetric systems and the connection between that structure and system properties.

IV. EXAMPLE

This section will complete the Example.

Example 4: Continuing Example 2, take as a Lyapunov function

$$V(x) = \sum_{i=1}^{N} V_i(x_i, x_{X_i}) = \sum_{i=1}^{N} \frac{1}{2} x_i^2$$

which is of the form required by Equation 15. Also, the nonautonomous input for each agent satisfies

$$\left\| \sum_{j \in \mathcal{N}_i} k_{i,j} \sin \omega_{i,j} t \right\| < 4\alpha.$$

Finally, for the ith agent, Equation 18

$$\dot{V}_i = -x_i^2 \leq -\frac{1}{2} x_i^2 - 4\alpha x_i$$

for $|x_i| > 8\alpha$. Hence, all the conditions required for boundedness are satisfied.

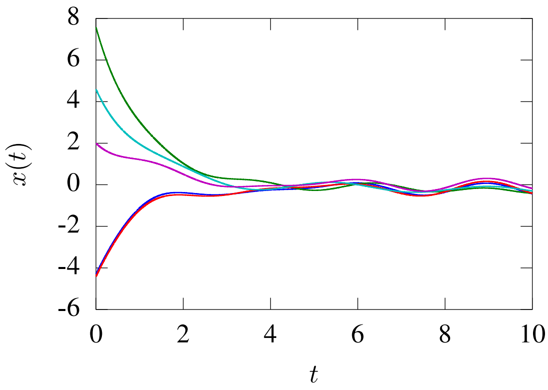

Fig. 3. Bounded solutions for five-agent system.

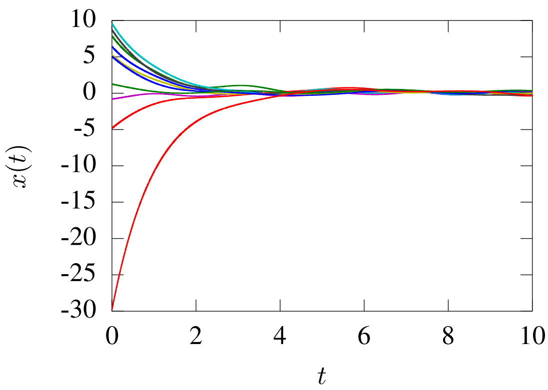

Fig. 4. Bounded solutions for ten-agent system.

Solutions for both 5 and 10 agent systems are illustrated in Figures 3 and 4, respectively. For the five-agent system, we used the same k_j and ω_j for each agent, specifically

$$k_i = \begin{bmatrix} 0.007 \\ 0.462 \\ -0.142 \\ 0.405 \\ 0.425 \end{bmatrix}, \qquad \omega_i = \begin{bmatrix} 1.79 \\ 1.68 \\ -1.33 \\ 2.42 \\ -0.07 \end{bmatrix}$$

and for the ten-agent system we used

$$k_i = \begin{bmatrix} 0.393 \\ -0.348 \\ 0.496 \\ -0.304 \\ 0.282 \\ 0.172 \\ -0.241 \\ -0.038 \\ 0.230 \\ 0.453 \end{bmatrix}, \qquad \omega_i = \begin{bmatrix} 0.37 \\ -1.73 \\ 2.97 \\ -2.60 \\ -0.91 \\ -2.54 \\ 0.56 \\ 1.30 \\ -0.82 \\ -1.86 \end{bmatrix}$$

Example 5: As a final example, consider a more extreme nonlinear coupling between the agents of the form

$$\dot{x}_i = -x_i + \frac{k_j}{1 + e^{x_j}} \sin \omega_j t.$$

In this case, the vector field $g_{i,j}$ is a nonlinear function of x_j. However, Equation 17 is still satisfied due to the form of

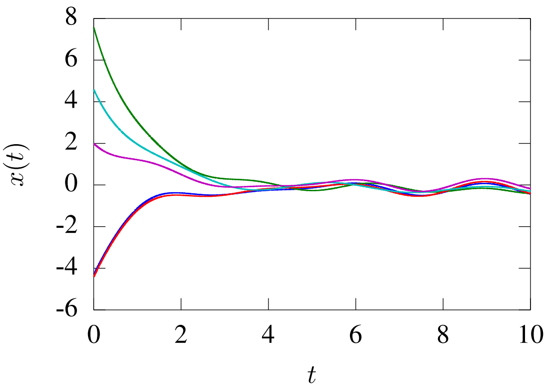

Fig. 5. Bounded solutions for eight agent system for Example 5.

$g_{i,j}$, and hence the solutions are correspondingly bounded. Simulation results are illustrated in Figure 5.

V. CONCLUSIONS AND FUTURE WORK

This paper extends the prior work of the author which considers stability of classes of symmetric systems to consider boundedness of symmetric systems with persistent nonautonomous inputs. The value of that prior work was that if a controls engineer checks the stability of only one symmetric system, then it is automatically the case that stability of all symmetric systems equivalent to it is guaranteed. This paper extends those results to *boundedness* wherein the nonautonomous term does not have to be symmetric. The main contribution of this work is that boundedness of a single system in an equivalence class of systems implies boundedness for the entire equivalence class as long as the nonautonomous components of the inputs are bounded.

The main thrust of future work related to this is directed toward a more useful result for the multi-agent formation control problem. Because formation problems normally do not have an equilibrium at the origin of the system, we will require a boundedness result for symmetric system relative to *sets of equilibria*. Such results are not unknown; however, they are beyond the scope of the common texts on Lyapunov stability such as [9]. These results pose computational difficulties because they naturally involve functions of distances from *sets*, which do not naturally lend themselves to common methods for bounding quantities.

VI. ACKNOWLEDGMENTS

The support of the US National Science Foundation under the CPS Large Grant No. CNS-1035655 is gratefully acknowledged. The author would also like to thank an anonymous reviewer of an earlier version of this paper, who provided many helpful comments and suggested corrections or clarifications.

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